Izračunati zapreminu tijela ograničenog valjkom $x^2+y^2=6x$ i ravnima $x-z=0$, $5x-z=0$.

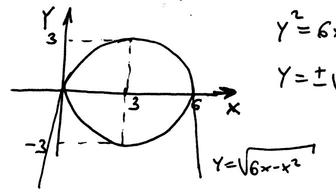
Rj. $V=\iiint\limits_{\Omega} dx\,dy\,dz$

$x^2+y^2=6x$

$x^2-2\cdot x\cdot 3+3^2-3^2+y^2=0$

$(x-3)^2+y^2=3^2$

projekcija valjka na xOy ravan izgleda



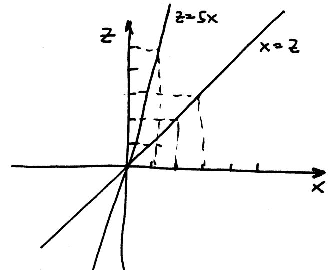
$Y^2=6x-x^2$

$Y=\pm\sqrt{6x-x^2}$

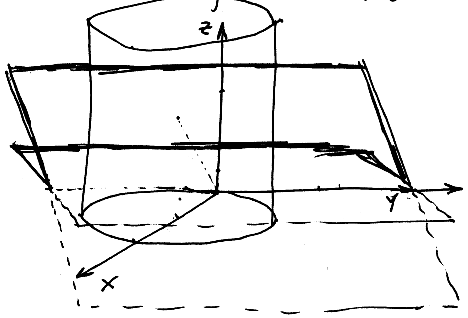
$x-z=0$ $\quad$ $5x-z=0$

$x=z$ $\quad$ $z=5x$

projekcije ravni $x-z=0$ i $5x-z=0$ na xOz ravan izgleda



Skica ovih figura u prostoru bi otprilike izgledala ovako



valjak presječen sa dvije ravni

na klasičan način

$$\Omega: \begin{cases} 0<x<6 \\ 0<y<\sqrt{6x-x^2}=\sqrt{9-(x-3)^2} \\ x\le z\le 5x \end{cases}$$

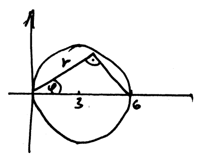
uvedimo cilindrične koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$dx\,dy=r\,dr\,d\varphi$



$\cos\varphi=\frac{r}{6}$

Primjetimo da je oblast Ω simetrična u odnosu na xOz ravan

$$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz \qquad \Omega': \begin{cases} 0\le\varphi\le\frac{\pi}{2} \\ 0\le r\le 6\cos\varphi \\ r\cos\varphi<z<5r\cos\varphi \end{cases}$$

$$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=2\int_0^{\pi/2}d\varphi\int_0^{6\cos\varphi}r\,dr\underbrace{\int_{r\cos\varphi}^{5r\cos\varphi}dz}_{=4r\cos\varphi}=8\int_0^{\pi/2}\cos\varphi\,d\varphi\int_0^{6\cos\varphi}r^2\,dr=8\int_0^{\pi/2}\frac{1}{3}r^3\Big|_0^{6\cos\varphi}\cos\varphi\,d\varphi$$

$$=8\cdot\frac{6^3}{3}\int_0^{\pi/2}\cos^4\varphi\,d\varphi=576\int_0^{\pi/2}\left(\frac{1}{2}(1+\cos2\varphi)\right)^2d\varphi=144\int_0^{\pi/2}(1+2\cos2\varphi+\cos^2 2\varphi)\,d\varphi=\underbrace{\dots}_{\text{za vježbu}}=108\pi$$

tražena zapremina

Izračunati zapreminu tijela ograničenog ravninom XOY, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

Rj. Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi $V=\iiint\limits_{\Omega} dx\,dy\,dz$. Pokušajmo skicirati tijelo čiju zapreminu tražimo.

valjak $x^2+y^2=2ax$

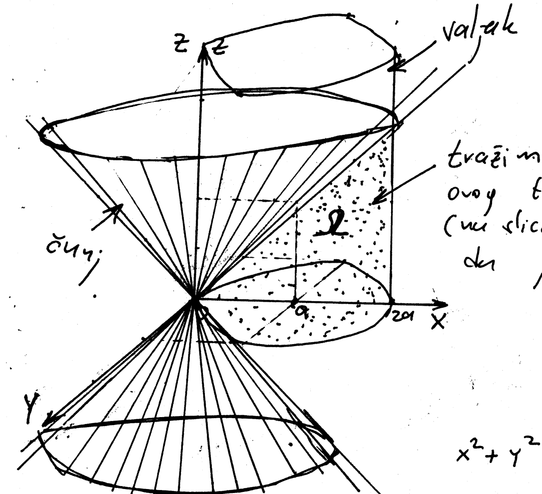
$x^2-2ax+y^2=0$

$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

$(x-a)^2+y^2=a^2$

valjak u presjeku sa XOY ravni je krug sa centrom u tački $(a,0)$ poluprečnika a

čunj $x^2+y^2=z^2$ u presjeku sa XOY ravni je tačka, a u presjeku sa YOZ ili sa XOZ su po dvije prave



tražimo zapreminu ovog tijela (na slici smo pretpostavili da je $a>0$)

Oblast Ω je najlakše projicirati na XOY ravan.

Uvodimo cilindrične koordinate

$x=a+r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$$\Omega': \begin{cases} dx\,dy\,dz=r\,dr\,d\varphi\,dz \\ 0\le r\le a \\ 0\le\varphi\le 2\pi \end{cases}$$

$x^2+y^2=z^2$

$z=\pm\sqrt{x^2+y^2}$ čunj

$x^2+y^2=(a+r\cos\varphi)^2+(r\sin\varphi)^2=$

$=a^2+2ar\cos\varphi+r^2\cos^2\varphi+r^2\sin^2\varphi=$

$=a^2+2ar\cos\varphi+r^2$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=\int_0^a dr\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2+2ar\cos\varphi+r^2}} r\,dz=\dots$$

ao je teško izračunati

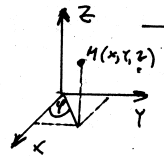
Pokušajmo uvesti drugačije smjene.

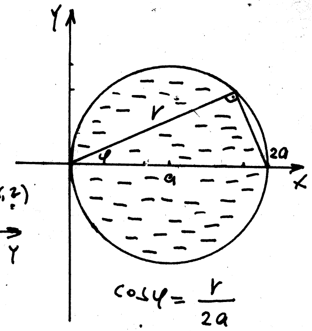
$x = r\cos\varphi$ $\qquad dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$y = r\sin\varphi$

$z = z$ $\qquad x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$

$$\Omega'': \begin{cases} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2a\cos\varphi \\ 0 \le z \le \sqrt{r^2} \end{cases}$$





$\cos\varphi = \frac{r}{2a}$

$r = 2a\cos\varphi$

$$V = \iiint_{\Omega} dx\,dy\,dz = \iiint_{\Omega''} r\,dr\,d\varphi\,dz =$$

$$= \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} dr \int_0^{r} r\,dz = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} \left(r z\Big|_0^r\right) dr = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} r^3\Big|_0^{2a\cos\varphi}\right) d\varphi = \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\varphi\,d\varphi = \frac{8}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi$$

$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\cos^2\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi(1-\sin^2\varphi)\,d\varphi = \left| \begin{matrix} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi = -\frac{\pi}{2} \Rightarrow t = -1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{matrix} \right|$$

$$= \int_{-1}^{1} (1-t^2)\,dt = t\Big|_{-1}^{1} - \frac{1}{3}t^3\Big|_{-1}^{1} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$$V = \frac{32}{9} a^3 \quad \text{tražena zapremina}$$

II način: $V = \iint_S f(x,y)\,dx\,dy$ uvedimo smjene $x = r\cos\varphi$, $y = r\sin\varphi$, $0 \le \varphi \le 2a\cos\varphi$, $-\frac{\pi}{2} \le r \le \frac{\pi}{2}$

$$V = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr$$

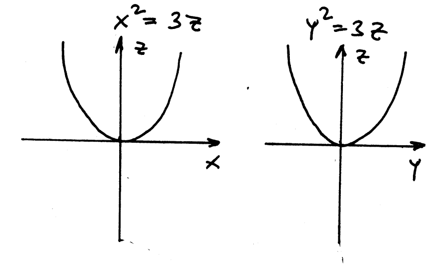
ZAVRŠITI ZA VJEŽBU

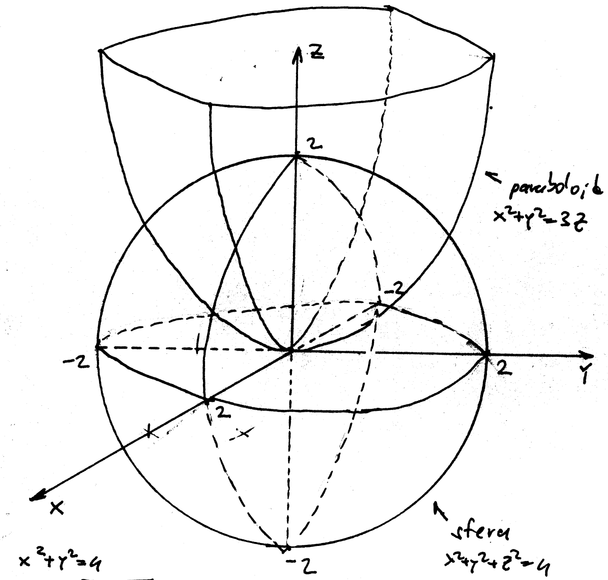
Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u $(0,0,0)$ poluprečnika 2

$x^2+y^2=3z$ je paraboloid

Skicirajmo ova dva tijela





$$V = \iiint_{\Omega} dx\,dy\,dz$$

Primjetimo da je tijelo dobijeno presjekom simetrično na ravni xOz i na yOz. Prema tome

$$V = 4\iiint_{\Omega_1} dx\,dy\,dz \quad \text{gdje je}$$

Ω_1 oblast u presjeku dva tijela u prvom oktantu

$x^2+y^2=4$

$x = \pm\sqrt{4-y^2}$

$y = \pm\sqrt{4-x^2}$

$$\Omega_1: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \\ 0 \le z \le \frac{1}{3}(x^2+y^2) \end{cases}$$

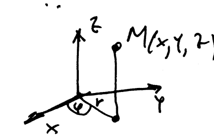
$$V = 4\int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{\frac{1}{3}(x^2+y^2)} dz = 4\int_0^2 dx \int_0^{\sqrt{4-x^2}} \frac{1}{3}(x^2-y^2)\,dy$$

$$= \frac{4}{3}\int_0^2 \left(x^2 y\Big|_0^{\sqrt{4-x^2}} + \frac{1}{3}y^3\Big|_0^{\sqrt{4-x^2}}\right) dx = \ldots = \frac{8\pi}{3}$$

komplikovano

II način:

Uvedimo cilindrične koordinate $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$ $\qquad dx\,dy\,dz = r\,dr\,d\varphi\,dz$



Oblast $\Omega_1 \xrightarrow{\text{transformacije}} \Omega_1' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \frac{1}{3} r^2 \end{cases}$

$$V = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2} dr \int_0^{\frac{1}{3}r^2} r\,dz = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2} r\cdot\frac{1}{3}r^2\,dr = \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4}r^4\Big|_0^2 d\varphi = \frac{1}{3}\cdot 16\cdot\frac{\pi}{2} = \frac{8\pi}{3}$$

$V = \frac{8\pi}{3}$ tražena zapremina

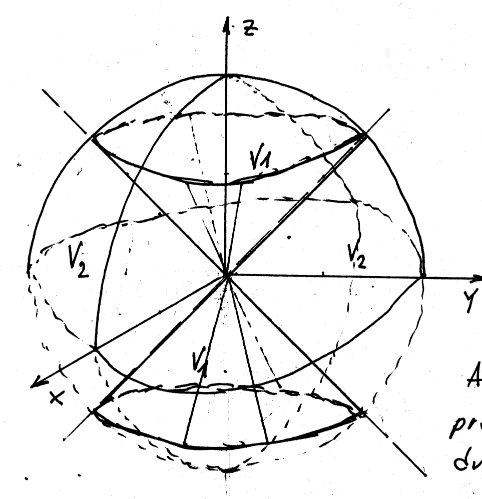
Izračunati zapreminu tijela koje je ograničeno površima $z^2 = x^2 + y^2$, $x^2+y^2+z^2=4$.

Rj.

$x^2+y^2+z^2=4$ je kugla sa centrom u $(0,0,0)$ poluprečnika $r=2$

$z^2 = x^2+y^2$ je konus

Skicirajmo ove dvije figure u prostoru.



Presjek konusa i kugle daje dva tijela za koje možemo računati zapreminu: prvo tijelo je određeno u presjeku unutrašnjosti konusa i kugle, a drugo tijelo je određeno dijelom lopte van konusa.

Ako sa V_1 označimo zapreminu prvog, a sa V_2 zapreminu drugog tijela, imamo da je

$V = V_1 + V_2 = \frac{4}{3} r^3 \pi$ (zapremina kugle)

Kako je $r=2 \Rightarrow V = \frac{4}{3}\cdot 8\pi = \frac{32\pi}{3}$.

$V = \iiint\limits_{\Omega} dx\,dy\,dz$ – zapremina tijela ograničenog sa oblašću Ω

Uvedimo sferne koordinate

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho\cos\varphi$

$dx\,dy\,dz = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

$z^2 = x^2+y^2$

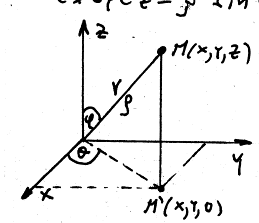
$\rho^2\cos^2\varphi = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta =$

$= \rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta) = \rho^2\sin^2\varphi$

$\Rightarrow \cos^2\varphi = \sin^2\varphi \quad /:\sin^2\varphi$

$tg^2\varphi = 1 \Rightarrow tg\varphi = \pm 1$

$x^2+y^2+z^2=4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = \pm 2$ tj. $\rho = 2$ udaljenost tačke



$\Omega : \begin{cases} z^2 = x^2+y^2 \\ x^2+y^2+z^2=4 \end{cases} \xrightarrow{\text{transformacije}} \Omega' : \begin{cases} tg\varphi = \pm 1 \\ \rho = 2 \end{cases}$

Odredimo granice za drugo tijelo $\Omega'_{V_2}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \end{cases}$

$$V_2 = \iiint\limits_{\Omega'_{V_2}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi \, d\varphi \int_0^2 \rho^2 \, d\rho =$$

$$= 2\pi \cdot (-\cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \frac{1}{3}\rho^3\Big|_0^2 = 2\pi\left(-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} =$$

$$= 2\pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 2\pi\sqrt{2} \cdot \frac{8}{3} = \frac{16\pi\sqrt{2}}{3}$$ traženo rješenje

Zapreminu V_1 sad možemo odrediti na dva načina

I način:

$$V = V_1 + V_2 = \frac{32\pi}{3} \Rightarrow V_1 = \frac{32\pi}{3} - V_2 = \frac{32\pi}{3} - \frac{16\pi\sqrt{2}}{3}$$

$$V_1 = \frac{16\pi}{3}(2 - \sqrt{2})$$ traženo rješenje

II način:

Ako uzmemo u obzir simetričnost date oblasti Ω' u odnosu na xOy-ravan, možemo računati polovinu zapremine V_1 za $z \ge 0$ i tada bi trebalo odabrati sljedeće granice $\Omega'_{V_1}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{cases}$

$$V_1 = \iiint\limits_{\Omega'_{V_1}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$\frac{1}{2}V_1 = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^2 \rho^2 \, d\rho = 2\pi(-\cos\varphi)\Big|_0^{\pi/4} \cdot \frac{\rho^3}{3}\Big|_0^2 =$$

$$= 2\pi\left(1 - \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3}$$

$$\Rightarrow V_1 = 4\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 4\pi \cdot \frac{2 - \sqrt{2}}{2} \cdot \frac{8}{3} = \frac{16\pi}{3}(2 - \sqrt{2}).$$

Izračunati zapreminu tijela koje je određeno oblašću $\Omega: |x+y+z| + |x-y+z| + |x+y-z| = 1$.

Rj. $V = \iiint\limits_{\Omega} dx\,dy\,dz$

uvedimo smjenu $\quad u = x+y+z, \quad v = x-y+z, \quad w = x+y-z \qquad dx\,dy\,dz = J\,du\,dv\,dw$

Jakobijan

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad J^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \frac{I_v + II_v}{II_v + III_v}$$

$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(-4) = 4 \Rightarrow$$

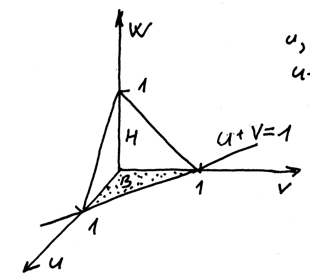
$$\Rightarrow J = \frac{1}{4}$$

pa je

$dx\,dy\,dz = \frac{1}{4}\,du\,dv\,dw$

$\Omega': |u| + |v| + |w| = 1$

$$V = \iiint\limits_{\Omega'} \frac{1}{4}\,du\,dv\,dw$$



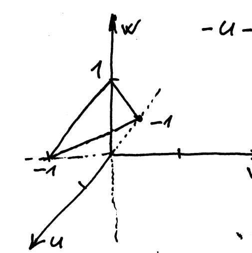
$u, v, w > 0$

$u + v + w = 1$

pored ovoga imamo još 7 slučajeva

npr. $u, v < 0 \quad w > 0$

$-u - v + w = 1$



Vidimo da je dovoljno oblast integrirati u 1. oktantu jer imamo simetričnu oblast po svim oktantima.

$$V = 8 \cdot \frac{1}{4} \iiint\limits_{\Omega''} du\,dv\,dw =$$

$$= 2\int_0^1 du \int_0^{1-u} dv \int_0^{1-u-v} dw = 2\int_0^1 du \int_0^{1-u} w\Big|_0^{1-u-v} dv =$$

$u + v = 1$

$v = 1 - u$

$u + v + w = 1$

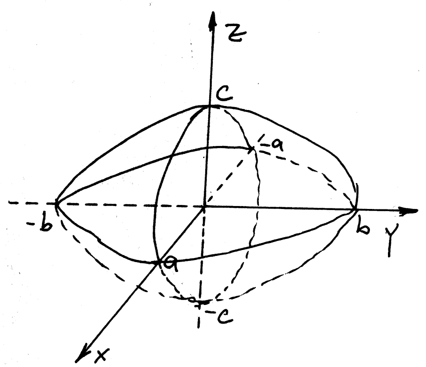
$w = 1 - u - v$

$$= 2\int_0^1 du \int_0^{1-u} (1 - u - v)\,dv = 2\int_0^1 \left(v\Big|_0^{1-u} - uv\Big|_0^{1-u} - \frac{1}{2}v^2\Big|_0^{1-u}\right) du = \ldots = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

Na drugi način: $V_1 = \frac{B \cdot H}{3} = \frac{\frac{1 \cdot 1}{2} \cdot H}{3} = \frac{\frac{1}{2} \cdot 1}{3} = \frac{1}{6}, \quad V = 2 \cdot \frac{1}{6} = \frac{1}{3}$ zapremina tijela

Izračunati zapreminu elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Rj.



$$V=\iiint_S dx\,dy\,dz$$

$$S:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

smjena: uopštene sferne koordinate

$x = a r \sin\varphi\cos\theta \qquad 0\le r\le 1$

$y = b r \sin\varphi\sin\theta \qquad 0\le\varphi\le\pi$

$z = c r \cos\varphi \qquad 0\le\theta\le 2\pi$

$$dx\,dy\,dz = J\,dr\,d\varphi\,d\theta$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta}\end{vmatrix}=\begin{vmatrix}a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\varphi & -cr\sin\varphi & 0\end{vmatrix}$$

$$=abc\begin{vmatrix}\text{ista determinanta}\\ \text{kao kod standardnih}\\ \text{sfernih koordinata}\end{vmatrix}=abc\,r^2\sin\varphi$$

$$V=\int_0^{\pi}d\varphi\int_0^1 dr\int_0^{2\pi}abc\,r^2\sin\varphi\,d\theta=\int_0^{\pi}\sin\varphi\,d\varphi\int_0^1 r^2\,dr\int_0^{2\pi}abc\,d\theta=$$

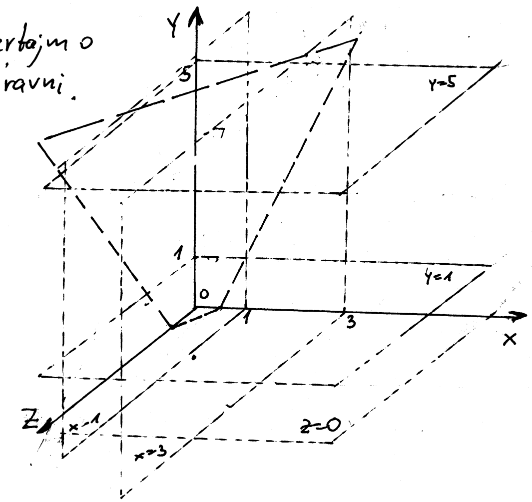
$$=abc\,\theta\Big|_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\int_0^1 r^2\,dr=2\pi abc\int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^1 d\varphi=$$

$$=\frac{2}{3}\pi abc\int_0^{\pi}\sin\varphi\,d\varphi=\frac{2}{3}\pi abc\left(-\cos\varphi\Big|_0^{\pi}\right)=\frac{2}{3}\pi abc\,(1+1)=\frac{4}{3}\pi abc$$

q.e.d.

Naći zapreminu tijela ograničenog ravnima $x=1$, $x=3$, $y=1$, $y=5$, $2x-y+z-1=0$, $z=0$.

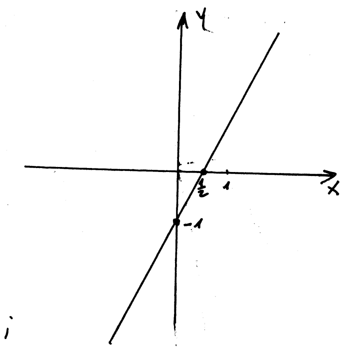
Rj. Nacrtajmo ove ravni.



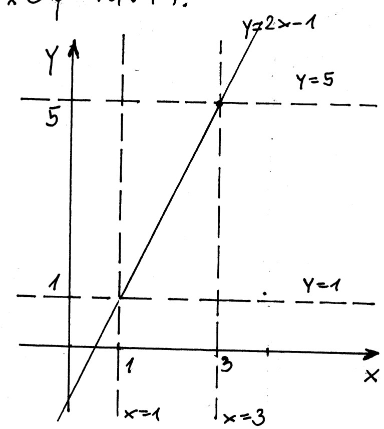
$2x-y+z-1=0$

$z=-2x+y+1$

projekcija ove ravni na xOy ravan



Slika u prostoru je komplikovana i sa nje ne možemo pročitati granice. Nacrtajmo projekcije ovih ravni na xOy ravan.



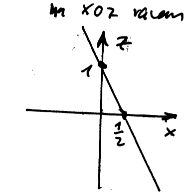
$2x-y-1=0$

$y=2x-1$

$x=3 \Rightarrow y=5$

$x=1 \Rightarrow y=1$

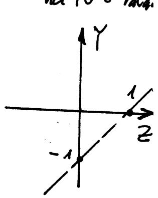
na xOz ravan



$2x+z-1=0$

$z=-2x+1$

na yOz ravan



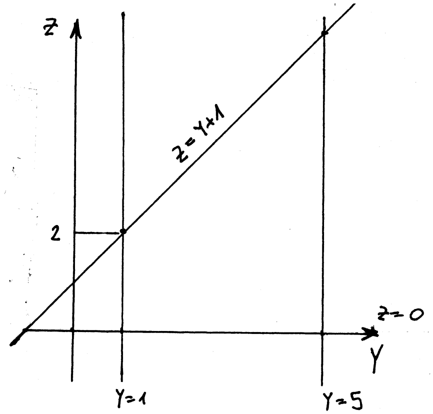
$-y+z-1=0$

$y=z-1$

Sad na osnovu slike u prostoru i projekcija na ravni možemo pročitati granice za tijelo

$$\Omega:\begin{cases}1\le x\le 3\\ 2x-1\le y\le 5\\ 0\le z\le -2x+y+1\end{cases}$$

Da su napisane granice ispravne provjerimo projekcijom ravni na yOz ravan.



$$-y+z-1=0$$
$$z=y+1$$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz =$$

$$=\int_1^3 dx\int_{2x-1}^{5} dy \underbrace{\int_0^{-2x+y+1} dz}_{z\Big|_0^{-2x+y+1}} =$$

$$=\int_1^3 dx\int_{2x-1}^{5}(-2x+y+1)\,dy = \int_1^3\left((-2x)\cdot y\Big|_{2x-1}^{5} + \frac{1}{2}y^2\Big|_{2x-1}^{5} + y\Big|_{2x-1}^{5}\right)dx =$$

$$=\int_1^3\left((-2x)(5-(2x-1)) + \frac{1}{2}\left(5^2-(2x-1)^2\right) + 5-(2x-1)\right)dx =$$

$$=\int_1^3\left((-2x)(6-2x)+\frac{1}{2}\left(25-(4x^2-4x+1)\right)+6-2x\right)dx =$$

$$=\int_1^3\left(-12x+4x^2+\overbrace{\frac{1}{2}(-4x^2+4x+24)}^{-2x^2+2x+12}+6-2x\right)dx = \int_1^3\left(2x^2-12x+18\right)dx$$

$$=\frac{2}{3}x^3\Big|_1^3-\frac{12}{2}x^2\Big|_1^3+18x\Big|_1^3=\frac{2}{3}\cdot 26-\overset{48}{6\cdot 8}+\overset{36}{18\cdot 2}=\frac{52}{3}-12=\frac{16}{3}$$

Zapremina tijela ograničenog spomenutim ravnima iznosi $\frac{16}{3}$.

(#) Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$, $a>0$ u I oktantu.

Rj.

Zapremina tijela ograničenog sa oblasti Ω se računa po formuli $V=\iiint\limits_{\Omega} dx\,dy\,dz$.

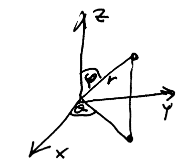
Datu površ $(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$ ne možemo skicirati.

Uvedimo sferne koordinate

$x=r\sin\varphi\cos\theta$

$y=r\sin\varphi\sin\theta$

$z=r\cos\varphi$



$dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$\Omega \xrightarrow{\text{transformacija}} \Omega'$

pa pokušajmo naći granice na osnovu date formule.

$$x^2+y^2+z^2=\underline{r^2\sin^2\varphi}\cos^2\theta+\underline{r^2\sin^2\varphi}\sin^2\theta+r^2\cos^2\varphi=r^2\sin^2\varphi+r^2\cos^2\varphi=r^2$$

$(x^2+y^2+z^2)^3=(r^2)^3=r^6$

$z^2=r^2\cos^2\varphi$

$x^2+y^2=r^2\sin^2\varphi$

$(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$

sad postaje $r^6=\frac{a^6r^2\cos^2\varphi}{r^2\sin^2\varphi}$

tj. $r^6=a^6\operatorname{ctg}^2\varphi$

$r=\sqrt[6]{a^6\operatorname{ctg}^2\varphi}$

$r=a\sqrt[3]{\operatorname{ctg}\varphi}$.

Na osnovu ove formule i znanja da je tijelo u I oktantu možemo zaključiti da je

$$\Omega':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le a\sqrt[3]{\operatorname{ctg}\varphi}\\ 0\le\theta\le\frac{\pi}{2}\end{cases}$$

$$V=\iiint\limits_{\Omega'} r^2\sin\varphi\,dr\,d\varphi\,d\theta=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\varphi\,d\varphi\int_0^{a\sqrt[3]{\operatorname{ctg}\varphi}}r^2dr=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\varphi\,\frac{r^3}{3}\Big|_0^{a\sqrt[3]{\operatorname{ctg}\varphi}}d\varphi$$

$$=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\frac{a^3}{3}\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}\,d\varphi=\frac{a^3}{3}\int_0^{\pi/2}d\theta\int_0^{\pi/2}\cos\varphi\,d\varphi=\frac{a^3}{3}\cdot\theta\Big|_0^{\frac{\pi}{2}}\cdot\sin\varphi\Big|_0^{\frac{\pi}{2}}=\frac{a^3\pi}{6}$$ tražena zapr.

Računanje težišta tijela

U slijedećim zadacima izračunajte koordinate težišta tijela (oblasti) Ω ograničenog datim površima!

1. $\Omega: z^2 = xy \wedge x = 5 \wedge y = 5 \wedge z = 0$.

Rješenje: najprije ćemo izračunati zapreminu date oblasti Ω. Očito je $0 \le z \le \sqrt{xy}$, a iz $z^2 = xy$ slijedi $xy \ge 0$, pa je $0 \le x \le 5 \wedge 0 \le y \le 5$. Zato je

$$V = \int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \int_0^5 \sqrt{x} dx \int_0^5 \sqrt{y} dy = \left(\int_0^5 \sqrt{x} dx \right)^2 = \frac{500}{9}.$$

Dalje imamo da je

$$\overline{x} = \frac{9}{500} \int_0^5 x\, dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \frac{9}{500} \int_0^5 x\sqrt{x}\, dx \int_0^5 \sqrt{y} dy = \frac{9}{500} \int_0^5 x^{\frac{3}{2}}\, dx \int_0^5 y^{\frac{1}{2}} dy = ... = 3.$$

Očigledno je $\overline{x} = \overline{y}$. Najzad,

$$\overline{z} = \frac{9}{500} \int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} z\, dz = \frac{9}{500} \cdot \frac{1}{2} \int_0^5 x dx \int_0^5 y dy = \frac{9}{1000} \left[\frac{x^2}{2} \Big|_0^5 \right]^2 = \frac{9}{1000} \cdot \frac{25}{2} \cdot \frac{25}{2} = \frac{45}{32}.$$

Dakle, težište ima koordinate $T\left(3, 3, \frac{45}{32}\right)$.

2. $\Omega: z = 3 - x^2 - y^2, z = 0.$

Rješenje: Uvešćemo cilindrične koordinate. Tada se Ω preslikava u oblast:
$\Omega': z = 3 - \rho^2, z = 0.$

U presjeku ove dvije površi se dobija kružnica $\rho^2 = 3 \Rightarrow \rho = \sqrt{3}$. Zato je $0 \le \varphi \le 2\pi,\ 0 \le \rho \le \sqrt{3},\ 0 \le z \le 3 - \rho^2$. Odatle slijedi:

$$V = \iiint_{\Omega'} \rho d\varphi\, d\rho\, dz = \int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} dz = 2\pi \int_0^{\sqrt{3}} \rho\left(3 - \rho^2\right) d\rho = 2\pi \int_0^{\sqrt{3}} \left(3\rho - \rho^3\right) d\rho =$$

$$= 2\pi \left(3\frac{\rho^2}{2} - \frac{\rho^4}{4} \right) \Bigg|_0^{\sqrt{3}} = 2\pi \left(\frac{9}{2} - \frac{9}{4} \right) = \frac{9\pi}{2}.$$

Sada možemo izračunati koordinate težišta tijela:

$$\overline{x} = \frac{1}{V} \iiint_{\Omega} x\, dx\, dy\, dz = \frac{2}{9\pi} \iiint_{\Omega'} \rho \cos\varphi \cdot \rho\, d\varphi\, d\rho\, dz = \frac{2}{9\pi} \int_0^{2\pi} \cos\varphi\, d\varphi \int_0^{\sqrt{3}} \rho^2 d\rho \int_0^{3-\rho^2} dz = 0,$$

jer je

$\int_0^{2\pi} \cos\varphi\, d\varphi = 0.$ Na isti način dobijamo da je $\overline{y} = 0.$ I najzad,

$$\overline{z} = \frac{1}{V} \iiint_{\Omega'} \rho z\, d\varphi\, d\rho\, dz = \frac{2}{9\pi} \int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} z\, dz = \frac{4\pi}{9\pi} \int_0^{\sqrt{3}} \rho \frac{\left(3-\rho^2\right)^2}{2} d\rho.$$

U posljednjem integralu zgodno je uzeti smjenu $3 - \rho^2 = t.$ Dobija se dalje da je

$$\overline{z} = \frac{4}{9} \int_3^0 \frac{t^2}{2} \cdot \left(\frac{-1}{2} \right) dt = ... = 1.$$ Znači, $T(0,0,1)$.

Napomena: U nekim slučajevima možemo i bez računanja odmah zaključiti da je neka od koordinata težišta jednaka nuli. Radi se o slučajevima kada su jednačine površi koje opisuju oblast Ω simetrične u odnosu na neku od promjenljivih *x*, *y* ili *z*. Tako npr. u posljednjem zadatku, ako označimo $f(x,y,z) = z - \left(3 - x^2 - y^2\right) = x^2 + y^2 - z - 3$, imamo da je $f(x,y,z) = f(-x,y,z)$ i $f(x,y,z) = f(x,-y,z)$, što znači da je funkcija $f(x,y,z)$ simetrična u odnosu na x i u odnosu na y. Zato smo dobili da je $\overline{x} = \overline{y} = 0$.

Zadaci za samostalan rad:

3. $\Omega: z = \dfrac{y^2}{2},\ x = 0,\ y = 0,\ z = 0,\ 2x + 3y - 12 = 0.$
4. $x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 = ax.$

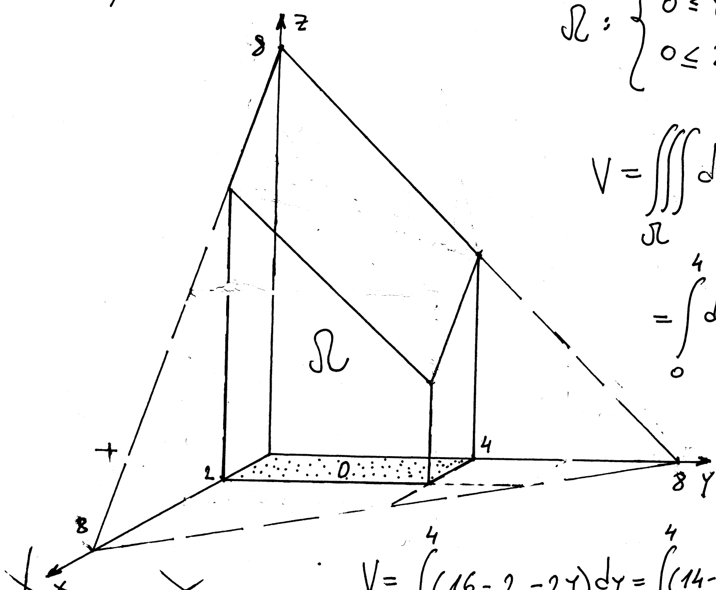
Naći težište homogenog tijela ograničenog sa ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasječen paralelopiped).

Rj. Težište $T(x_T, y_T, z_T)$ homogenog tijela ograničenog sa oblašću Ω tražimo po formulama

$$x_T=\frac{1}{V}\iiint_\Omega x\,dx\,dy\,dz,\quad y_T=\frac{1}{V}\iiint_\Omega y\,dx\,dy\,dz,\quad z_T=\frac{1}{V}\iiint_\Omega z\,dx\,dy\,dz$$

gdje je V zapremina tijela Ω.

Skicirajmo dato tijelo



$$\Omega:\begin{cases}0\le x\le 2\\ 0\le y\le 4\\ 0\le z\le 8-x-y\end{cases}$$

$$V=\iiint_\Omega dx\,dy\,dz=\iint_D (8-x-y)\,dx\,dy$$

$$=\int_0^4 dy\int_0^2 (8-x-y)\,dx=$$

$$=\int_0^4 \left(8x\Big|_0^2-\frac{1}{2}x^2\Big|_0^2-yx\Big|_0^2\right)d$$

$$V=\int_0^4 (16-2-2y)\,dy=\int_0^4 (14-2y)\,dy=14y\Big|_0^4-2\cdot\frac{1}{2}y^2\Big|_0^4$$

$$V=14\cdot 4-16=4(14-4)=40$$

$$V=40$$

$$\iiint_\Omega x\,dx\,dy\,dz=\int_0^4 dy\int_0^2 x\,dx\int_0^{8-x-y} dz=\int_0^4 dy\int_0^2 (8x-x^2-yx)\,dx=\int_0^4\left(4x^2\Big|_0^2-\frac{1}{3}x^3\Big|_0^2-y\frac{1}{2}x^2\Big|_0^2\right)dy$$

$$=\int_0^4\left(16-\frac{8}{3}-2y\right)dy=\int_0^4\left(\frac{40}{3}-2y\right)dy=\frac{40}{3}y\Big|_0^4-2\cdot\frac{1}{2}y^2\Big|_0^4=\frac{160}{3}-16=\frac{112}{3}$$

$$\iiint_\Omega y\,dx\,dy\,dz=\int_0^2 dx\int_0^4 y\,dy\int_0^{8-x-y} dz=\int_0^2 dx\int_0^4 y(8-x-y)\,dy=\int_0^2 dx\int_0^4 (8y-xy-y^2)\,dy=$$

$$=\int_0^2\left(8\frac{1}{2}y^2\Big|_0^4-x\frac{1}{2}y^2\Big|_0^4-\frac{1}{3}y^3\Big|_0^4\right)dx=\int_0^2\left(64-8x-\frac{64}{3}\right)dx=\int_0^2\left(\frac{128}{3}-8x\right)dx=$$

$$=\frac{128}{3}x\Big|_0^2-8\cdot\frac{1}{2}x^2\Big|_0^2=\frac{256}{3}-16=\frac{208}{3}$$

$$\iiint_\Omega z\,dx\,dy\,dz=\underset{\dots}{\text{završiti za vježbu}}=\frac{320}{3}$$

Prema tome, $x_T=\dfrac{1}{V}\displaystyle\iiint_\Omega x\,dx\,dy\,dz=\frac{1}{\cancel{40}_{\cancel{20}_{\cancel{10}_{5}}}}\cdot\frac{\cancel{112}^{\cancel{56}^{\cancel{28}^{14}}}}{3}=\frac{14}{15}$

$$y_T=\frac{1}{V}\iiint_\Omega y\,dx\,dy\,dz=\frac{1}{\cancel{40}_{5}}\cdot\frac{\cancel{208}^{25}}{3}=\frac{25}{15}$$

$$z_T=\frac{1}{V}\iiint_\Omega z\,dx\,dy\,dz=\frac{1}{\cancel{40}_{1}}\cdot\frac{\cancel{320}^{8}}{3}=\frac{8}{3}$$

Težište homogenog tijela je $T\left(\frac{14}{15},\frac{25}{15},\frac{8}{3}\right)$.

Zadaci za vježbu

Zapremina tela. II

U zadacima 3609 — 3625 pomoću trojnih integrala izračunati zapremine tela ograničenih datim površinama (parametre koji ulaze u uslove zadatka smatrati pozitivnim veličinama).

3609. Cilindrima $z=4-y^2$ i $z=y^2+2$ i ravnima $x=-1$ i $x=2$.

3610. Paraboloidima $z=x^2+y^2$ i $z=x^2+2y^2$ i ravnima $y=x$, $y=2x$ i $x=1$.

3611. Paraboloidima $z=x^2+y^2$ i $z=2x^2+2y^2$, cilindrom $y=x^2$ i ravni $y=x$.

3612. Cilindrima $z=\ln(x+2)$ i $z=\ln(6-x)$ i ravnima $x=0$, $x+y=2$ i $x-y=2$.

3613*. Paraboloidom $(x-1)^2+y^2=z$ i ravni $2x+z=2$.

3614*. Paraboloidom $z=x^2+y^2$ i ravni $z=x+y$.

3615*. Sferom $x^2+y^2+z^2=4$ i paraboloidom $x^2+y^2=3z$.

3616. Sferom $x^2+y^2+z^2=R^2$ i paraboloidom $x^2+y^2=R(R-2z)$ $(z \geqslant 0)$.

3617. Paraboloidom $z=x^2+y^2$ i konusom $z^2=xy$.

3618. Sferom $x^2+y^2+z^2=4Rz-3R^2$ i konusom $z^2=4(x^2+y^2)$ (misli se na deo loptine zapremine koji leži unutar konusa).

3619*. $(x^2+y^2+z^2)^2=a^3x$.

3620. $(x^2+y^2+z^2)^2=axyz$.

3621. $(x^2+y^2+z^2)^3=a^2z^4$. **3622.** $(x^2+y^2+z^2)^3=\dfrac{a^6z^2}{x^2+y^2}$,

3623. $(x^2+y^2+z^2)^3=a^2(x^2+y^2)^2$.

3624. $(x^2+y^2)^2+z^4=a^3z$.

3625. $x^2+y^2+z^2=1$, $x^2+y^2+z^2=16$, $z^2=x^2+y^2$, $x=0$, $y=0$, $z=0$ $(x>0,\ y>0,\ z \geqslant 0)$.

Težišta homogenih tela

U zadacima 3666 — 3672 naći težišta homogenih tela ograničenih datim površinama.

3666. Ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasečeni paralelepiped).

3667. Elipsoidom $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ i koordinatnim ravnima (misli se na deo elipsoida koji leži u prvom oktantu).

3668. Cilindrom $z=\dfrac{y^2}{2}$ i ravnima $x=0$, $y=0$, $z=0$ i $2x+3y-12=0$.

3669. Cilindrima $y=\sqrt{x}$, $y=2\sqrt{x}$ i ravnima $z=0$ i $x+z=0$.

3670. Paraboloidom $z=\dfrac{x^2+y^2}{2a}$ i sferom $x^2+y^2+z^2=3a^2$ $(z>0)$.

3671. Sferom $x^2+y^2+z^2=R^2$ i konusom $z\operatorname{tg}\alpha=\sqrt{x^2+y^2}$ (loptin isečak).

3672. $(x^2+y^2+z^2)^2=a^3z$.

Rješenja

3666. $\xi=\dfrac{14}{15}$, $\eta=\dfrac{26}{15}$, $\zeta=\dfrac{8}{3}$. **3667.** $\xi=\dfrac{3}{8}a$, $\eta=\dfrac{3}{8}b$, $\zeta=\dfrac{3}{8}c$.

3668. $\xi=\dfrac{6}{5}$, $\eta=\dfrac{12}{5}$, $\zeta=\dfrac{8}{5}$. **3669.** $\xi=\dfrac{18}{7}$, $\eta=\dfrac{15}{16}\sqrt{6}$, $\zeta=\dfrac{12}{7}$.

3670. $\xi=0$, $\eta=0$, $\zeta=\dfrac{5a}{83}(6\sqrt{3}+5)$.

3671. $\xi=0$, $\eta=0$, $\zeta=\dfrac{3R}{8}(1+\cos\alpha)$. **3672.** $\xi=0$, $\eta=0$, $\zeta=\dfrac{9a}{20}$.

Rješenja

3609. 8.

3610. $\dfrac{7}{12}$. **3611.** $\dfrac{3}{35}$.

3612. $4(4-3\ln 3)$.

3613*. $\dfrac{\pi}{2}$. Projekcija tela na ravan xOy je krug.

3614. $\dfrac{\pi}{8}$. Preneti koordinatni početak u tačku $\left(\dfrac{1}{2}, \dfrac{1}{2}, 0\right)$.

3615*. $\dfrac{19}{6}\pi$ i $\dfrac{15}{2}\pi$. Preći na cilindrične koordinate.

3616. $\dfrac{5}{12}\pi R^3$. **3617.** $\dfrac{\pi}{96}$.

3618. $\dfrac{92}{75}\pi R^2$.

3619*. $\dfrac{1}{3}\pi a^3$. Preći na sferne koordinate.

3620. $\dfrac{a^3}{360}$. **3621.** $\dfrac{4}{21}\pi a^3$.

3622. $\dfrac{4}{3}\pi a^3$. **3623.** $\dfrac{64}{105}\pi a^3$.

3624. $\dfrac{\pi^2a^3}{6}$. **3625.** $\dfrac{21(2-\sqrt{2})}{4}\pi$.

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Krivolinijski integral prve vrste (po luku)

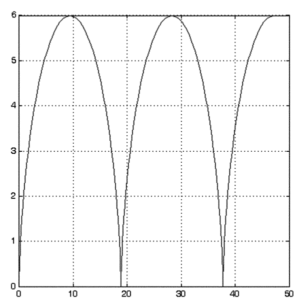
Ako je c kriva data u ravni opisana jednačinom $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_c f(x,y)\,ds = \int_a^b f(x,\eta(x))\underbrace{\sqrt{1+(\eta'(x))^2}\,dx}_{ds}$$

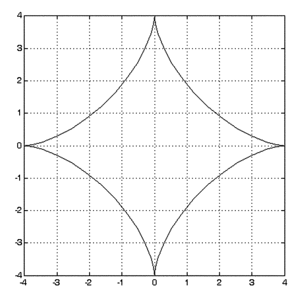
Ako je c kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada $\boxed{\mu'(t)=\frac{\partial \mu}{\partial t}}$

$$\int_c f(x,y)\,ds = \int_{t_1}^{t_2} f(\mu(t),\eta(t))\underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt}_{ds}$$

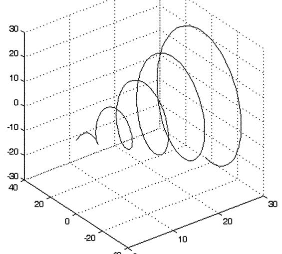
Krivolinijski integrali prve vrste f-ja triju promjenljivih $f(x,y,z)$ uzeti po prostornoj krivoj se računaju analogno. Krivolinijski integral prve vrste NE OVISI O SMJERU PUTA INTEGRACIJE.



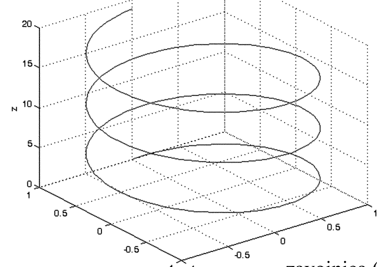
cikloida
$x=3(t-\sin t), y=3(1-\cos t)$, $0\le t\le 5\pi$.



funkcija $x=4\cos^3 t, y=4\sin^3 t$, $0\le t\le 2\pi$.



funkcija $x=t, y=t\cos t, z=t\sin t$, $0\le t\le 30$.



zavojnica (spirala)
$x=\sin t, y=\cos t, z=t$, $0\le t\le 6\pi$.

Izračunati krivolinijski integral $I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$ između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;

b) po liniji astroide $x=\cos^3 t$, $y=\sin^3 t$.

Rj. $I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$

Ovo je krivolinijski integral prve vrste. Prisjetimo se

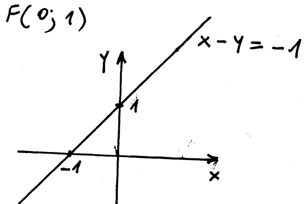
Ako je L kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$ tada

$$\int_L f(x,y)\,dl=\int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

Ako je L opisana parametarskim jednačinama $\begin{cases}x=\mu(t)\\ y=\theta(t)\end{cases}$ gdje $t_1\le t\le t_2$

$$\int_L f(x,y)\,dl=\int_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{(\mu'(t))^2+(\theta'(t))^2}\,dt$$

a) $E(-1;0)$
$F(0;1)$



$-y=-x-1$, $x\in[-1,0]$ tj. $y=x+1$

$y'=1$ $\Rightarrow$ $dl=\sqrt{1+1^2}\,dx=\sqrt{2}\,dx$

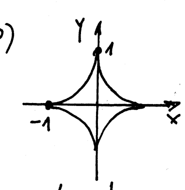
$$I=\int_L(4\sqrt[3]{x}-3\sqrt{y})\,dl=\int_{-1}^{0}(4x^{\frac13}-3(x+1)^{\frac12})\sqrt{2}\,dx$$

$$=4\sqrt{2}\int_{-1}^{0}x^{\frac13}dx-3\sqrt{2}\int_{-1}^{0}(x+1)\,dx=$$

$$=4\sqrt{2}\cdot\frac34 x^{\frac43}\Big|_{-1}^{0}-3\sqrt{2}\int_{-1}^{0}(x+1)^{\frac12}d(x+1)=3\sqrt{2}(0-1)-3\sqrt{2}\cdot\frac23(x+1)^{\frac32}\Big|_{-1}^{0}=-5\sqrt{2}$$

traženo rješenje

b)



astroida

$x=\cos^3 t$, $x'=-3\cos^2 t\sin t$

$y=\sin^3 t$, $y'=3\sin^2 t\cos t$

$$dl=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt$$

$$\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t} = 3\sqrt{\cos^2 t\sin^2(t)\underbrace{(\cos^2 t+\sin^2 t)}_{=1}} =$$

$$= 3\,|\sin t\cos t|$$

U našem slučaju t uzima vrijednost od $\frac{\pi}{2}$ do π, pa je

$dl = -3\sin t\cos t\,dt$

$$I=\int_L \left(4\sqrt[3]{x}-3\sqrt{y}\right)dl = \int_{\frac{\pi}{2}}^{\pi}\left(4\sqrt[3]{\cos^3 t}-3\sqrt{\sin^3 t}\right)(-3\sin t\cos t)\,dt$$

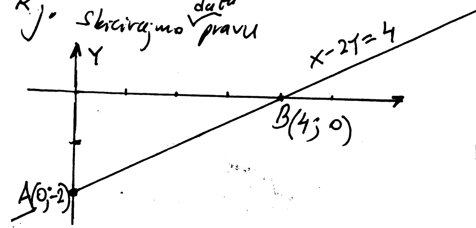
$$= -12\int_{\frac{\pi}{2}}^{\pi}\cos^2 t\sin t\,dt + 9\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac52}t\cos t\,dt =$$

$$= +12\int_{\frac{\pi}{2}}^{\pi}\cos^2 t\,d\cos t + 9\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac52} t\,d\sin t = 12\cdot\frac{\cos^3 t}{3}\Big|_{\frac{\pi}{2}}^{\pi} + 9\cdot\frac{\sin^{\frac72}t}{\frac72}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 4\left((-1)^3-0\right)+\frac{18}{7}\left(0-1^{\frac72}\right) = -4-\frac{18}{7} = -\frac{46}{7}$$ traženo rješenje

Izračunati krivolinijski integral $\int_{AB}\frac{dl}{\sqrt{x^2+y^2}}$ po odsječku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.

Rj: Skicirajmo datu pravu



Prisjetimo se kako se računa krivolinijski integral prvog tipa, ako je kriva integracije C u ravni opisana formulom $y=\eta(x)$, $a\le x\le b$

$$\int_C f(x,y)\,dl = \int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

I način:

$x-2y=4$

$2y=x-4$

$y=\frac12x-2$

$y'=\frac12$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int_0^4 \frac{\sqrt{1+\frac14}}{\sqrt{x^2+(\frac12x-2)^2}}\,dx = \frac{\sqrt5}{2}\int_0^4\frac{dx}{\sqrt{\frac{5x^2}{4}-2x+4}}$$

$$=\frac{\sqrt5}{2}\cdot\frac{1}{\sqrt{\frac54}}\int_0^4\frac{dx}{\sqrt{x^2-\frac85x+\frac{16}{5}}} = \left| \begin{array}{l} x^2-\frac85x+\frac{16}5 = \\ = x^2-2\cdot x\cdot\frac45+\frac{16}{25}-\frac{16}{25}+\frac{16\cdot5}{5\cdot5} \\ =\left(x-\frac45\right)^2+\frac{64}{25}\end{array}\right|$$

$$=\int_0^4\frac{d(x-\frac45)}{\sqrt{(x-\frac45)^2+\frac{64}{25}}} = \ln\left|x-\frac45+\sqrt{\left(x-\frac45\right)^2+\frac{64}{25}}\right|\Bigg|_0^4 = \ln\left(\frac{16}{5}+\sqrt{\frac{16\overset{4\cdot5}{(16+4)}}{25}}\right) -$$

$$- \ln\left(-\frac45+\sqrt{\frac{\underset{16(1+4)}{16+64}}{25}}\right) = \ln\left(\frac{16}5+\frac{8\sqrt5}{5}\right) - \ln\left(-\frac45+\frac{4\sqrt5}{5}\right)$$

$$=\ln\frac{\frac{16+8\sqrt5}{\cancel{5}}}{\frac{-4+4\sqrt5}{\cancel{5}}} = \ln\frac{4+2\sqrt5}{\sqrt5-1}\cdot\frac{(\sqrt5+1)}{(\sqrt5+1)} = \ln\frac{4+6\sqrt5+10}{5-1} = \ln\frac{7+3\sqrt5}{2}$$ traženo rješenje

II način

$x-2y=4$

$x=2y+4$

$\frac{\partial x}{\partial y}=2$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int_{-2}^0\frac{\sqrt{1+4}}{\sqrt{(2y+4)^2+y^2}}\,dy = \sqrt5\int_{-2}^0\frac{dy}{\sqrt{5y^2+16y+16}} = \dots$$

ZAVRŠITI ZA VJEŽBU

Izračunati krivoliniski integral $I=\int_c z\sqrt{x^2+y^2+2z^2}\,dS$

ako je c kriva $x=\frac{r\sqrt{2}}{2}\cos t$, $y=\frac{r\sqrt{2}}{2}\cos t$, $z=r\sin t$, $t\in[0,\pi]$.

Rj. Ako je c kriva opisana parametarskim jednačinama

$c:\begin{cases}x=\mu(t)\\ y=\eta(t)\\ z=\psi(t)\end{cases}$, $t_1\le t\le t_2$ tada

$$\int_c f(x,y,z)\,dS=\int_{t_1}^{t_2} f(\mu(t),\eta(t),\psi(t))\underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\psi'(t))^2}\,dt}_{dS}$$

$$x^2+y^2+2z^2=\frac{2r^2}{4}\cos^2t+\frac{2r^2}{4}\cos^2t+2r^2\sin^2t=r^2\cos^2t+2r^2\sin^2t$$

$$x'_t=-\frac{\sqrt{2}}{2}r\sin t,\quad y'_t=-\frac{\sqrt{2}}{2}r\sin t,\quad z'_t=r\cos t$$

$$(x'_t)^2+(y'_t)^2+(z'_t)^2=\frac{1}{2}r^2\sin^2t+\frac{1}{2}r^2\sin^2t+r^2\cos^2t=r^2\sin^2t+r^2\cos^2t$$
$$=r^2(\sin^2t+\cos^2t)=r^2$$

$$\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\psi'(t))^2}=\sqrt{r^2}=r$$

$$I=\int_c z\sqrt{x^2+y^2+2z^2}\,dS=\int_0^\pi r\sin t\sqrt{\underbrace{r^2\cos^2t+2r^2\sin^2t}_{r^2(\cos^2t+2\sin^2t)}}\,r\,dt=$$

$$=r^3\int_0^\pi \sin t\sqrt{\cos^2t+2\underbrace{\sin^2t}_{1-\cos^2t}}\,dt=r^3\int_0^\pi\sin t\sqrt{2-\cos^2t}\,dt=$$

$$=\left|\begin{matrix}\cos t=u\\ -\sin t\,dt=du\\ \sin t\,dt=-du\end{matrix}\quad t\Big|_0^\pi\Rightarrow u\Big|_1^{-1}\right|=r^3\int_{-1}^1\sqrt{2-t^2}\,dt=r^3\int_{-1}^1\frac{2-t^2}{\sqrt{2-t^2}}\,dt$$

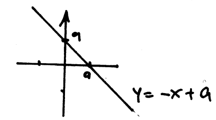
ZA VJEŽBU

$$\dots=r^3\cdot\frac{1}{2}t\sqrt{2-t^2}\Big|_{-1}^1+r^3\int_{-1}^1\frac{dt}{2-t^2}=\dots=\left(1+\frac{\pi}{2}\right)r^3$$

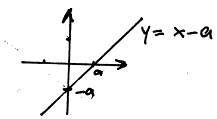
Izračunati integral po krivoj c $\int_c xy\,ds$ gdje je c kvadrat $|x|+|y|=a$, $a>0$.

Rj. Kako nacrtati kvadrat $|x|+|y|=a$?

1° $x>0$, $y>0$: $x+y=a$



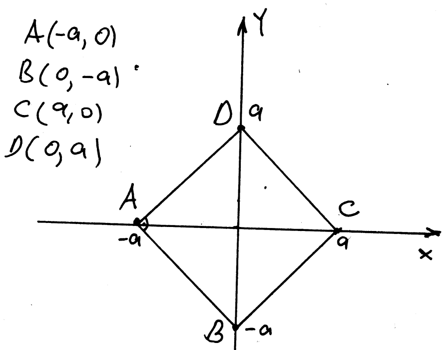
2° $x>0$, $y<0$: $x-y=a$



3° $x<0$, $y>0$: $-x+y=a$

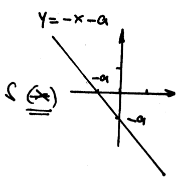
4° $x<0$, $y<0$: $-x-y=a$

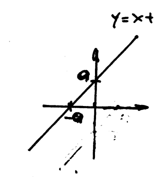
A(-a,0)
B(0,-a)
C(a,0)
D(0,a)



Kriva po kojoj se integrali mora biti glatka, ako ima čošak razbije se na dijelove.

$$\int_c xy\,ds=\int_{AB}xy\,ds+\int_{BC}xy\,ds+\int_{CD}xy\,ds+\int_{DA}xy\,ds\overset{(*)}{=}$$





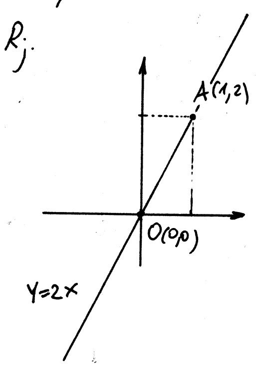
$$\boxed{\int_c f(x,y)\,ds=\int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx,\quad \text{gdje je } c:\ y=\varphi(x) \text{ kriva } x\in[a,b]}$$

$$\overset{(*)}{=}\int_{-a}^0 x\cdot(-x-a)\sqrt{1+(-1)^2}\,dx+\int_0^a x(x-a)\sqrt{1+1^2}\,dx+\int_0^a x(-x+a)\sqrt{1+(-1)^2}\,dx$$

$$+\int_{-a}^0 x(x+a)\sqrt{1+1^2}\,dx=\sqrt{2}\left(\int_{-a}^0(-x^2-ax+x^2+ax)\,dx+\int_0^a(x^2-ax-x^2+ax)\,dx\right)=0$$

\# Izračunati integral $\int_c \frac{ds}{\sqrt{x^2+y^2+4}}$ gdje je c duž koja spaja tačke $O(0,0)$ i tačku $A(1,2)$.

Rj.



$c: \; y=2x \qquad y'=2$

$$\boxed{\int_c f(x,y)\,ds = \int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx \quad \text{gdje je } y=\varphi(x) \text{ kriva, } x\in[a,b]}$$

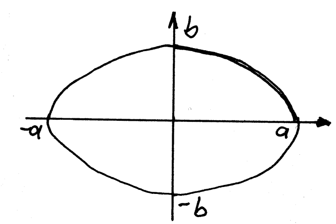
$$\int_c \frac{1}{\sqrt{x^2+y^2+4}}\,dS = \int_0^1 \frac{\sqrt{1+2^2}}{\sqrt{x^2+(2x)^2+4}}\,dx = \sqrt{5}\int_0^1 \frac{dx}{\sqrt{5x^2+4}} =$$

$$=\sqrt{5}\int_0^1 \frac{dx}{\sqrt{4\left(\frac{5}{4}x^2+1\right)}} = \frac{\sqrt{5}}{2}\int_0^1 \frac{d\left(\frac{\sqrt{5}}{2}x\right)}{\sqrt{\left(\frac{\sqrt{5}}{2}x\right)^2+1}}\cdot\frac{2}{\sqrt{5}} = \ln\left|\frac{\sqrt{5}x}{2}+\sqrt{\left(\frac{\sqrt{5}x}{2}\right)^2+1}\right|\Bigg|_0^1$$

$$= \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{5}{4}+1}\right| - \ln 1 = \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{9}{4}}\right| = \ln\frac{\sqrt{5}+3}{2}$$

\# Izračunati $\int_c xy\,ds$ gdje je c četvrtina elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ koja leži u prvom kvadrantu.

Rj. I način:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$
$$y^2=b^2-\frac{b^2}{a^2}x^2$$
$$y=\pm\sqrt{b^2\left(1-\frac{1}{a^2}x^2\right)}$$
$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$c: \begin{cases} y=\frac{b}{a}\sqrt{a^2-x^2} \\ 0\le x\le a \end{cases}$$

$$y'=\frac{b}{a}\cdot\frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}$$

$$\int_c xy\,ds = \int_0^a x\frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\left(\frac{b}{a}\cdot\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx = \ldots$$

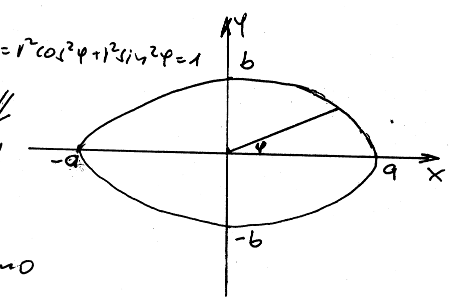
II način

Uvodimo poopštene polarne koordinate

$x=ar\cos\varphi \qquad x^2=a^2r^2\cos^2\varphi \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2\cos^2\varphi+r^2\sin^2\varphi=1$

$y=br\sin\varphi \qquad y^2=b^2r^2\sin^2\varphi$

$$\left.\begin{array}{l} \text{za } \varphi=0 \text{ imamo } x=a,\ y=0 \\ \text{za } \varphi=\frac{\pi}{2} \text{ imamo } x=0,\ y=b \end{array}\right\} \Rightarrow r=1$$



Sad elipsu možemo napisati u parametarskom obliku tj. imamo

$$c: \begin{cases} x=a\cos\varphi \\ y=b\sin\varphi \\ 0\le\varphi\le\frac{\pi}{2} \end{cases} \qquad \frac{\partial x}{\partial\varphi}=-a\sin\varphi \qquad \frac{\partial y}{\partial\varphi}=b\cos\varphi$$

$$\boxed{\int_c f(x,y)\,ds = \int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\psi}{\partial t}\right)^2}\,dt \quad \text{gdje je } \begin{array}{l} x=\varphi(t) \\ y=\psi(t) \\ \alpha\le t\le\beta \end{array}}$$

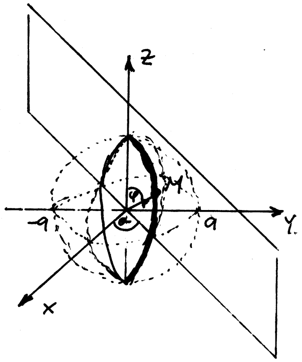
$$\int_c xy\,ds = \int_0^{\pi/2}(a\cos\varphi)(b\sin\varphi)\sqrt{a^2\underbrace{\sin^2\varphi}_{1-\cos^2\varphi}+b^2\cos^2\varphi}\,d\varphi = ab\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\sqrt{a^2+(b^2-a^2)\cos^2\varphi}\,d\varphi$$

$$=\left|\begin{array}{ll} a^2+(b^2-a^2)\cos^2\varphi=u & \varphi=0 \Rightarrow u=b^2 \\ (b^2-a^2)\,2\cos\varphi(-\sin\varphi)\,d\varphi=du & \varphi=\frac{\pi}{2} \Rightarrow u=a^2 \end{array}\right| = ab\cdot\frac{-1}{2(b^2-a^2)}\int_{b^2}^{a^2}\sqrt{u}\,du$$

$$=\frac{-ab}{2(b^2-a^2)}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{b^2}^{a^2} = \frac{-ab}{(b-a)(b+a)}\cdot\frac{1}{3}\cdot\underbrace{(a^3-b^3)}_{(a-b)(a^2+ab+b^2)} = \frac{ab}{3(a+b)}(a^2+ab+b^2)$$

(#) Izračunati $\int\limits_c \sqrt{2y^2+z^2}\, ds$ gdje je c krug dobijen presjekom sfere $x^2+y^2+z^2=a^2$ i ravni $x=y$.

Rj.



Kako ćemo opisati datu sferu parametarski? (sferne koordinate)

$$x = r\sin\varphi\cos\theta,\quad y = r\sin\varphi\sin\theta,\quad z = r\cos\varphi \qquad r=a,\quad 0\le\theta\le 2\pi,\quad 0\le\varphi\le\pi$$

Kako da parametarski opišemo krug dobijen presjekom sfere i ravni?

Za pravu $x=y$ znamo da je ugao između ove prave i x-ose $45°$. Prema tome $\theta=45°$, $(r=a)$:

$$C:\begin{cases} x = \frac{\sqrt{2}}{2}a\sin\varphi \\ y = \frac{\sqrt{2}}{2}a\sin\varphi \\ z = a\cos\varphi \\ 0\le\varphi\le 2\pi \end{cases}$$

a i r su fiksirani

$$2y^2+z^2 = 2\cdot\frac{2}{4}a^2\sin^2\varphi + a^2\cos^2\varphi = a^2$$

Ako je kriva c opisana sa $x=\mu(t)$, $y=\eta(t)$, $z=\xi(t)$, $\alpha<t<\beta$ onda je

$$\int\limits_c f(x,y,z)\,ds = \int\limits_\alpha^\beta f(\mu(t),\eta(t),\xi(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\xi'(t))^2}\,dt$$

$$\frac{\partial x}{\partial\varphi} = \frac{\sqrt{2}}{2}a\cos\varphi$$

$$\frac{\partial y}{\partial\varphi} = \frac{\sqrt{2}}{2}a\cos\varphi$$

$$\frac{\partial z}{\partial\varphi} = -a\sin\varphi$$

$$\int\limits_c \sqrt{2y^2+z^2}\,ds = \int\limits_0^{2\pi}\sqrt{a^2}\cdot\sqrt{\frac{2}{4}a^2\cos^2\varphi+\frac{2}{4}a^2\cos^2\varphi+a^2\sin^2\varphi}\,dt$$

$$= \int\limits_0^{2\pi} a\cdot\sqrt{a^2(\cos^2\varphi+\sin^2\varphi)}\,dt = a^2\int\limits_0^{2\pi} dt = 2a^2\pi$$

Zadaci za vježbu

U zadacima 3770—3775 izračunati date krivolinijske integrale.

3770. $\int\limits_L \frac{ds}{x-y}$, pri čemu je L odsečak na pravoj $y=\frac{1}{2}x-2$, koji leži između tačaka $A(0, -2)$ i $B(4, 0)$.

3771. $\int\limits_L xy\,ds$, pri čemu je L kontura pravougaonika čija su temena $A(0, 0)$, $B(4, 0)$, $C(4, 2)$ i $D(0, 2)$.

3772. $\int\limits_L y\,ds$, pri čemu je L luk parabole $y^2=2px$, koji leži unutar parabole $x^2=2py$.

3773. $\int\limits_L (x^2+y^2)^n\,ds$, pri čemu je L krug $x=a\cos t$, $y=a\sin t$.

3774. $\int\limits_L xy\,ds$, pri čemu je L četvrtina elipse koja leži u prvom kvadrantu.

3775. $\int\limits_L \sqrt{2y}\,ds$, pri čemu je L prvi svod cikloide $x=a(t-\sin t)$,

$$y=a(1-\cos t).$$

3776. Napisati obrazac za izračunavanje integrala $\int\limits_L F(x, y)\,ds$ u polarnim koordinatama, ako je kriva L zadata jednačinom $\rho=\rho(\varphi)$ $(\varphi_1\leqslant\varphi\leqslant\varphi_2)$.

3777*. Izračunati $\int\limits_L (x-y)\,ds$, po kružnoj liniji $x^2+y^2=ax$.

3778. Izračunati $\int\limits_L \sqrt{x^2-y^2}\,ds$ po krivoj $(x^2+y^2)^2=a^2(x^2-y^2)$ $(x\geqslant 0)$ (polovina lemniskate).

3779. Izračunati $\int\limits_L \operatorname{arctg}\frac{y}{x}\,ds$ po delu Arhimedove spirale $\rho=2\varphi$ koji leži unutar kruga poluprečnika R, čiji je centar u koordinatnom početku.

3780. Izračunati $\int\limits_L \frac{z^2\,ds}{x^2+y^2}$ po prvom zavoju zavojnice $x=a\cos t$, $y=a\sin t$, $z=at$.

3781. Izračunati $\int\limits_L xyz\,ds$ po delu kružne linije $x^2+y^2+z^2=R^2$, $x^2+y^2=\frac{R^2}{4}$, koji leži u prvom oktantu.

3782. Izračunati $\int\limits_L (2z-\sqrt{x^2+y^2})\,ds$ po prvom zavoju konusne zavojnice $x=t\cos t$, $y=t\sin t$, $z=t$.

3783. Izračunati $\int\limits_L (x+y)\,ds$ po delu kružne linije $x^2+y^2+z^2=R^2$, $y=x$, koji leži u prvom oktantu.

Rješenja

3770. $\sqrt{5}\ln 2$. **3771.** 24.

3772. $\frac{p^2}{3}(5\sqrt{5}-1)$. **3773.** $2\pi a^{2n+1}$.

3774. $\frac{ab(a^2+ab+b^2)}{3(a+b)}$. **3775.** $4\pi a\sqrt{a}$.

3776. $\int\limits_{\varphi_1}^{\varphi_2} F(\rho\cos\varphi, \rho\sin\varphi)\sqrt{\rho^2+\rho'^2}\,d\varphi$.

3777*. $\frac{\pi a^2}{2}$. Preći na polarne koordinate.

3778. $\frac{2a^3\sqrt{2}}{3}$. 3779. $\frac{1}{12}[(R^2+4)^{\frac{3}{2}}-8]$.

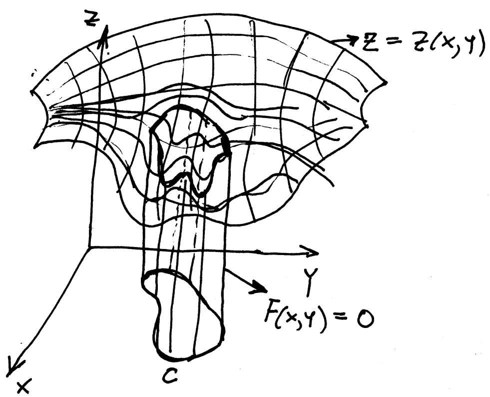
3780. $\frac{8a\pi^2\sqrt{2}}{3}$. 3781. $\frac{R^4\sqrt{3}}{32}$

3782. $\frac{2\sqrt{2}}{3}[(1+2\pi^2)^{\frac{3}{2}}-1]$. 3783. $R^2\sqrt{2}$.

Računanje površine cilindrične površi

Ako je S dio cilindrične površine $F(x,y)=0$ između xOy ravni i neke površine $z=z(x,y)$ tada se površina $P(S)$ površi S računa po formuli

$$P(S)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad C:\begin{cases}F(x,y)=0\\ z=0\end{cases}.$$



$P(S)$ - površina dijela cilindrične površi

\# Izračunati površinu eliptičkog valjka $9x^2+5y^2=45$ koji se nalazi između površi $z=0$ i $z=y$.

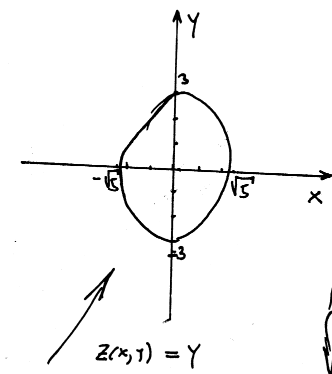
Rj:

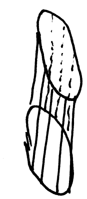
$$P(S)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad C:\begin{cases}F(x,y)=0\\ z=0\end{cases}.$$

Skicirajmo valjak $9x^2+5y^2=45.\ /:45$ u xOy ravni on izgleda

$\frac{x^2}{5}+\frac{y^2}{9}=1$ elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\sqrt{5}\approx 2{,}23$



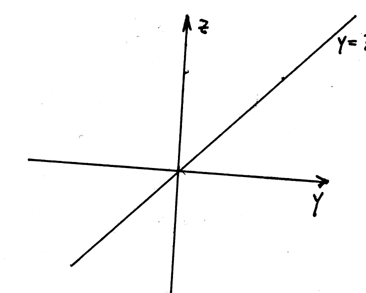


$z(x,y)=y$

$C:\begin{cases}9x^2+5y^2=45\\ z=0\end{cases}$ C je elipsa

$z=0$ je xOy ravan

$z=y$ u yOz ravni



$z=y$ je ravan koja sadrži x-osu a u yOz ravni sadrži $y=z$ pravu

Svedimo elipsu $\frac{x^2}{5}+\frac{y^2}{9}=1$ na parametarski oblik $\quad x=a\cos t,\ y=b\sin t$

U našem slučaju $\quad x=\sqrt{5}\cos t,\ y=3\sin t,\ 0\le t\le 2\pi$

površina figure koja se nalazi u I i II oktantu

$$\boxed{C:\begin{cases}x=\varphi(t)\\ y=\psi(t)\\ t_1\le t\le t_2\end{cases} \quad \int_C f(x,y)\,ds=\int_{t_1}^{t_2} f(\varphi(t),\psi(t))\underbrace{\sqrt{\varphi'(t)^2+\psi'(t)^2}\,dt}_{ds}}$$

$dS=\sqrt{5\sin^2 t+9\cos^2 t}\,dt$ kako se ravni $z=0$ i $z=y$ sijeku u x-osi to će parametar t uzimati vrijednosti od 0 do π

$$P(S)=\int_C y\,dS=\int_0^\pi 3\sin t\sqrt{5\sin^2 t+9\cos^2 t}\,dt=3\int_0^\pi \sin t\sqrt{5(1-\cos^2 t)+9\cos^2 t}\,dt=$$

$$=3\int_0^\pi \sin t\sqrt{5+4\cos^2 t}\,dt=\left|\begin{matrix}2\cos t=u & t=0\Rightarrow u=2\\ -2\sin t\,dt=du & t=\pi\Rightarrow u=-2\\ \sin t\,dt=-\frac{1}{2}du & \end{matrix}\right|=3\int_2^{-2}\left(-\frac{1}{2}\right)\underbrace{\sqrt{5+u^2}}_{\text{parna f-ja}}\,du=$$

$$=3\cdot\frac{1}{2}\cdot 2\int_0^2\sqrt{5+u^2}\,du\overset{\text{umjesto u stavimo x}}{=}3\int_0^2\frac{5+x^2}{\sqrt{5+x^2}}\,dx=3\int_0^2\frac{5}{\sqrt{5+x^2}}\,dx+3\int_0^2\frac{x^2}{\sqrt{5+x^2}}=\left|\begin{matrix}u=x & dv=\frac{x}{\sqrt{5+x^2}}\\ \text{ZAVRŠITI SAMI} & \ldots\end{matrix}\right|=\frac{15\ln 5}{4}+9$$

\# Izračunati površinu dijela valjka $x^2+y^2=1$ koji se nalazi između površi $z=0$ i $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

R: $P(S)=\int_c z(x,y)\,dS$ gdje je $c: \begin{cases} F(x,y)=0 \\ z=0 \end{cases}$

U ovom slučaju je $z(x,y)=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

$c: \begin{cases} x^2+y^2=1 \\ z=0 \end{cases}$ tj. $c: x^2+y^2=1$

Parametrizirajmo kružnicu: $x=r\cos\varphi$, $y=r\sin\varphi$, $0\le\varphi\le 2\pi$

U našem slučaju: $x=\cos\varphi$, $y=\sin\varphi$, $0\le\varphi\le 2\pi$

$$\boxed{c: \begin{cases} x=\mu(t) \\ y=\eta(t) \\ t_1\le t\le t_2 \end{cases} \qquad \int_c f(x,y)\,dS=\int_{t_1}^{t_2} f(\mu(t),\eta(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt}$$

$(\cos\varphi)'=-\sin\varphi$

$(\sin\varphi)'=\cos\varphi$

$dS=\sqrt{\sin^2\varphi+\cos^2\varphi}\,d\varphi=d\varphi$

$\sqrt{x^2+y^2}=1$

$\sqrt{1-x^2}=\cos\varphi$; $\sqrt{1-y^2}=\sin\varphi$

Definiciono područje f-je $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$ je $\{(x,y)\mid -1\le x\le 1\ ;\ -1\le y\le 1\}$

zbog simetričnosti četiri dijela $\frac{\pi}{2}$

$$P(S)=\int_c\left(\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}\right)dS=4\int_0^{\frac{\pi}{2}}(1+\sin\varphi+\cos\varphi)\,d\varphi=$$

$$=4\left[\varphi\Big|_0^{\frac{\pi}{2}}-\cos\varphi\Big|_0^{\frac{\pi}{2}}+\sin\varphi\Big|_0^{\frac{\pi}{2}}\right]=4\left(\frac{\pi}{2}+1+1\right)=2\pi+8$$

\# Izračunati površinu cilindra $x^2+y^2=R^2$ između ravni $z=0$ i površi $z=R+\frac{x^2}{R}$.

Zadaci za vježbu

U zadacima 3792 — 3797 izračunati površine datih cilindričnih omotača, koji leže između ravni *Oxy* i navedenih površina.

3792. $x^2+y^2=R^2$, $z=R+\frac{x^2}{R}$.

3793. $y^2=2px$, $z=\sqrt{2px-4x^2}$.

3794. $y^2=\frac{4}{9}(x-1)^3$, $z=2-\sqrt{x}$.

3795. $x^2+y^2=R^2$, $2Rz=xy$.

3796. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=kx$ i $z=0$ $(z\geqslant 0)$ („cilindrična potkovica")

3797. $y=\sqrt{2px}$, $z=y$ i $x=\frac{8}{9}p$.

3798. Izračunati površinu onog dela kružnog cilindra koji iz njega iseca drugi isti takav cilindar, ako im se ose seku pod pravim uglom a poluprečnici su im R (uporedi sa rešenjem zadatka 3642).

3799. Naći površinu onog dela cilindra $x^2+y^2=Rx$, koji leži unutar sfere $x^2+y^2+z^2=R^2$.

Rješenja

3792. $3\pi R^2$. **3793.** $\frac{\pi p^2}{4}$. **3794.** $\frac{11}{3}$. **3795.** R^2.

3796. $ka\left(a+\frac{b^2}{2c}\ln\frac{a+c}{a-c}\right)$, gde je $c=\sqrt{a^2-b^2}$. Za $a=b$ $S=2ka^2$.

3797. $\frac{98}{81}p^2$. **3798.** $8R^2$. **3799.** $4R^2$.

Krivoliniski integral druge vrste (po koordinatama)

Ako je c data kriva u ravni opisana jednačinom $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

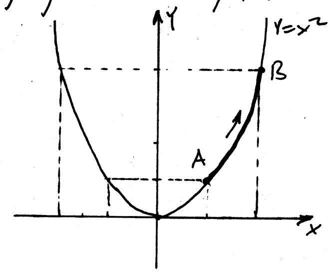
Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

Analogne formule vrijede za krivoliniski integral druge vrste uzete po prostornoj krivoj. Krivoliniski integral druge vrste OVISI O SMJERU PUTA INTEGRACIJE (bitna je orjentacija i u kom smjeru ide luk).

Izračunati krivoliniski integral $\int_c(x^2-2xy)\,dx+(2xy+y^2)\,dy$ gdje je c luk parabole $y=x^2$ od tačke A(1,1) do B(2,4).

Rj.



$y=x^2$

$\frac{\partial y}{\partial x}=2x$

$1\le x\le 2$

Ako je data kriva $y=\eta(x)$, $a\le x\le b$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

$$\int_c(x^2-2xy)\,dx+(2xy+y^2)\,dy=\int_1^2\left(x^2-2x^3+(2x^3+x^4)\,2x\right)dx=\int_1^2(2x^5+4x^4-2x^3+x^2)\,dx$$

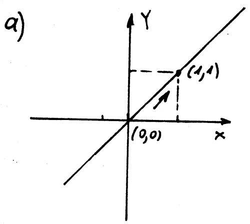
$$=2\cdot\frac{1}{6}x^6\Big|_1^2+4\cdot\frac{1}{5}x^5\Big|_1^2-2\cdot\frac{1}{4}x^4\Big|_1^2+\frac{1}{3}x^3\Big|_1^2=\frac{1}{3}\cdot63+\frac{4}{5}\cdot31-\frac{1}{2}\cdot15+\frac{1}{3}\cdot7=40+\frac{19}{30}$$

Izračunati krivoliniski integral $\int_{(0,0)}^{(1,1)}2xy\,dx+x^2dy$ ako prelazimo po liniji

a) $y=x$ b) $y=x^2$ c) $y=x^3$ d) $y^2=x$.

Rj.

a)



Ako je data kriva $y=\eta(x)$, $\alpha\le x\le\beta$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_\alpha^\beta\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

$y=x$

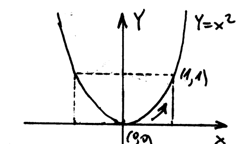
$y'=1$

$$\int_{(0,0)}^{(1,1)}2xy\,dx+x^2dy=\int_0^1(2x^2+x^2\cdot1)\,dx=3\int_0^1x^2dx=3\cdot\frac{1}{3}x^3\Big|_0^1=3\cdot\frac{1}{3}=1$$

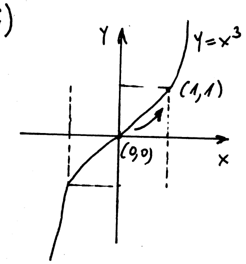
b) $y=x^2$

$y'=2x$

$$\int_{(0,0)}^{(1,1)}2xy\,dx+x^2dy=\int_0^1(2x\cdot x^2+x^2\cdot2x)\,dx=\int_0^1 4x^3dx=4\cdot\frac{1}{4}x^4\Big|_0^1=4\cdot\frac{1}{4}=1$$



c)



$y=x^3$

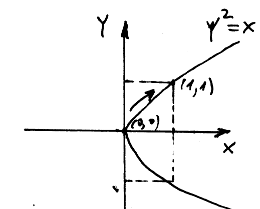
$y'=3x^2$

$$\int_{(0,0)}^{(1,1)}2xy\,dx+x^2dy=\int_0^1(2x\cdot x^3+x^2\cdot3x^2)\,dx=\int_0^1 5x^4dx=5\cdot\frac{1}{5}x^5\Big|_0^1=5\cdot\frac{1}{5}=1$$

d) $x=y^2$

$x'_y=2y$

$$\int_{(0,0)}^{(1,1)}2xy\,dx+x^2dy=\int_0^1\left(2\cdot y^2\cdot y\cdot2y+(y^2)^2\right)dy=\int_0^1(4y^4+y^4)\,dy=\int_0^1 5y^4dy=5\cdot\frac{1}{5}y^5\Big|_0^1=1$$



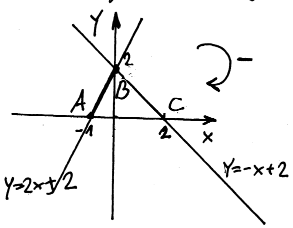
#) Izračunati krivolinijske integrale

a) $\oint\limits_{-l} 2x\,dx-(x+2y)\,dy$

b) $\oint\limits_{+l} y\cos x\,dx+\sin x\,dy$

gdje je l kontura trougla čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj. a) Nacrtajmo trougao ΔABC.



Provucimo pravu kroz tačke $B(\overset{x_1}{0};\overset{y_1}{2})$ i $C(\overset{x_2}{2};\overset{y_2}{0})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{x}{2}=\frac{y-2}{-2}\quad /\cdot 2$$

$$x=-y+2$$

$$y=-x+2$$

Provucimo pravu kroz $A(\overset{x_1}{-1};\overset{y_1}{0})$ i $B(\overset{x_2}{0};\overset{y_2}{2})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}\qquad \frac{x+1}{1}=\frac{y}{2}\qquad y=2x+2$$

$$\oint\limits_{-l} 2x\,dx-(x+2y)\,dy=\int\limits_{B(0;2)}^{C(2;0)}2x\,dx-(x+2y)\,dy+\int\limits_{C(2;0)}^{A(-1;0)}2x\,dx-(x+2y)\,dy+\int\limits_{A(-1;0)}^{B(0;2)}2x\,dx-(x+2y)\,dy$$

$$\int\limits_{(0;2)}^{(2;0)}2x\,dx-(x+2y)\,dy=\left|\begin{matrix}y=-x+2\\ dy=-dx\end{matrix}\right|=\int\limits_{(0;2)}^{(2;0)}\left[2x-(x+2(-x+2))(-1)\right]dx=$$

$$=\int\limits_{(0;2)}^{(2;0)}\left[2x+x-2x+4\right]dx=\int\limits_{(0;2)}^{(2;0)}(x+4)\,dx=\left(\frac{1}{2}x^2+4x\right)\Big|_0^2=2+8=10$$

$$\int\limits_{C(2;0)}^{A(-1;0)}2x\,dx-(x+2y)\,dy=\left|\begin{matrix}y=0\\ dy=0\end{matrix}\right|=\int\limits_{2}^{-1}2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=(1-4)=-3$$

$$\int\limits_{A(-1;0)}^{B(0;2)}2x\,dx-(x+2y)\,dy=\left|\begin{matrix}y=2x+2\\ dy=2\,dx\end{matrix}\right|=\int\limits_{-1}^{0}\left[2x-(x+2(2x+2))\,2\right]dx=$$

$$=\int\limits_{-1}^{0}(2x-2x-8x-8)\,dx=(-8)\int\limits_{-1}^{0}(x+1)\,dx=(-8)\left[\frac{1}{2}x^2\Big|_{-1}^{0}+x\Big|_{-1}^{0}\right]=$$

$$\underbrace{}_{0-1}\qquad\underbrace{}_{0+1}$$

$$=(-8)\left(-\frac{1}{2}+1\right)=-4$$

Prema tome $\oint\limits_{\Delta ABC}2x\,dx-(x+2y)\,dy=10-3-4=3$

b) $$\oint\limits_{+l}y\cos x\,dx+\sin x\,dy=\int\limits_{AC}y\cos x\,dx+\sin x\,dy+\int\limits_{CB}y\cos x\,dx+\sin x\,dy+\int\limits_{BA}y\cos x\,dx+\sin x\,dy$$

$$\int\limits_{A(-1;0)}^{C(2;0)}y\cos x\,dx+\sin x\,dy=\left|\begin{matrix}y=0\\ dy=0\end{matrix}\right|=\int\limits_{-1}^{2}0\,dx=0$$

$$\int\limits_{C(2;0)}^{B(0;2)}y\cos x\,dx+\sin x\,dy=\left|\begin{matrix}y=-x+2\\ dy=-dx\end{matrix}\right|=\int\limits_{2}^{0}\left[(-x+2)\cos x-\sin x\right]dx$$

$$=\left|\begin{matrix}u=-x+2 & dv=\cos x\\ du=-1 & v=\sin x\end{matrix}\right|=(-x+2)\sin x\Big|_2^0+\int\limits_2^0\sin x\,dx-\int\limits_2^0\sin x\,dx=0$$

$$\int\limits_{B(0;2)}^{A(-1;0)}y\cos x\,dx+\sin x\,dy=\left|\begin{matrix}y=2x+2\\ dy=2\,dx\end{matrix}\right|=\int\limits_{0}^{-1}\left[(2x+2)\cos x+2\sin x\right]dx=$$

$$=2\int\limits_{0}^{-1}\left[(x+1)\cos x+\sin x\right]dx=\left|\begin{matrix}u=x+1 & dv=\cos x\\ du=dx & v=\sin x\end{matrix}\right|=2(x+1)\sin x\Big|_0^{-1}-2\int\limits_0^{-1}\sin x\,dx$$

$$+2\int\limits_0^{-1}\sin x\,dx=0$$

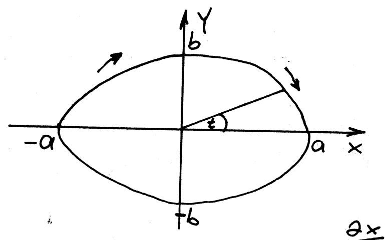
Prema tome

$$\oint\limits_{+l}y\cos x\,dx+\sin x\,dy=0$$

Izračunati krivoliniski integral $\int\limits_c y^2dx + x^2dy$

gdje je c gornja polovina elipse $x=a\cos t$, $y=b\sin t$ $(a>0, b>0)$, koja se prelazi u smislu pomjeranja kazaljke na satu.

Rj.



Ako je kriva c zadana parametarski $x=\varphi(t)$, $y=\psi(t)$ gdje $\alpha\le t\le\beta$ imamo

$$\int\limits_c P(x,y)\,dx + Q(x,y)\,dy = \int\limits_\alpha^\beta [P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t)]\,dt$$

$$\frac{\partial x}{\partial t} = -a\sin t \qquad \frac{\partial y}{\partial t} = b\cos t$$

$$\int\limits_c y^2dx+x^2dy = \int\limits_\pi^0 [b^2\sin^2 t\cdot(-a\sin t) + a^2\cos^2 t\cdot b\cos t]\,dt =$$

$$= -ab^2\int\limits_\pi^0 \sin^3 t\,dt + a^2b\int\limits_\pi^0 \cos^3 t\,dt \overset{(*)}{=} \frac{4}{3}ab^2$$

$$\int\limits_\pi^0 \sin^3 t\,dt = \int\limits_\pi^0 \sin t(1-\cos^2 t)\,dt = \left|\begin{array}{ll} \cos t = u & t=\pi \Rightarrow u=-1 \\ -\sin t\,dt = du & t=0 \Rightarrow u=1 \\ \sin t\,dt = -du & \end{array}\right| = -\int\limits_{-1}^{1}(1-u^2)\,du =$$

$$= -\left(u\Big|_{-1}^{1} - \frac{1}{3}u^3\Big|_{-1}^{1}\right) = -\left(2-\frac{1}{3}\cdot 2\right) = -\left(\frac{6-2}{3}\right) = -\frac{4}{3} \quad \ldots(*)$$

$$\int\limits_\pi^0 \cos^3 t\,dt = \int\limits_\pi^0 \cos t(1-\sin^2 t)\,dt = \left|\begin{array}{l} \sin t = u \\ \cos t\,dt = du \\ t=\pi \Rightarrow u=0 \\ t=0 \Rightarrow u=0 \end{array}\right| = \int\limits_0^0 (1-u^2)\,du = 0$$

Date ~~sa~~ tačke $A(3;-6;0)$ i $B(-2;4;5)$. Izračunati krivolinijski integral $I=\int\limits_c xy^2dx + yz^2dy - zx^2dz$ gdje je c:

a) duž koja spaja tačke O i B (O koordinatni početak)

b) kriva od A do B: kruga zadan jednačinama

$x^2+y^2+z^2=45$, $2x+y=0$.

Rj.

$$I=\int\limits_c xy^2dx+yz^2dy+zx^2dz$$

Ovo je krivoliniski integral druge vrste. Prisjetimo se:

Ako je c kriva u prostoru opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$, $z=\theta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int\limits_c P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz = \int\limits_{t_1}^{t_2} P(\mu(t),\eta(t),\theta(t))\,\mu'(t)\,dt +$$

$$+\int\limits_{t_1}^{t_2} Q(\mu(t),\eta(t),\theta(t))\,\eta'(t)\,dt + \int\limits_{t_1}^{t_2} R(\mu(t),\eta(t),\theta(t))\,\theta'(t)\,dt$$

Da bi smo opisali duž $\overline{OB}$ u prostoru prvo postavimo pravu kroz ove dvije tačke.

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

jednačina prave kroz dvije tačke $M_1(x_1,y_1,z_1)$ i $M_2(x_2,y_2,z_2)$

$O(0,0,0)$

$B(-2,4,5)$

$$\frac{x}{-2} = \frac{y}{4} = \frac{z}{5} \quad (=t)$$

$x=-2t$

$y=4t$

$z=5t$

Naše c je sada oblika

$$c:\begin{cases} x=-2t,\ y=4t,\ z=5t \\ 0<t<1 \end{cases}$$

$$I=\int\limits_c xy^2dx+yz^2dy-zx^2dz = \int\limits_0^1 \left((-2t)\,16t^2\cdot(-2) + 4t\cdot 25t^2\cdot 4 - 5t\cdot 4t^2\cdot 5\right)dt =$$

$$= \int_0^1 (64t^3 + 400t^3 - 100t^3)\,dt = 364\int_0^1 t^3\,dt = \frac{364}{4} = 91$$ traženo rješenje

b) Dat je krug u prostoru zadan jednačinama

$x^2+y^2+z^2=45$ (krug), $2x+y=0$ (ravan)

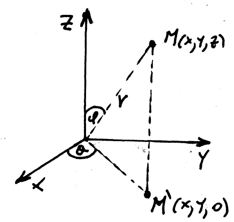
Da bi smo naš krug opisali u parametarskom obliku, veliku pomoć će odigrati sferne koordinate.

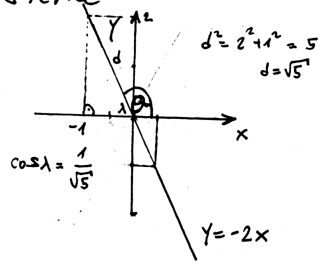
Sferne koordinate

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$





Da bi smo krug u prostoru opisali parametarski potrebno je u sfernim koordinatama fiksirati r i θ. U našem slučaju ugao θ nije moguće svesti na lijep oblik.

Pristupimo parametarizaciji kruga na drugi način:

$$\left.\begin{array}{l} 2x+y=0 \Rightarrow y=-2x \\ x^2+y^2+z^2=45 \Rightarrow z^2=45-x^2-y^2 \end{array}\right\} \rightarrow c:\begin{cases} x=t \\ y=-2t \\ z=\sqrt{45-t^2-4t^2}=\sqrt{45-5t^2} \\ 3\le t\le -2 \end{cases}$$

$$dx = dt,\quad dy=-2\,dt,\quad dz=\frac{1}{2}(45-5t^2)^{-\frac{1}{2}}\cdot(-10t) = -\frac{5t}{\sqrt{45-5t^2}}\,dt$$

$$I=\int_c xy^2dx+yz^2dy-zx^2dz = \int_3^{-2}\left(t\cdot 4t^2 + (-2t)(45-5t^2)\cdot(-2) - \cancel{\sqrt{45-5t^2}}\cdot t^2\,\frac{(-5t)}{\cancel{\sqrt{45-5t^2}}}\right)dt$$

$$=\int_3^{-2}(4t^3+180t-20t^3+5t^3)\,dt = \int_3^{-2}(-11t^3+180t)\,dt$$

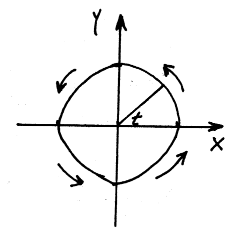
$$=-11\cdot\frac{1}{4}t^4\Big|_3^{-2} + 180\cdot\frac{1}{2}t^2\Big|_3^{-2} = -\frac{11}{4}\cdot(-65)+90\cdot(-5) = \frac{715-1800}{4} = \frac{-1\,085}{4}$$

$$= -271\frac{1}{4}$$ traženo rješenje

Izračunati krivoliniski integral $\int_c \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2}$

gdje je c krug $x^2+y^2=a^2$ koji se prelazi u smislu suprotnom pomjeranju kazaljke na satu.

R:



Krug $x^2+y^2=a^2$ napisan parametarski

$x = a\cos t$

$y = a\sin t$

$0\le t\le 2\pi$

$\frac{\partial x}{\partial t} = -a\sin t$

$\frac{\partial y}{\partial t} = a\cos t$

Ako je c kriva zadana parametarski $x=\mu(t)$, $y=\eta(t)$, $a\le t\le b$

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_a^b [P(\mu(t),\eta(t))\mu'(t) + Q(\mu(t),\eta(t))\eta'(t)]\,dt$$

$$\int_c \frac{(x+y)dx-(x-y)dy}{x^2+y^2} = \int_c \frac{x+y}{x^2+y^2}dx - \frac{x-y}{x^2+y^2}dy = \int_0^{2\pi}\left[\frac{a\cos t+a\sin t}{a^2}\cdot(-a\sin t) - \frac{a\cos t-a\sin t}{a^2}\cdot a\cos t\right]dt = \int_0^{2\pi}[(\cos t+\sin t)\cdot(-\sin t)-(\cos t-\sin t)\cdot\cos t]\,dt$$

$$=\int_0^{2\pi}(\underline{-\sin t\cos t}-\sin^2 t-\cos^2 t\,\underline{+\sin t\cos t})\,dt = \int_0^{2\pi}(-1)\,dt = -2\pi$$

1. Izračunaj krivolinijski integral $I = \int\limits_L (xy-1)dx + x^2 y dy$ od tačke A(1,0) do tačke B(0,2).

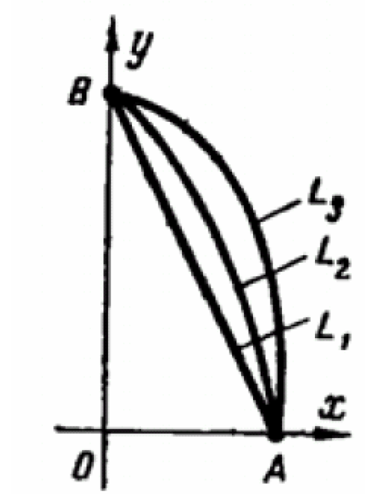
a) po pravoj 2x+y=2

b) duž parabole $4x + y^2 = 4$

c) duž elipse x=cost ; y= 2sint

Rješenja:

a) Skicirajmo datu pravu (uputa vidi sliku desno).



2x+y=2
y = 2 - 2x
dy = -2dx

$$I = \int\limits_{L_1} (xy-1)dx + x^2 y dy =$$

$$= \int\limits_1^0 [x(2-2x)-1]dx + x^2(2-2x)(-2dx) =$$

$$= \int\limits_1^0 (2x-2x^2-1)dx + (-4x^2+4x^3)dx =$$

$$= \int\limits_1^0 (4x^3-6x^2+2x-1)dx = 4\cdot\frac{x^4}{4}\Big|_1^0 - 6\cdot\frac{x^3}{3}\Big|_1^0 + 2\cdot\frac{x^2}{2}\Big|_1^0 - x\Big|_1^0 = -1+2-1+1 = 1$$

b) Skicirajmo parabolu (uputa: vidi sliku iznad).

$$4x + y^2 = 4 \;\Rightarrow\; x = 1 - \frac{y^2}{4} \;\Rightarrow\; dx = -\frac{y}{2}dy$$

$$I = \int\limits_{L_2} (xy-1)dx + x^2 y dy = \int\limits_0^2 \left[\left(1-\frac{y^2}{4}\right)y - 1\right]\left(-\frac{y}{2}dy\right) + \left(1-\frac{y^2}{4}\right)^2 y dy =$$

$$= \int\limits_0^2 \left(y - \frac{y^3}{4} - 1\right)\left(-\frac{y}{2}dy\right) + \left(1 - \frac{y^2}{2} + \frac{y^4}{16}\right) y dy =$$

$$= \int\limits_0^2 \left(-\frac{y^2}{2} + \frac{y^4}{8} + \frac{y}{2}\right)dy + \left(y - \frac{y^3}{2} + \frac{y^5}{16}\right)dy =$$

$$= \int\limits_0^2 \left(\frac{y^5}{16} + \frac{y^4}{8} - \frac{y^3}{2} - \frac{y^2}{2} + \frac{3y}{2}\right)dy = \frac{y^6}{96}\Big|_0^2 + \frac{y^5}{40}\Big|_0^2 - \frac{y^4}{8}\Big|_0^2 - \frac{y^3}{6}\Big|_0^2 + \frac{3y^2}{4}\Big|_0^2 =$$

$$= \frac{64}{96} + \frac{32}{40} - \frac{16}{8} - \frac{8}{6} + \frac{12}{4} = \frac{2}{3} + \frac{4}{5} - 2 - \frac{4}{3} + 3 = \frac{10+12-30-20+45}{15} = \frac{17}{15}.$$

c) Skicirajmo elipsu (uputa: vidi sliku sa prethodne stranice).

x = cost y = 2sint

dy = 2costdt

$$L_3 : \begin{cases} x = \cos t \\ y = \sin t \\ 0 \le t \le \frac{\pi}{2} \end{cases}$$

$$I = \int\limits_{L_3} (xy-1)dx + x^2 y dy = \int\limits_0^{\frac{\pi}{2}} (\cos t \cdot 2\sin t - 1)\cdot(-\sin t dt) + \cos^2 t \cdot 2\sin t \cdot 2\cos t dt =$$

$$= \int\limits_0^{\frac{\pi}{2}} (-2\sin^2 t\cos t + \sin t)dt + 4\cos^3 t\sin t dt = \int\limits_0^{\frac{\pi}{2}} (4\cos^3 t\sin t + \sin t - 2\sin^2 t\cos t)dt =$$

$$= 4\int\limits_0^{\frac{\pi}{2}} \cos^3 t\sin t dt + \int\limits_0^{\frac{\pi}{2}} \sin t dt - 2\int\limits_0^{\frac{\pi}{2}} \sin^2 t\cos t dt \overset{*}{=}$$

$$\int \cos^3 t \sin t dt = \left| \begin{array}{l} \cos t = u \\ -\sin t dt = du \\ \sin t dt = -du \end{array} \right| = -\int u^3 du = -\frac{u^4}{4} + c = -\frac{\cos^4 t}{4} + c$$

$$\int \sin\, t\cos t dt = \left| \begin{array}{l} \sin t = u \\ \cos t dt = du \end{array} \right| = \int u^2 du = \frac{u^3}{3} + c = \frac{\sin^3 t}{3} + c$$

$$\overset{*}{=} 4\cdot\left(-\frac{\cos^4 t}{4}\right)\Bigg|_0^{\frac{\pi}{2}} - \cos t\Big|_0^{\frac{\pi}{2}} - 2\left(\frac{\sin^3 t}{3}\right)\Bigg|_0^{\frac{\pi}{2}} = 4\cdot\frac{1}{4} + 1 - 2\cdot\frac{1}{3} = 2 - \frac{2}{3} = \frac{4}{3}$$

Izračunati krivolinijski integral $\int_C x^3dx + 3zy^2dy - x^2y\,dz$ gdje je c dio prave od tačke $A(3,2,1)$ do tačke $O(0,0,0)$.

Rj.

jednačina prave kroz dvije tačke $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

$p(O,A): \quad \frac{x}{3} = \frac{y}{2} = \frac{z}{1} \quad (=t)$

$$\begin{cases} x=3t \\ y=2t \\ z=t \end{cases} \qquad \begin{array}{l} dx=3\,dt \\ dy=2\,dt \\ dz=dt \end{array}$$

Trebaju nam još granice za t

$A(3,2,1)$ $\quad \begin{array}{l} x=3t \\ y=2t \\ z=t \end{array} \quad \Rightarrow \quad t=1$

$O(0,0,0)$ $\quad \begin{array}{l} x=3t \\ y=2t \\ z=t \end{array} \quad \Rightarrow \quad t=0$

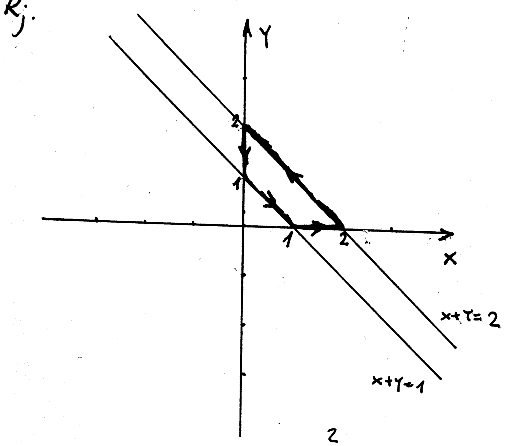
$$\int_C x^2dx + 3zy^2dy - x^2y\,dz =$$

$$= \int_1^0 \left[(3t)^3\cdot 3 + 3\cdot t\cdot(2t)^2\cdot 2 - (3t)^2\cdot 2t\cdot 1\right]dt$$

$$= \int_1^0 (81t^3 + 24t^3 - 18t^3)\,dt = -\int_0^1 87t^3\,dt = -87\cdot\frac{1}{4}t^4\Big|_0^1 = -\frac{87}{4}$$

Izračunati krivolinijski integral $I = \int_C (x^2+y^2)\,dx + x^2y\,dy$ gdje je c kontura trapeza koja obrazuju prave $x=0$, $y=0$, $x+y=1$, $x+y=2$.

Rj.



Ako je $c: y=\varphi(x), \ a\le x\le b$

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \left[P(x,\varphi(x)) + Q(x,\varphi(x))\cdot\varphi'(x)\right]dx$$

U našem slučaju postoje 4 krive

$C_1: \ y=0, \quad 1\le x\le 2$

$C_1: \ y=-x+2, \quad 2\ge x\ge 0$

$C_3: \ x=0, \quad 2\ge y\ge 1$

$C_4: \ y=-x+1, \quad 0\le x\le 1$

$I = I_1+I_2+I_3+I_4$, $\quad I_1 = \int_1^2 (x^2 + x^2\cdot 0)\,dx = \int_1^2 x^2\,dx = \frac13 x^3\Big|_1^2 = \frac13(8-1) = \frac73$

$$I_2 = \int_2^0 \left(x^2+(-x+2)^2 + x^2(-x+2)\cdot(-1)\right)dx = \int_2^0 (x^2+x^2-4x+4+x^3-2x^2)\,dx =$$

$$= \int_2^0 (x^3-4x+4)\,dx = \frac14 x^4\Big|_2^0 - 4\cdot\frac12 x^2\Big|_2^0 + 4x\Big|_2^0 = -4+8-8 = -4$$

$$I_3 = \int_2^1 (y^2\cdot 0 + 0y)\,dy = 0$$

$$I_4 = \int_0^1 \left(x^2+(-x+1)^2 + x^2(-x+1)\cdot(-1)\right)dx = \int_0^1 (x^2+x^2-2x+1+x^3-x^2)\,dx =$$

$$= \int_0^1 (x^3+x^2-2x+1)\,dx = \frac14 x^4\Big|_0^1 + \frac13 x^3\Big|_0^1 - 2\cdot\frac12 x^2\Big|_0^1 + x\Big|_0^1 = \frac14+\frac13-1+1 = \frac{7}{12}$$

$$I = I_1+I_2+I_3+I_4 = \frac73 + (-4) + \frac{7}{12} = \frac{28-48+7}{12} = -\frac{13}{12}$$ vrijednost krivolinijskog integrala

II način: Greenova formula ...

Zadaci za vježbu

U zadacima 3806 — 3821 izračunati date krivolinijske integrale.

3806. $\int\limits_L x\,dy$ po konturi trougla koji obrazuju koordinatne ose i prava $\frac{x}{2}+\frac{y}{3}=1$, — u pozitivnom smeru obilaženja (tj. nasuprot kretanju satne kazaljke).

3807. $\int\limits_L x\,dy$ po odsečku prave $\frac{x}{a}+\frac{y}{b}=1$, od tačke preseka sa apscisnom do tačke preseka sa ordinatnom osom.

3808. $\int\limits_L (x^2-y^2)\,dx$ po delu parabole $y=x^2$ od koordinatnog početka do tačke (2, 4).

3809. $\int\limits_L (x^2+y^2)\,dy$ po konturi četvorougla čija su temena (navedena po redu obilaženja): $A(0, 0)$, $B(2, 0)$, $C(4, 4)$ i $D(0, 4)$.

3810. $\int\limits_{(0,0)}^{(\pi, 2\pi)} -x\cos y\,dx+y\sin x\,dy$ duž pravolinijskog odsečka koji spaja tačke (0, 0) i $(\pi, 2\pi)$.

3811. $\int\limits_{(0,0)}^{(1,1)} xy\,dx+(y-x)\,dy$ duž krive 1) $y=x$, 2) $y=x^2$, 3) $y^2=x$, 4) $y=x^3$.

3812. $\int\limits_{(0,0)}^{(1,1)} 2xy\,dx+x^2\,dy$ duž krive 1) $y=x$, 2) $y=x^2$, 3) $y=x^2$, 4) $y^2=x$.

3813. $\int\limits_L y\,dx+x\,dy$ po delu kruga $x=R\cos t$, $y=R\sin t$, od $t_1=0$ do $t_2=\frac{\pi}{2}$.

3814. $\int\limits_L y\,dx-x\,dy$ po elipsi $x=a\cos t$, $y=b\sin t$, u pozitivnom smeru obilaženja.

3815. $\int\limits_L \frac{y^2\,dx-x^2\,dy}{x^2+y^2}$, po polukrugu $x=a\cos t$, $y=a\sin t$ od $t_1=0$ do $t_2=\pi$.

3816. $\int\limits_L (2a-y)\,dx-(a-y)\,dy$ duž prvog (računajući od koordinatnog početka) svoda cikloide $x=a(t-\sin t)$, $y=a(1-\cos t)$.

3817. $\int\limits_L \frac{x^2\,dy-y^2\,dx}{x^{\frac{5}{3}}+y^{\frac{5}{3}}}$, pri čemu je L deo astroide $x=R\cos^3 t$, $y=R\sin^3 t$ od tačke $(R, 0)$ do tačke $(0, R)$.

3818. $\int\limits_L x\,dx+y\,dy+(x+y-1)\,dz$ duž pravolinijskog odsečka od tačke (1, 1, 1) do tačke (2, 3, 4).

3819. $\int\limits_L yz\,dx+z\sqrt{R^2-y^2}\,dy+xy\,dz$ po zavojnici $x=R\cos t$, $y=R\sin t$, $z=\frac{at}{2\pi}$, od njenog preseka sa ravni $z=0$ do preseka sa ravni $z=a$.

3820. $\int\limits_{(1,1,1)}^{(4,4,4)} \frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2-x-y+2z}}$ duž prave linije.

3821. $\int\limits_L y^2\,dx+z^2\,dy+x^2\,dz$ duž krive po kojoj se seku sfera $x^2+y^2+z^2=R^2$ i cilindar $x^2+y^2=Rx$ $(R>0,\ z\geqslant 0)$, pri čemu je smer obilaženja po konturi, posmatran iz koordinatnog početka, suprotan kretanju satne kazaljke.

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Rješenja

3806. 3. **3807.** $\frac{ab}{2}$.

3808. $-\frac{56}{15}$. **3809.** $37\frac{1}{3}$.

3810. 4π. **3811.** 1) $\frac{1}{3}$; 2) $\frac{1}{12}$; 3) $\frac{17}{30}$; 4) $-\frac{1}{20}$.

3812. U sva četiri slučaja vrednost integrala je 1.

3813. 0. **3814.** $-2\pi ab$.

3815. $\frac{4}{3}a$. **3816.** πa^2.

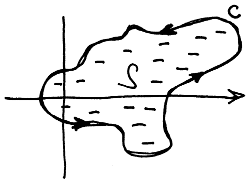
3817. $\frac{3}{16}\pi R\sqrt[3]{R}$

3818. 13. **3819.** $-\frac{a\pi R^4}{2}$

3820. $3\sqrt{3}$. **3821.** $-\frac{\pi R^3}{4}$.

Greenova formula za ravan

Ako je c po djelovima glatka granica područja S, a f-je $P(x,y)$ i $Q(x,y)$ neprekidne zajedno sa svojim parcijalnim izvodima prvog reda u zatvorenom području S+C, onda vrijedi Greenova formula



$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

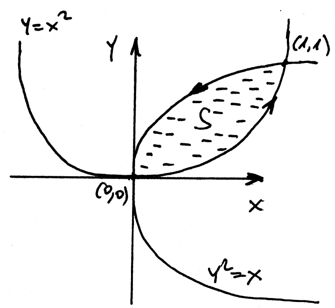
c - zatvorena kontura

S - oblast ograničena konturom

Izračunati integral $\int_C (2xy - x^2)\,dx + (x + y^2)\,dx$

gdje je c kontura površine ograničene sa $y = x^2$ i $y^2 = x$.

Rj.



$P(x,y) = 2xy - x^2 \qquad \frac{\partial P}{\partial y} = 2x$

$Q(x,y) = x + y^2 \qquad \frac{\partial Q}{\partial x} = 1.$

$$\int_C P\,dx - Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

formula Greena

$$\int_C (2xy - x^2)\,dx + (x + y^2)\,dx = \iint_S (1 - 2x)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (1 - 2x)\,dy\right] dx =$$

$$= \int_0^1 \left(y\Big|_{x^2}^{\sqrt{x}} - 2xy\Big|_{x^2}^{\sqrt{x}} \right) dx = \int_0^1 \left(\sqrt{x} - x^2 - 2x(\sqrt{x} - x^2)\right) dx =$$

$$= \int_0^1 \left(2x^3 - x^2 - 2x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) = 2 \cdot \frac{1}{4} x^4\Big|_0^1 - \frac{1}{2}x^3\Big|_0^1 - 2 \cdot \frac{2}{5} x^{\frac{5}{2}}\Big|_0^1 + \frac{2}{3} x^{\frac{3}{2}}\Big|_0^1 = \frac{1}{30}$$

Izračunati krivoliniske integrale

a) $\oint_{-l} 2x\,dx - (x + 2y)\,dy$ i b) $\oint_{+l} y\cos x\,dx + \sin x\,dy$

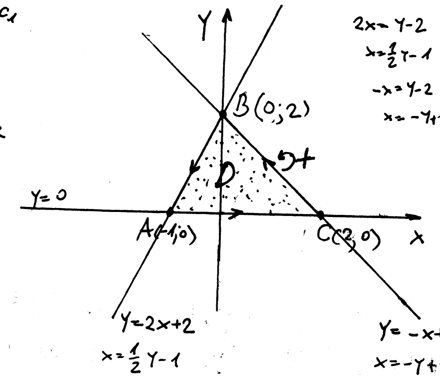
po krivoj l, gdje je l trougao čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj.

$\int_C P(x,y)\,dx + Q(x,y)\,dy$ je krivoliniski integral druge vrste.

Ako je kriva c data u obliku $y = \eta(x)$, $c_1 \le x \le c_2$ dati integral se računa po formuli

$$\int_{c_1}^{c_2} \left(P(x,\eta(x)) + Q(x,\eta(x))\,\eta'(x)\right) dx$$



$2x = y - 2$
$x = \frac{1}{2}y - 1$
$-x = y - 2$
$x = -y + 2$

Skicirajmo tačke u xOy ravni

prava koja prolazi kroz tačke A i B je

$\frac{x}{-1} + \frac{y}{2} = 1 \quad /\cdot 2$

$-2x + y = 1$

$y = 2x + 2 \quad \Rightarrow \quad y' = 2$

prava koja prolazi kroz tačke B i C

$\frac{x}{2} + \frac{y}{2} = 1 \quad /\cdot 2$

$x + y = 2$

$y = -x + 2 \quad \Rightarrow \quad y' = -1$

a)
$$\oint_{-l} 2x\,dx - (x+2y)\,dy = \int_{AB} 2x\,dx - (x+2y)\,dy + \int_{BC} 2x\,dx - (x+2y)\,dy + \int_{CA} 2x\,dx - (x+2y)\,dy$$

AB po pravoj $y = 2x + 2$; BC po pravoj $y = -x + 2$; CA po pravoj $y = 0$

$$\int_{AB} 2x\,dx - (x+2y)\,dy = \int_{-1}^{0} \Big[2x - \underbrace{(x + 2(2x+2))\,2}_{\substack{(x+4x+4)\cdot 2 \\ 10x+8}}\Big] dx = \int_{-1}^{0} (-8x - 8)\,dx = -8 \cdot \frac{1}{2}x^2\Big|_{-1}^{0} - 8x\Big|_{-1}^{0}$$

$$= (-4)(-1) - 8 = -4$$

$$\int_{BC} 2x\,dx-(x+2y)\,dy=\int_0^2\Big[2x-\big(x+2(-x+2)\big)(-1)\Big]dx=\int_0^2 (x+4)\,dx=\frac{1}{2}x^2\Big|_0^2+$$

$$\underbrace{(x-2x+4)(-1)}_{x-4}$$

BC po pravoj $y=-x+2$

$$+4x\Big|_0^2=2+8=10$$

$$\int_{CA} 2x\,dx-(x+2y)\,dy=\int_2^{-1}\Big[2x-\big(x+2(0)\big)0\Big]dx=\int_2^{-1}2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=1-4=-3$$

CA po pravoj $y=0$

$$\oint_{-\ell} 2x\,dx-(x+2y)\,dy=-4+10-3=3$$ traženo rješenje

b) Možemo upotrebiti Greenovu formulu

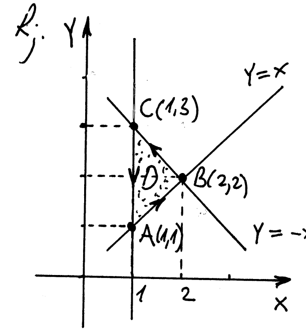
$$\int_C P(x,y)\,dx+Q(x,y)\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$ gdje je D oblast ograničena konturom c

$$\oint_{+\ell} y\cos x\,dx+\sin x\,dy=\left|\begin{array}{ll} Q(x,y)=\sin x & P(x,y)=y\cos x\\ \frac{\partial Q}{\partial x}=\cos x & \frac{\partial P}{\partial y}=\cos x\\ \text{D-vidi sliku (tačkasti dio sa slike)} & \end{array}\right|=$$

$$=\iint_D(\cos x-\cos x)\,dx\,dy=\iint_D 0\,dx\,dy=0$$ traženo rješenje

(#) Izračunati $\int_C 2(x^2+y^2)\,dx+(x+y)^2\,dy$ gdje je c kontura trougla ΔABC pozitivno orjentisana $(A(1,1),\ B(2,2),\ C(1,3))$.

R:



$$P(x,y)=2(x^2+y^2)=2x^2+2y^2$$
$$Q(x,y)=(x+y)^2=x^2+2xy+y^2$$

$$\int_C P(x,y)\,dx+Q(x,y)\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$ formula Grina

$$y-y_1=\frac{x_2-x_1}{y_2-y_1}(x-x_1)$$
$$y-2=\frac{-1}{1}(x-2)$$
$$y-2=-x+2\ \Rightarrow\ y=-x+4$$

$$\frac{\partial P}{\partial y}=4y\qquad \frac{\partial Q}{\partial x}=2x+2y\qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=2x+2y-4y=2x-2y$$

$$D:\begin{cases}1\le x\le 2\\ x\le y\le 4-x\end{cases}\qquad \int_C 2(x^2+y^2)\,dx+(x+y)^2\,dy=\iint_D(2x-2y)\,dx\,dy=$$

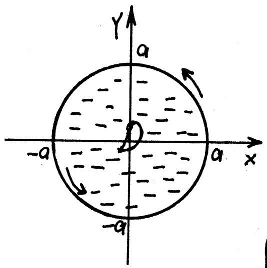
$$=\int_1^2\Big[\int_x^{4-x}(2x-2y)\,dy\Big]dx=\int_1^2\Big(\underbrace{2xy\Big|_x^{4-x}}_{4-x-x}-2\cdot\frac{1}{2}\underbrace{y^2\Big|_x^{4-x}}_{(4-x)^2-x^2}\Big)dx=$$

$$=\int_1^2\Big(2x(4-2x)-(16-8x)\Big)dx=\int_1^2(8x-4x^2-16+8x)\,dx=\int_1^2(-4x^2+16x-16)\,dx$$

$$=-4\cdot\frac{1}{3}x^3\Big|_1^2+16\cdot\frac{1}{2}x^2\Big|_1^2-16x\Big|_1^2=-\frac{4}{3}\cdot 7+8\cdot 3-16=8-\frac{28}{3}=-\frac{4}{3}$$

Izračunati $\int\limits_c xy^2\,dy - x^2y\,dx$ gdje je c krug $x^2+y^2=a^2$. Integraciju izvesti u pozitivnom smjeru.

Rj.



$P(x,y) = -x^2y \qquad \frac{\partial P}{\partial y} = -x^2$

$Q(x,y) = xy^2 \qquad \frac{\partial Q}{\partial x} = y^2 \qquad D: x^2+y^2 \le a^2$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + x^2 = x^2+y^2$$

$$\int\limits_c P(x,y)\,dx + Q(x,y)\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Formula Greena

polarne koordinate $x = r\cos\varphi$, $y = r\sin\varphi$ $\Rightarrow$ $D': \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$ $\quad dx\,dy = r\,dr\,d\varphi$

$$\int\limits_c xy^2\,dy - x^2y\,dx = \iint\limits_D (x^2+y^2)\,dx\,dy = \iint\limits_{D'} (r^2\cos^2\varphi + r^2\sin^2\varphi)\cdot r\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\left[\int_0^a r^3\,dr\right]d\varphi = \int_0^{2\pi} \frac{1}{4} r^4\Big|_0^a\,d\varphi = \frac{a^4}{4}\cdot\varphi\Big|_0^{2\pi} = \frac{\pi a^4}{2}$$

Izračunati krivoliniski integral

$$I = \int\limits_c (xy+x+y)\,dx + (xy+x-y)\,dy \quad \text{ako je } c: x^2+y^2=3x.$$

Rj.

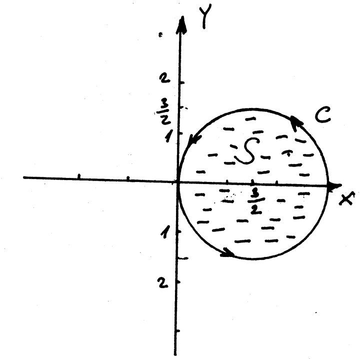
$x^2+y^2=3x$

$x^2-3x+y^2=0$

$x^2 - 2\cdot x\cdot\frac{3}{2} + \frac{9}{4} - \frac{9}{4} + y^2 = 0$

$\left(x-\frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

C: Krug sa centrom u tački $\left(\frac{3}{2}, 0\right)$ poluprečnika $r = \frac{3}{2}$



I način: Greenova formula za ravan

$$\int\limits_c P\,dx + Q\,dy = \iint\limits_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

c - zatvorena kontura
S - oblast ograničena konturom

$P = xy+x+y$, $\frac{\partial P}{\partial y} = x+1$, $Q = xy+x-y$, $\frac{\partial Q}{\partial x} = y+1$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y+1-(x+1) = y-x$$

Kako je c krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate $x = \frac{3}{2} + r\cos\varphi$, $y = r\sin\varphi$, $dx\,dy = r\,dr\,d\varphi$

$D: \begin{cases} 0 \le r \le \frac{3}{2} \\ 0 \le \varphi \le 2\pi \end{cases}$

$$I = \int\limits_c (xy+x+y)\,dx + (xy+x-y)\,dy = \iint\limits_S (y-x)\,dx\,dy = \iint\limits_D \left(r\sin\varphi - \left(\frac{3}{2} + r\cos\varphi\right)\right)\cdot r\,dr\,d\varphi$$

$$= \int_0^{3/2}\left[\int_0^{2\pi}\left(r^2\sin\varphi - \frac{3}{2}r - r^2\cos\varphi\right)d\varphi\right]dr = \int_0^{3/2}\left(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0} - \frac{3r}{2}\varphi\Big|_0^{2\pi} \underbrace{- r^2\sin\varphi\Big|_0^{2\pi}}_{=0}\right)dr$$

$$= \int_0^{3/2} -3\pi r\,dr = -3\pi\frac{r^2}{2}\Big|_0^{3/2} = -\frac{3}{2}\pi\cdot\frac{9}{4} = -\frac{27}{8}\pi$$

II način: Klasičan način

C kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$

$$\int_C P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_C P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

U našem slučaju c je kružnica. Parametrizirajmo kružnicu

$x=\frac{3}{2}+r\cos\varphi$

$y=r\sin\varphi$

U našem slučaju $r=\frac{3}{2}$ a umjesto promjenjive φ stavimo promjenjivu t

$\frac{\partial x}{\partial t}=-\frac{3}{2}\sin t$ $\qquad x=\frac{3}{2}+\frac{3}{2}\cos t$

$\frac{\partial y}{\partial t}=\frac{3}{2}\cos t$ $\qquad y=\frac{3}{2}\sin t$ $\qquad$ gdje $0\le t\le 2\pi$

$$I=\int_c (xy+x+y)\,dx+(xy+x-y)\,dy=\int_0^{2\pi}\Big[\Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big)+\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)+$$

$$+\big(\tfrac{3}{2}\sin t\big)\Big)\big(-\tfrac{3}{2}\sin t\big)+\Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big)+\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)-$$

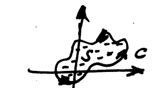
$$-\big(\tfrac{3}{2}\sin t\big)\Big)\tfrac{3}{2}\cos t\Big]dt=\dots$$

na klasičan način ovo je komplikovano ali se može izračunati

$$I=-\frac{27}{8}\pi$$

Pomoću Greenove formule izračunati integral $I=\int_C (xy+x+y)\,dx+(xy+x-y)\,dy$, ako je c kontura kružnice $x^2+y^2=ax$ pređena u pozitivnom smislu.

Rj: Greenova formula $\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$



$P(x,y)=xy+x+y$ $\qquad \frac{\partial P}{\partial y}=x+1$, $\frac{\partial Q}{\partial x}=y+1$

$Q(x,y)=xy+x-y$

$x^2+y^2=ax$

$x^2-ax+y^2=0$

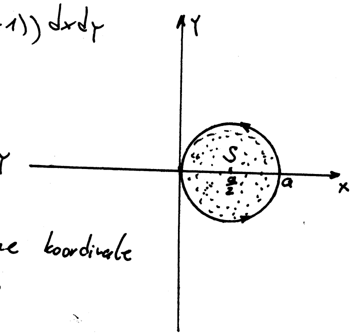
$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

krug sa centrom u $\left(\frac{a}{2},0\right)$ poluprečnika $\frac{a}{2}$

$I=\iint_S (y+1-(x+1))\,dx\,dy$

$I=\iint_S (y-x)\,dx\,dy$



uvedimo polarne koordinate

$x=\frac{a}{2}+r\sin\varphi$

$y=r\cos\varphi$

$dx\,dy=r\,dr\,d\varphi$

$S \xrightarrow{\text{transformiši}} S':\begin{cases}0\le r\le \frac{a}{2}\\ 0\le\varphi\le 2\pi\end{cases}$

$$I=\iint_{S'}\left(r\cos\varphi-\frac{a}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint_{S'}\left(r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right)dr\,d\varphi=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\frac{a}{2}}\left[r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right]dr=\int_0^{2\pi}\left[\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}(\cos\varphi-\sin\varphi)-\frac{a}{2}\cdot\frac{1}{2}r^2\Big|_0^{\frac{a}{2}}\right]d\varphi$$

$$=\frac{a^3}{24}\int_0^{2\pi}(\cos\varphi-\sin\varphi)\,d\varphi-\frac{a^3}{16}\int_0^{2\pi}d\varphi=\frac{a^3}{24}\Big(\underbrace{\sin\varphi\Big|_0^{2\pi}}_{=0}+\underbrace{\cos\varphi\Big|_0^{2\pi}}_{1-1}\Big)-\frac{a^3}{16}2\pi=$$

$$=-\frac{a^3\pi}{8}$$ traženo rješenje

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Zadaci za vježbu

U zadacima 3822—3823 krivolinijske integrale po zatvorenim konturama L, uzete u pozitivnom smeru obilaženja, transformisati u dvojne integrale po oblastima, ograničenim tim konturama.

3822. $\int_L (1-x^2)\, y\, dx + x\,(1+y^2)\, dy.$

3823. $\int_L (e^{xy} + 2x\cos y)\, dx + (e^{xy} - x^2 \sin y)\, dy.$

3824. Izračunati integral u zadatku 3822, ako je kontura integracije L krug $x^2+y^2=R^2$, na dva načina:

1) neposredno;
2) primenom Grinove formule.

3825. Izračunati $\int_L (xy+x+y)\, dx + (xy+x-y)\, dy$, pri čemu je kontura integracije L: 1) elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; 2) krug $x^2+y^2=ax$, a integral se uzima oba puta u pozitivnom smeru obilaženja. (Račun izvesti na dva načina: 1) neposredno, i 2) primenom Grinove formule).

3826. Dokazati da je integral $\int_L (yx^3+e^y)\, dx + (xy^3+x\,e^y-2y)\, dy$ jednak nuli ako je putanja integracije L zatvorena kriva simetrična u odnosu na koordinatni početak.

3827. Primenom Grinove formule izračunati razliku integrala

$$I_1 = \int_{AmB} (x+y)^2\, dx - (x-y)^2\, dy$$

i

$$I_2 = \int_{AnB} (x+y)^2\, dx - (x-y)^2\, dy,$$

pri čemu je AmB pravolinijski odsečak koji spaja tačke $A(0,\,0)$ i $B(1,\,1)$, a AnB je luk parabole $y=x^2$.

3828. Pokazati da je vrednost integrala $\int_L \{x\cos(N,\,x)+y\sin(N,\,x)\}\, dS$, u kojem je $(N,\,x)$ ugao između spoljne normale krive L i pozitivnog smera apscisne ose, uzetog u pozitivnom smeru obilaženja po zatvorenoj krivoj L, jednaka dvostrukoj površini oblasti ograničene zatvorenom krivom L.

3829. Dokazati da integral $\int_L (2xy-y)\, dx + x^2\, dy$, uzet po zatvorenoj krivoj L, izražava površinu oblasti ograničene tom krivom.

3830. Dokazati da je integral $\int_L \varphi(y)\, dx + [x\varphi'(y)+x^3]\, dy$ jednak trostrukom momentu inercije homogene ravne figure ograničene konturom L, u odnosu na ordinatnu osu.

Rješenja

3822. $\iint_D (x^2+y^2)\, dx\, dy.$

3823. $\iint_D (y-x)\, e^{xy}\, dx\, dy.$

3824. $\frac{\pi R^4}{2}.$

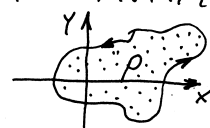
3825. 1) 0; 2) $-\frac{\pi a^3}{8}.$

3827. $\frac{1}{3}.$

Računanje površine ravne figure

Površinu figure ograničenu zatvorenom linijom c računamo po formuli:

$$P=\frac{1}{2}\int_c x\,dy - y\,dx .$$



Podrazumjeva se da po liniji c prelazimo u pozitivnom smjeru

Pokazati da se površina ograničena jednostruko zatvorenom krivom (konturom) c računa po formuli

$$\frac{1}{2}\int_c x\,dy - y\,dx$$

Rj. U formuli Greena stavimo $P(x,y)=-y$, $Q(x,y)=x$. Tada

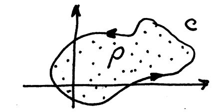
$$\int_c x\,dy - y\,dx = \iint_S\left(\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right)dx\,dy = 2\iint_S dx\,dy = 2\cdot P$$

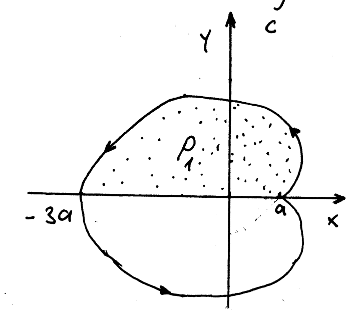
gdje je P tražena površina. Prema tome $P=\frac{1}{2}\int_c x\,dy-y\,dx$

Uz pomoć krivoliniskog integrala druge vrste, izračunati površinu, ograničenu kardioidom $x=2\cos t-\cos 2t$, $y=2\sin t-\sin 2t$.

Rj. Prisjetimo se, površina figure ograničene krivom c se računa po formuli

$$P=\frac{1}{2}\oint_c x\,dy-y\,dx$$





kardioida
$x=2a\cos t-a\cos 2t$
$y=2a\sin t-a\sin 2t$
$t=0:\ x=a,\ y=0$
$t=\pi:\ x=-3a,\ y=0$

Primjetimo da je kardioida kriva linija koja je simetrična u odnosu na x-osu, pa da bi izračunali površinu ograničenu kardioidom dovoljno je izračunati površinu iznad x-ose

Da bismo opisali kardioidu parametar t uzima vrijednosti od 0 do 2π.

Prisjetimo se, ako je kriva c data u parametarskom obliku $x=\mu(t)$, $y=\eta(t)$, $t_1\le t\le t_2$ tada se krivoliniski integral računa po formuli

$$\int_c P(x,y)dx+Q(x,y)dy=\int_c\left(P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right)dt$$

$$P=\frac{1}{2}\oint_c x\,dy-y\,dx=\left|\begin{array}{l} x=2\cos t-\cos 2t\\ dx=(-2\sin t+2\sin 2t)dt\\ y=2\sin t-\sin 2t\\ dy=(2\cos t-2\cos 2t)dt\end{array}\right|=\frac{1}{2}\int_0^{2\pi}(2\cos t-\cos 2t)\cdot(2\cos t-2\cos 2t)$$

$$=\frac{1}{2}\int_0^{2\pi}\left[(2\cos t-\cos 2t)(2\cos t-2\cos 2t)-(2\sin t-\sin 2t)(-2\sin t+2\sin 2t)\right]dt=2P_1$$

$$=\int_0^{\pi}\left(\underline{4\cos^2 t}-6\cos t\cos 2t+\underline{\underline{2\cos^2 2t}}+\underline{4\sin^2 t}-6\sin t\sin 2t+\underline{\underline{2\sin^2 2t}}\right)dt=$$

$$=\int_0^{\pi}(6-6\cos t\cos 2t-6\sin t\sin 2t)\,dt=6\int_0^{\pi}(1-\cos(t-2t))\,dt=\ldots=6\pi$$

(#) Izračunati pomoću krivolinijskog integrala II vrste površinu ravne figure ograničene konturom

$$C: \begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \\ 0 \le t \le 2\pi . \end{cases}$$

Rj. Površina figure ograničenu zatvorenom linijom C računamo po formuli $P = \frac{1}{2}\int_C x\,dy - y\,dx$.

$x = a(t-\sin t) \qquad y = a(1-\cos t)$

$dx = a(1-\cos t) \qquad dy = a\sin t$

$$x\,dy - y\,dx = a(t-\sin t)\cdot a\sin t - a(1-\cos t)\cdot a(1-\cos t)$$
$$= a^2 t\sin t - a^2\sin^2 t - a^2(1-\cos t)^2$$
$$= a^2(t\sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t)$$
$$= a^2(t\sin t + 2\cos t - 2)$$

$$P = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a^2(t\sin t + 2\cos t - 2))\,dt =$$

$$= \frac{a^2}{2}\left(\int_0^{2\pi} t\sin t\,dt + 2\int_0^{2\pi}\cos t\,dt - 2\int_0^{2\pi} dt\right) = \dots = \frac{a^2}{2}(-2\pi + 0 - 4\pi) = 3a^2\pi$$

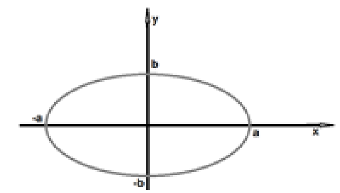
1. Izračunati površinu figure koja je ograničena krivom:

a) elipsom $x = a\cos t,\ \ y = b\sin t$;

b) petljom Dekartovim listom $x^3 + y^3 - 3axy = 0$.

Rješenja:

a)



Slika 1: elipsa

Koristit ćemo sljedeću formulu:

$$P = \frac{1}{2}\oint_{C_1} x\,dy - y\,dx,$$

gdje je (vidi sliku 1)

$$C_1 = \begin{cases} x = a\cos t \\ y = b\sin t \\ 0 \le t \le 2\pi \end{cases}$$

Izračunajmo izvode od x i y:

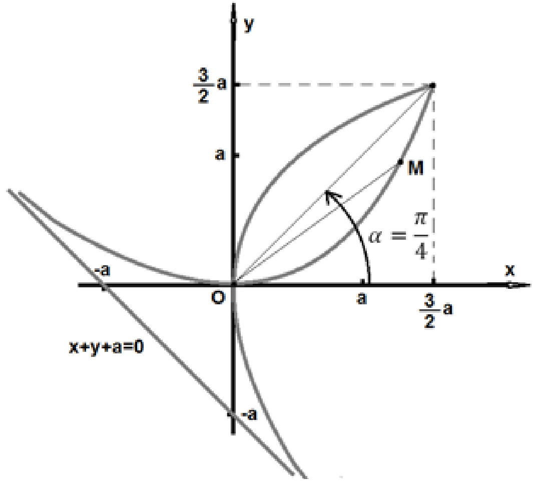
$$dx = -a\sin t\,dt$$
$$dy = b\cos t\,dt$$

Uvrstimo u formulu:

$$P = \frac{1}{2}\oint_{C_1} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}\left(a\cos t\, b\cos t - b\sin t\,(-a)\sin t\right)dt$$
$$= \frac{1}{2}\int_0^{2\pi}(ab\cos^2 t + ab\sin^2 t)dt = \frac{1}{2}ab\int_0^{2\pi}\left(\cos^2 t + \sin^2 t\right)dt$$
$$= \frac{1}{2}ab\int_0^{2\pi} 1\,dt = \frac{1}{2}ab\left(t\Big|_0^{2\pi}\right) = \frac{1}{2}ab(2\pi - 0) = \boldsymbol{ab\pi}$$

Konačno rješenje: $\boldsymbol{P = ab\pi}$.

b)



Slika 2: Dekartov list

Da bismo koristili formulu

$$P=\frac{1}{2}\oint_{C_1} xdy-ydx,$$

moramo preći na parametarsku jednačinu krive uzevši:

$$y=tx,\ t=\frac{y}{x}$$

Vidimo da polarni radijus OM (vidi sliku 2), gdje je O(0,0) i M(x,y), opisuje cijelu petlju krive kada t ide od 0 do +∞.

Uvrstimo smjenu $y=tx$ u $x^3+y^3-3axy=0$ te na dobiveni rezultat unijeti i smjenu $x=\frac{y}{t}$

pa ćemo imati:

$$x^3+(tx)^3-3ax(tx)=0$$

$$x^3\left(1+t^3\right)-3tax^2=0\ /:x^2$$

$$\frac{x^3\ (1+t^3\ )-3tax^2}{x^2}=0$$

$$x\left(1+t^3\right)-3ta=0$$

$$x\left(1+t^3\right)=3ta$$

$$\boldsymbol{x}=\frac{3\boldsymbol{ta}}{1+\boldsymbol{t}^3}$$

$$x=\frac{3ta}{1+t^3}$$

$$\frac{y}{t}=\frac{3ta}{1+t^3}$$

$$\frac{y}{t}=\frac{3ta}{1+t^3}$$

Pa dalje računamo izvod za x:

$$dx=\frac{3a\left(1+t^3\right)-3ta\ (3t^2)}{(1+t^3)^2}dt$$

$$dx=3a\frac{1+t^3-t(3t^2)}{(1+t^3)^2}dt$$

$$\boldsymbol{dx}=3\boldsymbol{a}\frac{1-2\boldsymbol{t}^3}{(1+\boldsymbol{t}^3)^2}\ \boldsymbol{dt}$$

te i za y :

$$dy=\frac{6at\left(1+t^3\right)-3at^2(3t^2)}{(1+t^3)^2}dt$$

$$dy=3at\frac{2\left(1+t^3\right)-t(3t^2)}{(1+t^3)^2}dt$$

$$dy=3at\frac{2+2t^3-3t^3}{(1+t^3)^2}dt$$

$$\boldsymbol{dy}=3\boldsymbol{at}\frac{2-\boldsymbol{t}^3}{(1+\boldsymbol{t}^3)^2}\ \boldsymbol{dt}$$

Pomnožimo izvode sa dx i dy sa y i x, redom

$$x\,dy=\frac{3ta}{\left(1+t^3\right)}3at\frac{2-t^3}{(1+t^3)^2}dt$$

$$\boldsymbol{x\ dy}=9\boldsymbol{a}^2\boldsymbol{t}^2\frac{2-\boldsymbol{t}^3}{\left(1+\boldsymbol{t}^3\right)^3}\ \boldsymbol{dt}$$

$$y\,dx=\frac{3t^2a}{1+t^3}3a\frac{1-2t^3}{(1+t^3)^2}dt$$

$$\boldsymbol{y\ dx}=9\boldsymbol{a}^2\boldsymbol{t}^2\frac{1-2\boldsymbol{t}^3}{(1+\boldsymbol{t}^3)^3}\ \boldsymbol{dt}$$

Sad uvrstimo dobijene rezultate:

$$P=\frac{1}{2}\oint_C x\,dy-y\,dx=\frac{1}{2}\int_0^\infty\left(9a^2t^2\frac{2-t^3}{\left(1+t^3\right)^3}-9a^2t^2\frac{1-2t^3}{(1+t^3)^3}\right)dt$$

$$=\frac{1}{2}\int_0^\infty 9a^2t^2\frac{2-t^3-1+2t^3}{\left(1+t^3\right)^3}dt=\frac{9a^2}{2}\int_0^\infty t^2\frac{1+t^3}{\left(1+t^3\right)^3}dt=$$

$$= \frac{9a^2}{2}\int_0^{\infty}\frac{t^2}{\left(1+t^3\right)^2}dt = \left| \begin{array}{ll} 1+t^3=u & \\ 3t^2dt=du & \begin{array}{l} u_1=1 \\ u_2=\infty \end{array} \\ t^2dt=\frac{1}{3}du & \end{array} \right| = \frac{3a^2}{2}\int_1^{\infty}\frac{1}{u^2}du$$

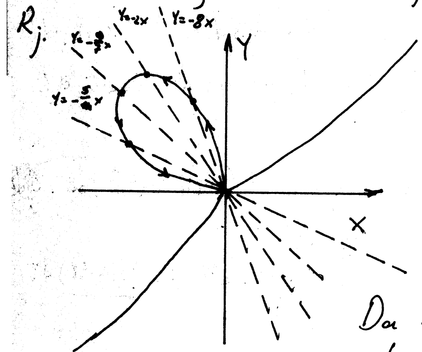
$$= \frac{3a^2}{2}\int_1^{\infty}u^{-2}du = \frac{3a^2}{2}\left(\frac{1}{-1}\right)\left(u^{-1}\Big|_1^{\infty}\right) = -\frac{3a^2}{2}(0-1) = \frac{3\boldsymbol{a}^2}{2}$$

Prema tome

$$\boldsymbol{P} = \frac{3\boldsymbol{a}^2}{2}.$$

\# Uz pomoć krivolinijskog integrala izračunati površinu Dekartovog lista dobijen petljom $x^3+y^3-3axy=0$.

Rj.



$$P = \frac{1}{2}\int_C x\,dy - y\,dx$$

Da bismo upotrebili ovu formulu potrebno je parametrizovati krivu.

Da bismo parametrizovali datu petlju, stavimo $y=tx$. Tada iz jednačine krive dobijamo:

$$x^3+y^3-3axy=0$$

$$x^3+t^3x^3-3atx^2=0 \quad /:x^2$$

$$x(1+t^3)=3at$$

$$x=\frac{3at}{1+t^3}$$

(Pokušajte sa slike shvatiti zašto smo stavili $y=tx$!!!)

$$y=tx$$

$$y=\frac{3at^2}{1+t^3}$$

$$dx = 3a\,d\left(\frac{t}{1+t^3}\right) = 3a\frac{1+t^3-t\cdot 3t^2}{(1+t^3)^2}dt = 3a\frac{1-2t^3}{(1+t^3)^2}dt$$

$$dy = 3a\,d\left(\frac{t^2}{1+t^3}\right) = 3a\frac{2t(1+t^3)-t^2\cdot 3t^2}{(1+t^3)^2} = 3at\frac{2+2t^3-3t^3}{(1+t^3)^2} = 3at\frac{2-t^3}{(1+t^3)^2}$$

$$x\,dy = 3at\cdot\frac{1}{1+t^3}\cdot 3at\cdot\frac{2-t^3}{(1+t^3)^2}dt = (3at)^2\frac{2-t^3}{(1+t^3)^3}dt$$

$$y\,dx = 3at\,\frac{t}{1+t^3}\cdot 3a\frac{1-2t^3}{(1+t^3)^2}dt = (3at)^2\frac{1-2t^3}{(1+t^3)^3}dt$$

$$P = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_{-\infty}^{0} 9a^2t^2\frac{\overbrace{2-t^3-1+2t^3}^{1+t^3}}{(1+t^3)^3}dt = \frac{9a^2}{2}\int_{-\infty}^{0}\frac{t^2}{(1+t^3)^2}dt =$$

$$= \left| \begin{array}{l} 1+t^3=u \\ 3t^2dt=du \\ t^2dt=\frac{1}{3}du \end{array} \right| = \frac{3a^2}{2}\int_{-\infty}^{0}\frac{du}{u^2} = \frac{3a^2}{2}\cdot\frac{u^{-1}}{-1} = -\frac{3a^2}{2}\cdot\frac{1}{1+t^3}\Bigg|_{-\infty}^{0}$$

$$= -\frac{3a^2}{2}(1-0) = -\frac{3a^2}{2}$$

Površina je uvijek pozitivna $\quad P=\frac{3a^2}{2}$

(#) Izračunati površinu figure koja je ograničena krivom $x=a\cos^3 t$, $y=a\sin^3 t$, $0\le t\le 2\pi$.

Rj. $P=\frac{1}{2}\oint_C x\,dy-y\,dx$, $\quad C:\begin{cases} x=a\cos^3 t\\ y=a\sin^3 t\\ 0\le t\le 2\pi\end{cases}$ $\quad dx=3a\cos^2 t\cdot(-\sin t)\,dt$, $\quad dy=3a\sin^2 t\cos t\,dt$

$$P=\frac{1}{2}\oint_C x\,dy-y\,dx=\frac{1}{2}\int_0^{2\pi}\left(a\cos^3 t\cdot 3a\sin^2 t\cos t-a\sin^3 t\cdot 3a\cos^2 t\cdot(-\sin t)\right)dt$$

$$=\frac{1}{2}\cdot 3a^2\int_0^{2\pi}\left(\sin^2 t\cos^4 t+\sin^4 t\cos^2 t\right)dt=\frac{3}{2}a^2\int_0^{2\pi}\sin^2 t\cos^2 t\underbrace{(\cos^2 t+\sin^2 t)}_{1}dt$$

$$=\frac{3}{2}a^2\int_0^{2\pi}\frac{1}{4}\underbrace{(2\sin t\cos t)}_{\sin 2t}{}^2\,dt=\frac{3}{8}a^2\int_0^{2\pi}\sin^2 2t\,dt\overset{(*)}{=}\frac{3}{8}a^2\int_0^{2\pi}\frac{1}{2}(1-\cos 4t)\,dt$$

$1=\sin^2 2t+\cos^2 2t$
$\cos 4t=\cos^2 2t-\sin^2 2t$ $\Rightarrow$ $1-\cos 4t=2\sin^2 2t$ ...(*)

$$=\frac{3}{16}a^2\left(t\Big|_0^{2\pi}-\frac{1}{4}\sin 4t\Big|_0^{2\pi}\right)$$

$$=\frac{3}{16}a^2(2\pi-0)=\frac{3}{8}a^2\pi$$

Zadaci za vježbu

U zadacima 3861 — 3868 pomoću krivolinijskog integrala izračunati površinu oblasti ograničene datim zatvorenim krivama.

3861. Elipsom $x=a\cos t$, $y=b\sin t$.

3862. Astroidom $x=a\cos^3 t$, $y=a\sin^3 t$.

3863. Kardioidom $x=2a\cos t-a\cos 2t$, $y=2a\sin t-a\sin 2t$.

3864*. Petljom dekartova lista $x^3+y^3-3axy=0$.

3865. Petljom krive $(x+y)^3=xy$.

3866. Petljom krive $(x+y)^4=x^2y$.

3867*. Bernulijevom lemniskatom $(x^2+y^2)^2=2a^2(x^2-y^2)$.

3868. Petljom krive $(\sqrt{x}+\sqrt{y})^{12}=xy$.

Rješenja

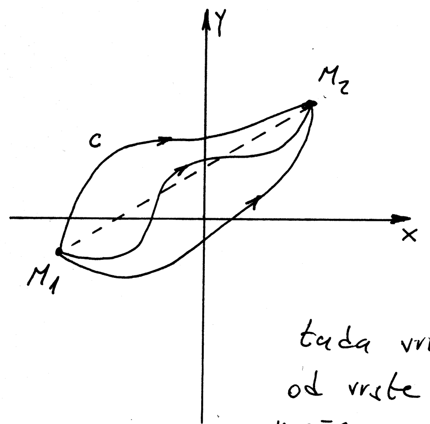
3861. πab. **3862.** $\frac{3}{8}\pi a^2$. **3863.** $6\pi a^2$.

3864*. $\frac{3}{2}a^2$. Preći na parametarske jednačine krive, stavljajući $y=tx$.

3865. $\frac{1}{60}$. **3866.** $\frac{1}{210}$. **3867*.** $2a^2$. Staviti $y=x\,\mathrm{tg}\,t$.

3868*. $\frac{1}{30}$. Staviti $y=xt^2$.

Nezavisnost krivoliniskog integrala od vrste krive linije. Određivanje primitivnih f-ja



Ako je data kriva linija c koja spaja tačke $M_1(a,b)$ i $M_2(c,d)$ (pri čemu je M_1 početak a M_2 kraj krive linije c) i krivoliniski integral

$$I=\int_c P(x,y)\,dx+Q(x,y)\,dy$$

kod kojeg vrijedi $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

tada vrijednost integrala I ne zavisi od vrste krive linije c (za krivu liniju c možemo uzeti bilo koju krivu koja spaja tačke M_1 i M_2).

Vrijednost integrala obično tražimo tako što nađemo f-ju $u=u(x,y)$ za koju vrijedi $du(x,y)=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=P(x,y)dx+Q(x,y)dy$

pa imamo

$$I=\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_c du(x,y)=u(x,y)\Big|_{(a,b)}^{(c,d)}=u(c,d)-u(a,b)$$

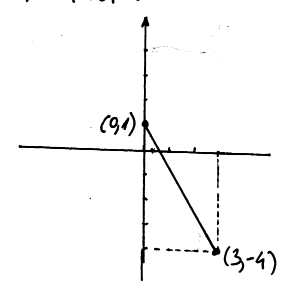
Izračunati krivoliniski integral $\int_{(0,1)}^{(3,-4)} x\,dx+y\,dy$.

Rj: Integral $I=\int_c P\,dx+Q\,dy$ kod kojeg vrijedi $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, vrijednost integrala I ne zavisi od vrste krive linije c.

U našem slučaju $P(x,y)=x$, $Q(x,y)=y$ $\qquad \frac{\partial P}{\partial y}=0,\ \frac{\partial Q}{\partial x}=0$

Prema tome vrijednost ne zavisi od vrste izbora krive linije c koja spaja tačke $(0,1)$ i $(3,-4)$.

I način



Ako je c data kriva u ravni opisana jednačinom $y=\eta(x)$ $(x\in[a,b])$ tada

$$\int_c P\,dx+Q\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

$y-y_1=k(x-x_1)$

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$A(0,1)$ (x_1 y_1), $B(3,-4)$

$y-1=\frac{-5}{3}(x-0)$

$y=-\frac{5}{3}x+1$

$y'=-\frac{5}{3}$

$$\int_{(0,1)}^{(3,-4)} x\,dx+y\,dy=\int_0^3\left(x+\left(-\frac{5}{3}x+1\right)\cdot\left(-\frac{5}{3}\right)\right)dx=\int_0^3\left(x+\frac{25}{9}x-\frac{5}{3}\right)dx=$$

$$=\left(1+\frac{25}{9}\right)\frac{x^2}{2}\Big|_0^3-\frac{5}{3}x\Big|_0^3=\frac{\cancel{34}^{17}}{\cancel{9}}\cdot\frac{\cancel{9}}{\cancel{2}}-\frac{5}{3}\cdot 3=17-5=12$$

II način

$P(x,y)dx+Q(x,y)dy=0$ i $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ je egzaktna diferencijalna jednačina

Riješimo diferenc. jedn. $\quad x\,dx+y\,dy=0$

$u=u(x,y)$

$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$

$\frac{\partial u}{\partial x}=x \quad \frac{\partial u}{\partial y}=y \quad \dots(1)$

$\partial u=x\,\partial x$

$u=\int x\,dx+\varphi(y)=\frac{1}{2}x^2+\varphi(y)$

$\frac{\partial u}{\partial y}=\varphi'(y)\quad\dots(2)$

(1) i (2) $\Rightarrow \varphi'(y)=y$

$\varphi(y)=\frac{1}{2}y^2$

$u=\frac{1}{2}x^2+\frac{1}{2}y^2$

Prema tome $u(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2$

$$\int_{(0,1)}^{(3,-4)} x\,dx+y\,dy=\int_{(0,1)}^{(3,-4)} du(x,y)=\frac{1}{2}x^2\Big|_{(0,1)}^{(3,-4)}+\frac{1}{2}y^2\Big|_{(0,1)}^{(3,-4)}=\frac{1}{2}(9-0)+\frac{1}{2}(16-1)$$

$$=\frac{9}{2}+\frac{15}{2}=\frac{24}{2}=12$$

Izračunati integral $\int_{(-2,-1)}^{(3,0)} (x^4+4xy^3)\,dx+(6x^2y^2-5y^4)\,dy$

Rj. Označimo sa $P(x,y)=x^4+4xy^3$; $Q(x,y)=6x^2y^2-5y^4$

$$\frac{\partial P}{\partial y}=12xy^2 \qquad \frac{\partial Q}{\partial x}=12xy^2$$

$P(x,y)\,dx+Q(x,y)\,dy=0$; $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ egzaktna diferencijalna jednačina

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=P(x,y)=x^4+4xy^3 \qquad \frac{\partial u}{\partial y}=Q(x,y)=6x^2y^2-5y^4 \quad \ldots(*)$$

$$\partial u=P(x,y)\,\partial x$$

$$u=\int(x^4+4xy^3)\,dx=\frac{1}{5}x^5+4\cdot\frac{1}{2}x^2y^3+\varphi(y)=\frac{1}{5}x^5+2x^2y^3+\varphi(y)$$

$$\frac{\partial u}{\partial y}=6x^2y^2+\varphi'(y)\ \ldots(**) \qquad \text{Iz } (*) \text{ i } (**) \Rightarrow \varphi'(y)=-5y^4$$

$$\varphi(y)=-5\int y^4\,dy=-y^5$$

Prema tome $u(x,y)=\frac{1}{5}x^5+2x^2y^3-y^5$

$$\int_{(-2,-1)}^{(3,0)} (x^4+4xy^3)\,dx+(6x^2y^5-5y^4)\,dy=\int_{(-2,1)}^{(3,0)} du(x,y)=\left(\frac{1}{5}x^5+2x^2y^3-y^5\right)\Big|_{(-2,1)}^{(3,0)}$$

$$=\left(\frac{3^5}{5}+0+0\right)-\left(\frac{(-2)^5}{5}+2\cdot4-1\right)=\frac{243}{5}+\frac{32}{5}-\frac{40}{5}+\frac{5}{5}=\frac{240}{5}$$

$$=48$$

Dokazati da integral $\int_L f(xy)(y\,dx + x\,dy)$ po zatvorenoj konturi L ima vrijednost 0 (nula) bez obzira na tip f-je uključen u integrand.

Rj. $\int_L f(xy)(y\,dx+x\,dy) = \int_L y f(xy)\,dx + x f(xy)\,dy$

Označimo sa $P(x,y)=y f(xy)$; $Q(x,y)=x f(xy)$. Imamo

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= f(xy) + y\cdot\frac{\partial f}{\partial (xy)}\cdot x = f(xy) + xy\cdot\frac{\partial f}{\partial (xy)} \\ \frac{\partial Q}{\partial x} &= f(xy) + x\,\frac{\partial f}{\partial (xy)}\cdot y = f(xy) + xy\,\frac{\partial f}{\partial (xy)} \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy \quad \text{formula Greena}$$

$$\int_L f(xy)(y\,dx + x\,dy) = \iint_S \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = 0 \quad \text{bez obzira na } L.$$

q.e.d.

Izračunati krivolinijski integral $\int_C \cos 2y\,dx - 2x\sin 2y\,dy$ gdje je c neka kriva koja spaja tačke $A(1,\frac{\pi}{6})$ i $B(2,\frac{\pi}{4})$.

Rj. Označimo sa $P(x,y)=\cos 2y$; $Q(x,y)=(-2x)\sin y$

$$\frac{\partial P}{\partial y} = -2\sin 2y \qquad \frac{\partial Q}{\partial x} = -2\sin y \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

vrijednost integrala ne zavisi od vrste konture

I način:

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C du(x,y) = u(x,y)\Big|_{(a,b)}^{(c,d)} \quad \text{gdje je}$$

$du(x,y)=P(x,y)\,dx+Q(x,y)\,dy$, tačka (a,b) početak a (c,d) kraj konture c

Odredimo f-ju $u=u(x,y)$

$$\frac{\partial u}{\partial x} = P(x,y) = \cos 2y, \qquad \frac{\partial u}{\partial y} = Q(x,y) = -2x\sin 2y$$

$P(x,y)\,dx + Q(x,y)\,dy = 0$; $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ovo je egzaktna diferencijalna jednačina

$$\frac{\partial u}{\partial x} = \cos 2y$$

$$\partial u = \cos 2y\,\partial x \qquad\qquad \frac{\partial u}{\partial y} = -2x\sin 2y \quad \dots(*)$$

$$u = \int \cos 2y\,dx = x\cos 2y + \varphi(y) \quad \Rightarrow \quad \frac{\partial u}{\partial y} = x\cdot(-\sin 2y)\cdot 2 + \varphi'(y) = -2x\sin 2y + \varphi'(y) \quad \dots(**)$$

Sad imamo $\overset{\text{iz } (*) \text{ i } (**)}{\varphi'(y)=0} \Rightarrow \varphi(y) = c$

$$u(x,y) = x\cos 2y + c$$

$$\int_C \cos 2y\,dx - 2x\sin 2y\,dy = \int_C d(x\cos 2y + c) = x\cos 2y\Big|_{(1,\frac{\pi}{6})}^{(2,\frac{\pi}{4})} + c\Big|_{(1,\frac{\pi}{6})}^{(2,\frac{\pi}{4})} =$$

$$= 2\cos\frac{\pi}{2} - \cos\frac{\pi}{3} + (c-c) = -\frac{1}{2}$$

II način (za vježbu): standardno rješavamo krivolinijski integral s tim da izaberemo pogodnu konturu koja spaja date tačke

\# Izračunati integral po glatkom luku koji spaja tačke A i B

$$\int_{\widehat{AB}} \left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz$$

$A(1,1,1)$, $B(1,2,3)$, $\widehat{AB} \subseteq \{(x,y,z) \mid x>0,\ y>0,\ z>0\}$.

Rj. Označimo sa $P(x,y,z)=1-\frac{1}{y}+\frac{y}{z}$, $Q(x,y,z)=\frac{x}{z}+\frac{x^2}{y^2}$, $R(x,y,z)=-\frac{xy}{z^2}$, i izračunajmo $\frac{\partial^2 P}{\partial y\partial z}$, $\frac{\partial^2 Q}{\partial x\partial z}$ i $\frac{\partial^2 R}{\partial x\partial y}$

$$\frac{\partial P}{\partial y}=-(-1)y^{-2}+\frac{1}{z} \qquad \frac{\partial Q}{\partial x}=\frac{1}{z}+\frac{1}{y^2} \qquad \frac{\partial R}{\partial x}=-\frac{y}{z^2}$$

$$\frac{\partial^2 P}{\partial y\partial z}=-\frac{1}{z^2} \qquad \frac{\partial^2 Q}{\partial x\partial z}=-\frac{1}{z^2} \qquad \frac{\partial^2 R}{\partial x\partial y}=-\frac{1}{z^2}$$

Kako je $\frac{\partial^2 P}{\partial y\partial z}=\frac{\partial^2 Q}{\partial x\partial z}=\frac{\partial^2 R}{\partial x\partial y}$ to integral ne zavisi od vrste krive linije koja spaja tačke A i B.

Odredimo f-ju $u=u(x,y,z)$ za koju vrijedi da je

$$du=\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz$$

$$\frac{\partial u}{\partial x}=1-\frac{1}{y}+\frac{y}{z};$$

$$u=\int\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\varphi(y,z)$$

$$u=x-\frac{x}{y}+\frac{xy}{z}+\varphi(y,z)$$

$$\frac{\partial u}{\partial y}=\frac{x}{y^2}+\frac{x}{z}+\varphi'_y(y,z)$$

$$\frac{\partial u}{\partial y}=\frac{x}{y^2}+\frac{x}{z}$$

$$\varphi'_y(y,z)=0$$

$$\varphi(y,z)=C+\psi(z) \quad \ldots(1)$$

$$\frac{\partial u}{\partial z}=-\frac{xy}{z^2}+\varphi'_z$$

$$\frac{\partial u}{\partial z}=-\frac{xy}{z^2}$$

$$\varphi'_z=0 \quad \ldots(2)$$

(1) i (2) $\Rightarrow$ $\psi(z)=0$ $\Rightarrow$ $\varphi(y,z)=C$

$$u=x-\frac{x}{y}+\frac{xy}{z}+C$$

$$\int_{\widehat{AB}}\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz=\int_{\widehat{AB}}du=\left(x-\frac{x}{y}+\frac{xy}{z}\right)\Bigg|_{(1,1,1)}^{(1,2,3)}=1-\frac{1}{2}+\frac{2}{3}-1=\frac{1}{6}$$

tražena vrijednost

\# Izračunati krivolinijski integral $\displaystyle\int_{(2,1)}^{(1,2)}\frac{y\,dx-x\,dy}{x^2}$ duž putanje koja ne siječe osu Oy.

Rj. Vrijednost integrala $I=\int_c P(x,y)dx+Q(x,y)dy$ ne zavisi od vrste konture c ako je $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

U našem slučaju
$$I=\int_{(2,1)}^{(1,2)}\frac{y}{x^2}dx-\frac{1}{x}dy \qquad P(x,y)=\frac{y}{x^2},\quad Q(x,y)=-\frac{1}{x}$$

$$\frac{\partial P}{\partial y}=\frac{1}{x^2},\quad \frac{\partial Q}{\partial x}=\frac{1}{x^2}$$

Prema tome vrijednost integrala ne zavisi od vrste krive linije c koja spaja tačke $(2,1)$ i $(1,2)$.

I način: Odredićemo primitivnu f-ju $u=u(x,y)$

$$P(x,y)dx+Q(x,y)dy=0$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

ovo je egzaktna dif. jednačina

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$du=\frac{y}{x^2}dx-\frac{1}{x}dy$$

$$\partial u=\frac{y}{x^2}\partial x \qquad \frac{\partial u}{\partial x}=\frac{y}{x^2},\quad \frac{\partial u}{\partial y}=-\frac{1}{x} \quad \ldots(1)$$

$$u=\int\frac{y}{x^2}dx+\varphi(y)=y\,\frac{x^{-1}}{-1}+\varphi(y)=-\frac{y}{x}+\varphi(y)$$

$$\frac{\partial u}{\partial y}=-\frac{1}{x}+\varphi'(y) \quad \ldots(2)$$

(1) i (2) $\Rightarrow$ $\varphi'(y)=0$, $\varphi(y)=C$

$$u=-\frac{y}{x}+C$$

$$\int_{(2,1)}^{(1,2)}\frac{y\,dx-x\,dy}{x^2}=\int_{(2,1)}^{(1,2)}du=-\frac{y}{x}\Bigg|_{(2,1)}^{(1,2)}=-\frac{2}{1}-\left(-\frac{1}{2}\right)=\frac{1}{2}-2=-\frac{3}{2}$$

II način: Spojimo tačke $(2,1)$ i $(1,2)$ nekom krivom (ili pravom) ili izlomljenom pravom linijom i izračunamo integral na klasičan način.

(#) Izračunati krivoliniski integral $\displaystyle\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ duž puta koji ne prolazi kroz koordinatni početak.

R: Ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ tada vrijednost integrala $\int_C P\,dx + Q\,dy$ ne zavisi od vrste izbora puta integracije.

$$I = \int_{(1,0)}^{(6,8)} \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy \Rightarrow \left. \begin{aligned} P(x,y) &= \frac{x}{\sqrt{x^2+y^2}} \\ Q(x,y) &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned} \right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -\frac{xy}{\sqrt{(x^2+y^2)^3}}$$

Prema tome vrijednost integrala ne zavisi od izbora krive kojom ćemo spojiti tačke (1,0) i (6,8).

I način: Odrediti ćemo primitivnu f-ju u.

$$u = u(x,y)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$$

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \quad \dots(1)$$

$$\partial u = \frac{x}{\sqrt{x^2+y^2}}\,\partial x$$

$$u = \int \frac{x}{\sqrt{x^2+y^2}}dx + \varphi(y) = \left| \begin{aligned} x^2+y^2 &= t^2 \\ 2x\,dx &= 2t\,dt \\ x\,dx &= t\,dt \end{aligned} \right| = \int \frac{t}{\sqrt{t^2}}dt + \varphi(y)$$

$$= t + \varphi(y) = \sqrt{x^2+y^2} + \varphi(y)$$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} + \varphi'(y) \quad \dots(2)$$

$$(1) \text{ i } (2) \Rightarrow \varphi'(y) = 0 \Rightarrow u = \sqrt{x^2+y^2}$$

$$\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \int_{(1,0)}^{(6,8)} du = u\Big|_{(1,0)}^{(6,8)} = \sqrt{x^2+y^2}\Big|_{(1,0)}^{(6,8)} = \sqrt{36+64} - \sqrt{1+0} = 9$$

II način: Spojimo tačke (1,0) i (6,8) nekom krivom koja ne prolazi kroz koordinatni početak i izračunamo integral na klasičan način.

Zadaci za vježbu

U zadacima 3831 — 3835 uveriti se da su vrednosti datih integrala, uzetih po zatvorenim konturama, jednake nuli bez obzira na oblik funkcija koje ulaze u podintegralni izraz.

3831. $\int_L \varphi(x)\,dx + \psi(y)\,dy.$ **3832.** $\int_L f(xy)\,(y\,dx + x\,dy).$

3833. $\int_L f\left(\frac{y}{x}\right)\frac{x\,dy - y\,dx}{x^2}.$

3834. $\int_L [f(x+y) + f(x-y)]\,dx + [f(x+y) - f(x-y)]\,dy.$

3835. $\int_L f(x^2+y^2+z^2)\,(x\,dx + y\,dy + z\,dz).$

3836*. Dokazati da integral $\int_L \frac{x\,dy - y\,dx}{x+y}$, uzet u pozitivnom smeru obilaženja po bilo kojoj zatvorenoj konturi koja obuhvata koordinatni početak, ima vrednost 2π.

3837. Izračunati $\int_L \frac{x\,dy - y\,dx}{x^2+4y^2}$ duž kruga $x^2+y^2=1$ u pozitivnom smeru obilaženja.

U zadacima 3838—3844 izračunati krivolinijske integrale totalnih diferencijala.

3838. $\int_{(-1,2)}^{(2,3)} y\,dx + x\,dy.$ **3839.** $\int_{(0,0)}^{(2,1)} 2xy\,dx + x^2\,dy.$

3840. $\int_{(3,4)}^{(5,12)} \frac{x\,dx + y\,dy}{x^2+y^2}$ (koordinatni početak ne leži na putanji integracije).

3841. $\int_{(P_1)}^{(P_2)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$, pri čemu tačke P_1 i P_2 leže na koncentričnim krugovima čiji je zajednički centar u koordinatnom početku, a poluprečnici su im R_1 i R_2 (koordinatni početak ne leži na putanji integracije).

3842. $\int_{(1,-1,2)}^{(2,1,3)} x\,dx - y^2\,dy + z\,dz.$

3843. $\int_{(1,2,3)}^{(3,2,1)} yz\,dx + zx\,dy + xy\,dz.$

3844. $\int_{(7,2,3)}^{(5,3,1)} \frac{zx\,dy + xy\,dz - yz\,dx}{(x-yz)^2}$ (putanja integracije ne preseca površinu $z = \frac{x}{y}$).

U zadacima 3845—3852 naći funkcije čiji su totalni diferencijali zadati.

3845. $du = x^2\,dx + y^2\,dy.$

3846. $du = 4(x^2-y^2)(x\,dx - y\,dy).$

3847. $du = \frac{(x+2y)\,dx + y\,dy}{(x+y)^2}.$

3848. $du = \frac{x}{y\sqrt{x^2+y^2}}\,dx - \left(\frac{x^2+\sqrt{x^2+y^2}}{y^2\sqrt{x^2+y^2}}\right)dy.$

3849. $du = \left[\frac{x-2y}{(y-x)^2} + x\right]dx + \left[\frac{y}{(y-x)^2} - y^2\right]dy.$

3850. $du = (2x\cos y - y^2\sin x)\,dx + (2y\cos x - x^2\sin y)\,dy.$

3851. $du = \frac{2x(1-e^y)}{(1+x^2)^2}\,dx + \left(\frac{e^y}{1+x^2} + 1\right)dy.$

Rješenja

3836*. Primeniti Grinovu formulu na dvostruko povezanu oblast, ograničenu zatvorenom konturom L i bilo kakvim krugom čiji je centar u koordinatnom početku i koji ne preseca konturu L.

3837. π. **3838.** 8.

3839. 4. **3840.** $\ln\frac{13}{5}$.

3841. $R_2 - R_1$. **3842.** $\frac{10}{3}$.

3843. 0. **3844.** $-\frac{9}{2}$.

3845. $u = \frac{x^3+y^3}{3} + C.$

3846. $u = (x^2-y^2)^2 + C.$

3847. $u = \ln|x+y| - \frac{y}{x+y} + C.$

3848. $u = \frac{\sqrt{x^2+y^2}+1}{y} + C.$

3849. $u = \ln|x-y| + \frac{y}{x-y} + \frac{x^2}{2} - \frac{y^3}{3} + C.$

3850. $u = x^2\cos y + y^2\cos x + C.$

3851. $u = \frac{e^y-1}{1+x^2} + y + C.$

3852. $du = \dfrac{(3y-x)\,dx + (y-3x)\,dy}{(x+y)^3}$.

3853. Odrediti broj n tako da izraz $\dfrac{(x-y)\,dx + (x+y)\,dy}{(x^2+y^2)^n}$ bude totalni diferencijal, i naći odgovarajuću primitivnu funkciju.

3854. Odrediti konstante a i b tako da izraz

$$\frac{(y^2+2xy+ax^2)\,dx - (x^2+2xy+by^2)\,dy}{(x^2+y^2)^2}$$

bude totalan diferencijal, i naći odgovarajuću primitivnu funkciju.

U zadacima 3855 — 3860 naći funkcije čiji su totalni diferencijali zadati.

3855. $du = \dfrac{dx+dy+dz}{x+y+z}$. **3856.** $du = \dfrac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$.

3857. $du = \dfrac{yz\,dx + xz\,dy + xy\,dz}{1+x^2y^2z^2}$.

3858. $du = \dfrac{2(zx\,dy + xy\,dz - yz\,dx)}{(x-yz)^2}$.

3859. $du = \dfrac{dx - 3\,dy}{z} + \dfrac{3y - x + z^3}{z^2}\,dz$.

3860. $du = e^{\frac{y}{z}}\,dx + \left(\dfrac{e^{\frac{y}{z}}(x+1)}{z} + ze^{yz}\right)dy + \left(-\dfrac{e^{\frac{y}{z}}(x+1)\,y}{z^2} + ye^{yz} + e^{-z}\right)dz$.

Rješenja

3852. $u = \dfrac{x-y}{(x+y)^2} + C$. **3853.** $n=1$, $u = \dfrac{1}{2}\ln(x^2+y^2) + \operatorname{arctg}\dfrac{y}{x} + C$.

3854. $a=b=-1$, $u = \dfrac{x-y}{x^2+y^2} + C$. **3855.** $u = \ln|x+y+z| + C$.

3856. $u = \sqrt{x^2+y^2+z^2} + C$. **3857.** $u = \operatorname{arctg} xyz + C$.

3858. $u = \dfrac{2x}{x-yz} + C$. **3859.** $u = \dfrac{x-3y}{z} + \dfrac{z^2}{2} + C$.

3860. $u = e^{\frac{y}{z}}(x+1) + e^{yz} - e^{-z}$.

Površinski integral prve vrste

Trebamo izračunati integral $\iint\limits_S f(x,y,z)\,dS$ gdje je S - površ u prostoru.

I način:

Ako je D projekcija površi S: $z=z(x,y)$ na xOy ravan tada

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

II način:

L je projekcija površi S: $y=y(x,z)$ na xOz ravan

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_L f(x,y(x,z),z)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz$$

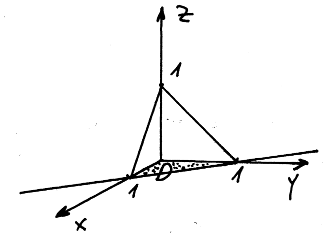
III način:

Neka je C projekcija površi S: $x=x(y,z)$ na yOz ravan

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_C f(x(y,z),y,z)\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$$

Izračunati površinski integral $I=\iint\limits_S xyz\,dS$, ako je S dio ravni $x+y+z=1$ u I oktantu.

Rj. $x+y+z=1$ je ravan koja na x, y i z osi odsijeca 1.

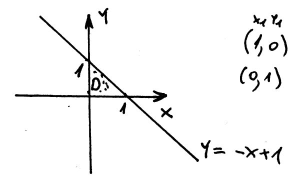


Ako je S data površ opisana jednačinom $z=z(x,y)$ i ako je D projekcija površi S na xOy ravan tada:

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

U našem slučaju $z=1-x-y$, $\frac{\partial z}{\partial x}=-1$, $\frac{\partial z}{\partial y}=-1$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+1+1}=\sqrt{3}$$



$x_1\ y_1$

$(1,0)$

$(0,1)$

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y=-x+1$$

$$D:\begin{cases} 0\le x\le 1\\ 0\le y\le -x+1\end{cases}$$

1
1 1
1 2 1
1 3 3 1

Sad imamo

$$I=\iint\limits_S xyz\,dS=\sqrt{3}\iint\limits_D x\cdot y\cdot(1-x-y)\,dx\,dy=\sqrt{3}\int_0^1 x\,dx\int_0^{-x+1}(y-xy-y^2)\,dy=$$

$$=\sqrt{3}\int_0^1 x\left(\frac{1}{2}y^2\Big|_0^{-x+1}-x\cdot\frac{1}{2}y^2\Big|_0^{(-x+1)}-\frac{1}{3}y^3\Big|_0^{(-x+1)}\right)dx=$$

$$=\sqrt{3}\int_0^1\left(\frac{1}{2}x\overbrace{(-x+1)^2}^{x^2-2x+1}-\frac{1}{2}x^2\overbrace{(-x+1)^2}^{x^2-2x+1}-\frac{1}{3}x\overbrace{(-x+1)^3}^{-x^3+3x^2-3x+1}\right)dx=$$

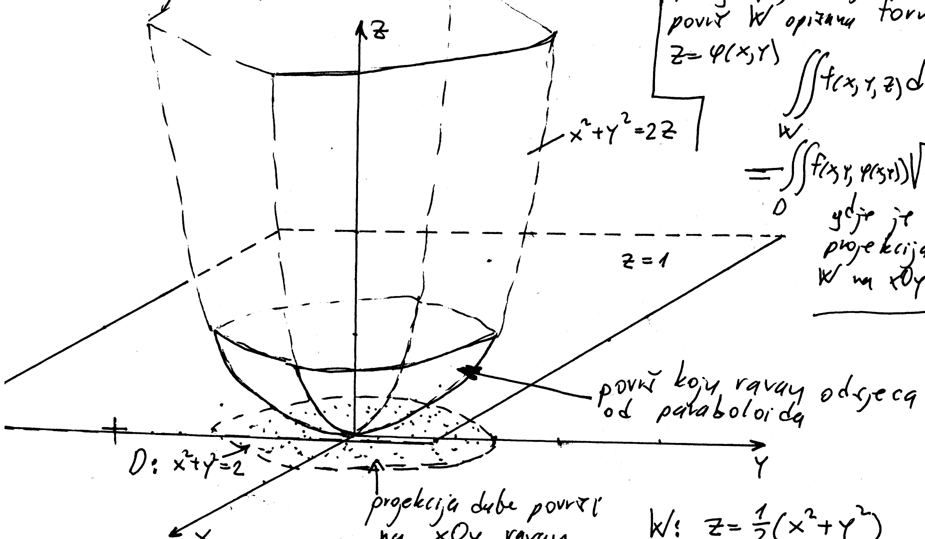
$$=\sqrt{3}\int_0^1\left(\frac{1}{2}x^3-\cancel{x^2}+\frac{1}{2}x-\frac{1}{2}x^4+\cancel{x^3}-\frac{1}{2}x^2+\frac{1}{3}x^4-\cancel{x^3}+\cancel{x^2}-\frac{1}{3}x\right)dx$$

$$=\sqrt{3}\int_0^1\left(-\frac{1}{6}x^4+\frac{1}{2}x^3-\frac{1}{2}x^2+\frac{1}{6}x\right)dx=\sqrt{3}\underbrace{\left(-\frac{1}{6}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{2}\right)}_{\frac{1}{120}}=\frac{\sqrt{3}}{120}$$

rješenje

Izračunati površinski integral prvog tipa $\iint\limits_W (x^2+y^2)\,dS$, gdje je W-površina dijela paraboloida $x^2+y^2=2z$ koju odsjeca ravan $z=1$ (dio paraboloida ispod date ravni).

Rj. Skicirajmo paraboloid i ravan



Primjetimo se kako se računa površinski integral prvog tipa, ako je data površ W opisana formulom $z=\varphi(x,y)$

$$\iint\limits_W f(x,y,z)\,dS = \iint\limits_D f(x,y,\varphi(x,y))\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dx\,dy$$

gdje je D projekcija površi W na xOy ravan

$W: z=\frac{1}{2}(x^2+y^2)$

$\frac{\partial z}{\partial x}=x$, $\frac{\partial z}{\partial y}=y$

$$\iint\limits_W (x^2+y^2)\,dS = \iint\limits_D (x^2+y^2)\sqrt{1+x^2+y^2}\,dx\,dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x=r\cos\varphi \\ y=r\sin\varphi \\ dx\,dy=r\,dr\,d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D': \begin{cases} 0\le r\le\sqrt{2} \\ 0\le\varphi\le 2\pi \end{cases} \quad x^2+y^2=r^2 \right|$$

$$= \iint\limits_{D'} r^2\sqrt{1+r^2}\,r\,dr\,d\varphi = \int_0^{2\pi}d\varphi\int_0^{\sqrt{2}} r^2\sqrt{1+r^2}\,r\,dr = \left| \begin{array}{ll} 1+r^2=t^2 & r^2=t^2-1 \\ 2r\,dr=2t\,dt & \\ r\,dr=t\,dt & r\Big|_0^{\sqrt{2}} \Rightarrow t\Big|_1^{\sqrt{3}} \end{array} \right|$$

$$= \int_0^{2\pi}d\varphi\int_1^{\sqrt{3}}(t^2-1)\sqrt{t^2}\,t\,dt = \int_0^{2\pi}d\varphi\int_1^{\sqrt{3}}(t^4-t^2)\,dt = \varphi\Big|_0^{2\pi}\cdot\left(\frac{1}{5}t^5\Big|_1^{\sqrt{3}}-\frac{1}{3}t^3\Big|_1^{\sqrt{3}}\right) =$$

$$= 2\pi\left(\frac{9\sqrt{3}-1}{5}-\frac{3\sqrt{3}-1}{3}\right) = 2\pi\,\frac{27\sqrt{3}-3-15\sqrt{3}+5}{15} = 2\pi\,\frac{12\sqrt{3}+2}{15} = \frac{(24\sqrt{3}+4)\pi}{15}$$ traženo rješenje

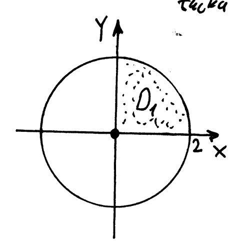
Izračunati površinski integral $\iint\limits_{(S)}\sqrt{-x^2+4}\,dS$, gdje je (S) omotač površi $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0\le z\le 3$.

Rj. Skicirajmo površ $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0\le z\le 3$

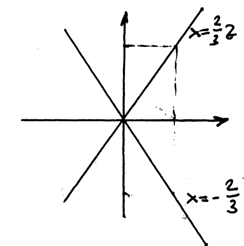
u xOy ravni

$\frac{x^2}{4}+\frac{y^2}{4}=0$

za $z=0$, $x^2+y^2=0$ tačka $(0,0)$



u xOz ravni



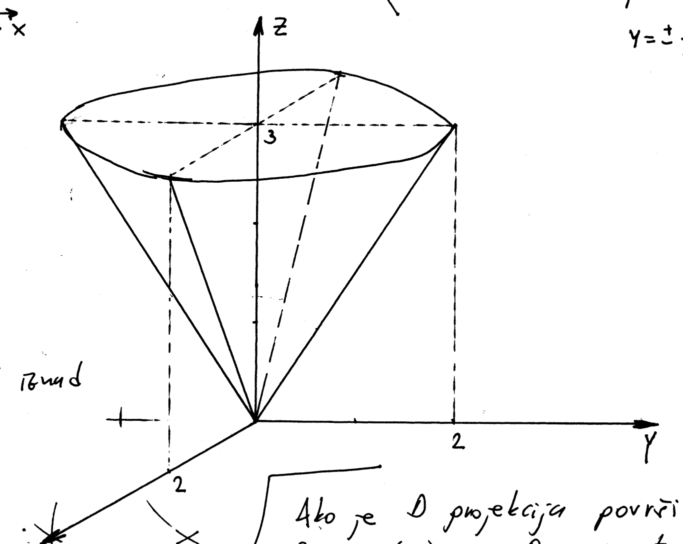
$\frac{x^2}{4}=\frac{z^2}{9}$

$x^2=\frac{4}{9}z^2$

$x=\pm\frac{2}{3}z$

yOz ravan

$y=\pm\frac{2}{3}z$



Za $z=3$ $x^2+y^2=4$

$\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$

$z^2=\frac{9}{4}(x^2+y^2)$

Kako je data površ iznad xOy ravni

$z=\frac{3}{2}\sqrt{x^2+y^2}$

$z'_x=\frac{3}{2}\cdot\frac{2x}{2\sqrt{x^2+y^2}} = \frac{3x}{2\sqrt{x^2+y^2}}$ $\qquad z'_y=\frac{3y}{2\sqrt{x^2+y^2}}$

Ako je D projekcija površi S $z=\eta(x,y)$ na xOy ravan tada

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,\eta(x,y))\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$1+(z'_x)^2+(z'_y)^2 = 1+\frac{9x^2}{4(x^2+y^2)}+\frac{9y^2}{4(x^2+y^2)} = \frac{13x^2+13y^2}{4(x^2+y^2)} = \frac{13}{4}$$

Primjetimo da je data površ (S) simetrična u odnosu na xOz ravan i yOz ravan pa možemo pisati

$$\iint_{(S)} \sqrt{-x^2+4}\, dS = \frac{\sqrt{13}}{2} \iint_{D} \sqrt{-x^2+4}\, dx\,dy = 4 \cdot \frac{\sqrt{13}}{2} \iint_{D_1} \sqrt{4-x^2}\, dx\,dy$$

gdje je $D_1 : \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$

$$\iint_{(S)} \sqrt{-x^2+4}\, dS = 2\sqrt{13} \int_0^2 \sqrt{4-x^2}\, dx \int_0^{\sqrt{4-x^2}} dy = 2\sqrt{13} \int_0^2 (4-x^2)\, dx =$$

$$= 2\sqrt{13}\left(4x\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2\right) = 2\sqrt{13}\left(8 - \frac{8}{3}\right) = 2\sqrt{13} \cdot \frac{\overset{16}{24-8}}{3}$$

$$= \frac{32}{3}\sqrt{13}$$ traženo rješenje

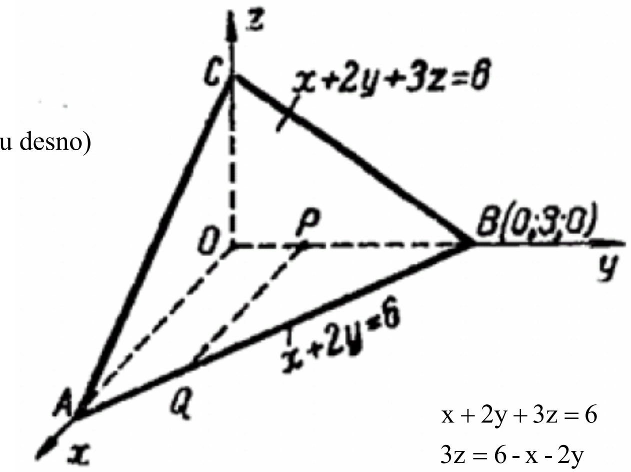
1. Izračunati površinski integral:

a) $I = \iint_{\sigma} (6x + 4y + 3z)ds$, gdje je σ oblast ravni x+2y+3z = 6, u prvom oktantu;

b) $K = \iint_{W} (y + z + \sqrt{a^2 - x^2})ds$, gdje je W površina cilindra $x^2 + y^2 = a^2$, koja se nalazi između ravni z = 0 i z = h.

Rješenja:

a) Skicirajmo oblast σ (vidi sliku desno)



x+2y+3z = 6/:6

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

segmentni oblik jednačine ravni

$$x + 2y + 3z = 6$$
$$3z = 6 - x - 2y$$
$$z = \frac{1}{3}(6 - x - 2y)$$

$$\iint_{\sigma} f(x,y,z)ds = \iint_{D} f(x,y,z(x,y)) \cdot \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

$$\frac{\partial z}{\partial x} = -\frac{1}{3}$$
$$\frac{\partial z}{\partial y} = -\frac{2}{3}$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{14}{9}} = \frac{\sqrt{14}}{3}$$

Projekcija na xOy ravan izgleda: Nacrtati projekciju (uputa:vidi xOy ravan sa slike iznad).

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{x-6}{0-6}=\frac{y-0}{3-0}$$

$$\frac{x-6}{-6}=\frac{y}{3}$$

$$3x-18=-6y$$

$$3x=18-6y$$

$$x=6-2y$$

$$D:\begin{cases}0\le y\le 3\\ 0\le x\le 6-2y\end{cases}$$

$$I=\iint_{\sigma}(6x+4y+3z)ds=\frac{\sqrt{14}}{3}\iint_{D}(6x+4y+6-x-2y)dxdy=\frac{\sqrt{14}}{3}\iint_{D}(5x+2y+6)dxdy=$$

$$\frac{\sqrt{14}}{3}\int_0^3 dy\int_0^{6-2y}(5x+2y+6)dx=\frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}x^2\Big|_0^{6-2y}+2xy\Big|_0^{6-2y}+6x\Big|_0^{6-2y}\right)dy=$$

$$=\frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}(6-2y)^2+2\cdot(6-2y)\cdot y+6\cdot(6-2y)\right)dy=$$

$$=\frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}(36-24y+4y^2)+12y-4y^2+36-12y\right)dy=$$

$$=\frac{\sqrt{14}}{3}\int_0^3(6y^2-60y+126)dy=2\sqrt{14}\int_0^3(y^2-10y+21)dy=$$

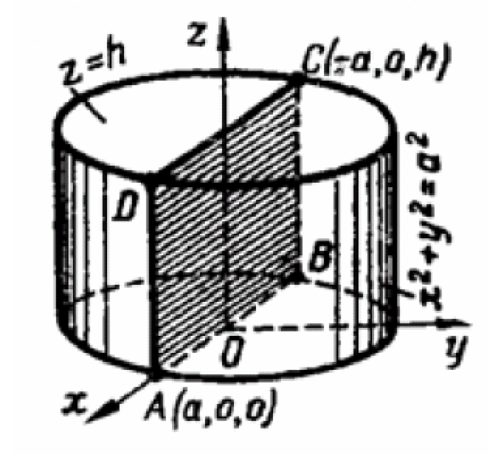
$$=2\sqrt{14}\cdot\left(\frac{y^3}{3}\Big|_0^3-10\frac{y^2}{2}\Big|_0^3+21y\Big|_0^3\right)=2\sqrt{14}\cdot(9-45+63)=54\sqrt{14}$$

---

b) $K=\iint_{W}(y+z+\sqrt{a^2-x^2})ds$ $\qquad x^2+y^2=a^2 \quad$ z = 0 i z = h

Skicirajmo oblast W (vidi sliku na sljedećoj stranici)

$$\iint_{W}f(x,y,z)ds=\iint_{D}f(x,y(x,z),z)\cdot\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dxdy$$



$$x^2+y^2=a^2$$

$$y^2=a^2-x^2$$

$$|y|=\sqrt{a^2-x^2}$$

$$y=\sqrt{a^2-x^2}$$

i

$$y=-\sqrt{a^2-x^2}$$

$$K=K_1+K_2$$

$$\frac{\partial y}{\partial x}=-\frac{x}{\sqrt{a^2-x^2}}$$

$$\frac{\partial y}{\partial z}=0$$

$$D:\begin{cases}-a\le x\le a\\ 0\le z\le h\end{cases}$$

$$ds=\sqrt{1+\left(-\frac{x}{\sqrt{a^2-x^2}}\right)^2}dxdz=\sqrt{1+\frac{x^2}{a^2-x^2}}dxdz=\frac{adxdz}{\sqrt{a^2-x^2}}$$

$$K_1=\iint_{W}\left(y+z+\sqrt{a^2-x^2}\right)ds=\iint_{D}\left(\sqrt{a^2-x^2}+z+\sqrt{a^2-x^2}\right)\frac{adxdz}{\sqrt{a^2-x^2}}=$$

$$=a\iint_{D}\left(2+\frac{z}{\sqrt{a^2-x^2}}\right)dxdz=2a\int_{-a}^{a}dx\int_0^h dz+a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\int_0^h zdz=$$

$$=2a\cdot 2a\cdot h+\frac{ah^2}{2}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=\left|\begin{array}{l}x=a\sin t\\ dx=a\cos tdt\\ x=a\Rightarrow t=\dfrac{\pi}{2}\\ y=-a\Rightarrow t=-\dfrac{\pi}{2}\end{array}\right|=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{\sqrt{a^2-a^2\sin^2 t}}=$$

$$=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{a\sqrt{1-\sin^2 t}}=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos tdt}{\cos t}=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dt=$$

$$=4a^2h+\frac{a^2h}{2}\cdot t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=4a^2h+\frac{ah^2\pi}{2}$$

$$y=-\sqrt{a^2-x^2}$$

$$\frac{\partial y}{\partial x}=\frac{x}{\sqrt{a^2-x^2}}$$

$$\frac{\partial y}{\partial z}=0$$

$$ds=\frac{adxdy}{\sqrt{a^2-x^2}}$$

$$K_2=\iint_W\left(y+z+\sqrt{a^2-x^2}\right)ds=\iint_D\left(-\sqrt{a^2-x^2}+z+\sqrt{a^2-x^2}\right)\frac{adxdz}{\sqrt{a^2-x^2}}=$$

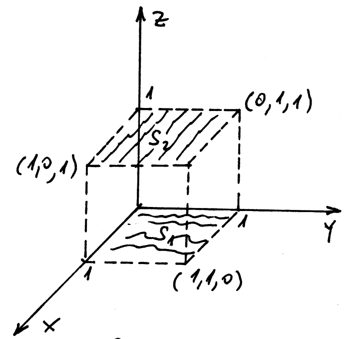
$$=\iint_D z\frac{adxdz}{\sqrt{a^2-x^2}}=a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\int_0^h zdz=a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\cdot\frac{z^2}{2}\Big|_0^h=\frac{ah^2}{2}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=$$

$$=\left|\begin{array}{l} x=a\sin t\\ dx=a\cos tdt\\ x=a\Rightarrow t=\frac{\pi}{2}\\ x=-a\Rightarrow t=-\frac{\pi}{2}\end{array}\right|=\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{\sqrt{a^2-a^2\sin^2 t}}=\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dt=\frac{ah^2}{2}\cdot t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{ah^2\pi}{2}$$

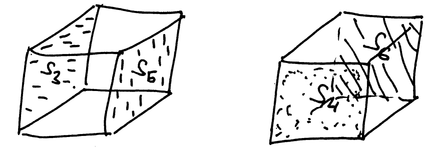
$$K=4a^2h+\frac{ah^2\pi}{2}+\frac{ah^2\pi}{2}=4a^2h+ah^2\pi=ah(4a+\pi h)$$

\# Izračunati površinski integral $\iint_S (x+y+z)\,dS$ gdje je S površina kocke $0\le x\le 1$, $0\le y\le 1$ i $0\le z\le 1$.

Rj.



Ako strane kocke označim sa S_1, S_2, S_3, S_4, S_5 i S_6



imamo:

$$\iint_S (x+y+z)\,dS=\iint_{S_1}(x+y+z)\,dx\,dy+\iint_{S_2}(x+y+z)\,dx\,dy+\iint_{S_3}(x+y+z)\,dx\,dz$$

$$+\iint_{S_4}(x+y+z)\,dy\,dz+\iint_{S_5}(x+y+z)\,dx\,dz+\iint_{S_6}(x+y+z)\,dy\,dz$$

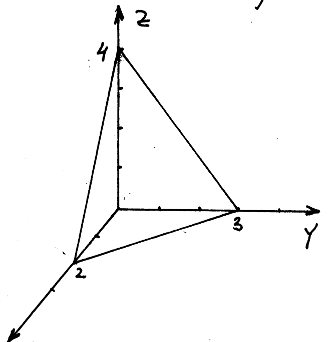
$$\iint_{S_1}(x+y+z)dx\,dy+\iint_{S_2}(x+y+z)\,dx\,dy=\int_0^1\left[\int_0^1(x+y)\,dy\right]dx+\int_0^1\left[\int_0^1(x+y+1)\,dy\right]dx=$$

$$\int_0^1\left(xy\Big|_0^1+\frac{1}{2}y^2\Big|_0^1\right)dx+\int_0^1\left(xy\Big|_0^1+\frac{1}{2}y^2\Big|_0^1+y\Big|_0^1\right)dx=\frac{1}{2}x^2\Big|_0^1+\frac{1}{2}x\Big|_0^1+\frac{1}{2}x\Big|_0^1$$

$$+\frac{1}{2}x\Big|_0^1+x\Big|_0^1=3 \quad \text{Prema tome: } \iint_S (x+y+z)\,dS=9.$$

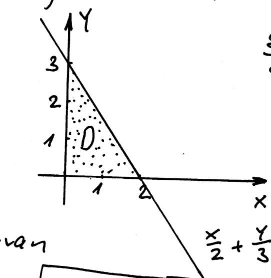
(#) Izračunati površinski integral $\iint (z+2x+\frac{4}{3}y)\,dS$ gdje je S dio ravni $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ u prvom oktantu.

Rj. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ segmentni oblik jednačine ravni $\qquad \frac{z}{4}=1-\frac{x}{2}-\frac{y}{3} \quad /\cdot 4$



$z=4-2x-\frac{4}{3}y$

Projekcija na xOy ravan izgleda



$\frac{\partial z}{\partial x}=-2$

$\frac{\partial z}{\partial y}=-\frac{4}{3}$

$\frac{x}{2}+\frac{y}{3}=1$

S' projekcija površi S na xOy ravan

$$I=\iint_S f(x,y,z)\,dS=\iint_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

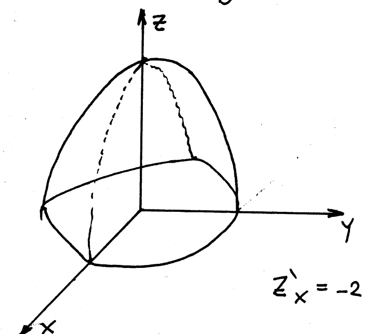
$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+4+\frac{16}{9}}=\sqrt{\frac{61}{9}}=\frac{\sqrt{61}}{9}$$

$$\iint_S (z+2x+\frac{4}{3}y)\,dS=\iint_D (4-2x-\frac{4}{3}y+2x+\frac{4}{3}y)\cdot\frac{\sqrt{61}}{3}\,dx\,dy=\frac{4\sqrt{61}}{3}\underbrace{\iint_D dx\,dy}_{\text{površina oblasti } D}$$

$$=\frac{4\sqrt{61}}{3}\cdot\frac{2\cdot 3}{2}=4\sqrt{61}.$$

(#) Izračunati $\iint_S U(x,y,z)\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy ravni i $U(x,y,z)$ je jednako a) 1 b) x^2+y^2 c) $3z$.

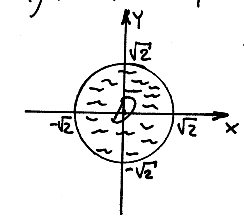
Rj. $\iint_S U(x,y,z)\,dS=\iint_D U(x,y,z)\sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$ gdje je oblast D projekcija površi S na xOy ravan



$z=2-(x^2+y^2)$

Projekcija na xOy ravan

$x^2+y^2=2$



$z'_x=-2x$

$z'_y=-2y$

$$\iint_S U(x,y,z)\,dS=\iint_D U(x,y,z)\sqrt{1+4x^2+4y^2}\,dx\,dy$$

a) $U(x,y,z)=1 \qquad I=\iint_D \sqrt{1+4x^2+4y^2}\,dx\,dy$

Da izračunamo ovo transformišim na polarne koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$

$$D':\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ 0\le\varphi\le 2\pi \\ 0\le r\le\sqrt{2}\end{cases} \qquad dx\,dy=r\,d\varphi\,dr$$

$$I=\iint_{D'}\sqrt{1+4r^2}\cdot r\,dr\,d\varphi=$$

$$=\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\sqrt{1+4r^2}\cdot r\,dr\right]d\varphi=\left|\begin{matrix} 1+4r^2=t^2 & r=0\Rightarrow t=1 \\ 8r\,dr=2t\,dt & r=\sqrt{2}\Rightarrow t=3 \\ r\,dr=\frac{1}{4}t\,dt & \end{matrix}\right|=$$

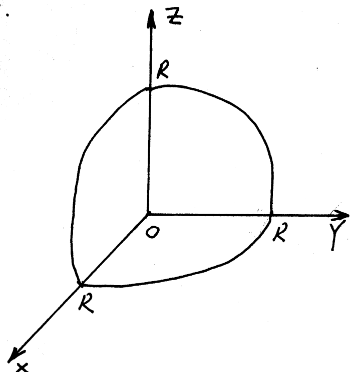
$$=\int_0^{2\pi}\left[\int_1^3 t\cdot\frac{1}{4}t\,dt\right]d\varphi=\frac{1}{4}\int_0^{2\pi}\frac{1}{3}t^3\Big|_1^3\,d\varphi=\frac{1}{12}\cdot\varphi\Big|_0^{2\pi}\cdot 26=\frac{13}{6}\cdot 2\pi=\frac{13\pi}{3}$$

b) vježbu $I=\iint_D (x^2+y^2)\sqrt{1+4x^2+4y^2}=\iint_{D'} r^3\sqrt{1+4r^2}\,dr\,d\varphi=\frac{149}{30}$

c) vjež. $I=\frac{111\pi}{10}$

Izračunati integral $I=\iint\limits_S x\,dS$ gdje je S dio sfere $x^2+y^2+z^2=R^2$ u prvom oktantu.

Rj.



$$x^2+y^2+z^2=R^2$$

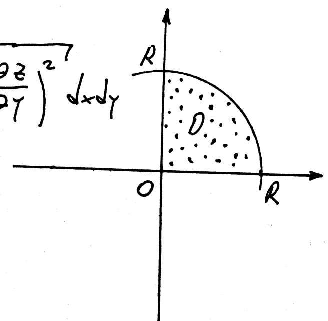
$$z^2=R^2-x^2-y^2$$

$$z=\pm\sqrt{R^2-x^2-y^2}$$

$$z\ge 0 \qquad z=\sqrt{R^2-x^2-y^2}$$

Projekcija površi na xOy ravan

$$\iint\limits_S f(x,y,z)\,dS=\iint\limits_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$



S' projekcija površi S na xOy ravan

$$\frac{\partial z}{\partial x}=\frac{-2x}{2\sqrt{R^2-x^2-y^2}}=\frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{R^2-x^2-y^2}} \qquad \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}}=\frac{R}{\sqrt{R^2-x^2-y^2}}$$

$$I=\iint\limits_S x\,dS=\iint\limits_D x\cdot\frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$ u našem slučaju

$$D':\begin{cases} x=r\cos\varphi\\ y=r\sin\varphi\\ 0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le R\end{cases} \qquad dx\,dy=r\,dr\,d\varphi \qquad x^2+y^2=r^2$$

$$\iint\limits_D \frac{xR}{\sqrt{R^2-x^2-y^2}}\,dx\,dy=\iint\limits_{D'}\frac{r\cos\varphi\, R\cdot r}{\sqrt{R^2-r^2}}\,dr\,d\varphi$$

$$=R\int_0^{\pi/2}\cos\varphi\left[\int_0^R\frac{r^2}{\sqrt{R^2-r^2}}\,dr\right]d\varphi=\begin{vmatrix} r=R\sin t\\ r=0\Rightarrow t=0\\ r=R\Rightarrow t=\frac{\pi}{2}\\ dr=R\cos t\,dt\end{vmatrix}=R\int_0^{\pi/2}\cos\varphi\left[\int_0^{\pi/2}\frac{1}{R}\frac{R^2\sin^2 t}{\sqrt{1-\sin^2 t}}R\cos t\,dt\right]d\varphi$$

$$=R^3\int_0^{\pi/2}\cos\varphi\left[\int_0^{\pi/2}\sin^2 t\,dt\right]d\varphi=R^3\int_0^{\pi/2}\cos\varphi\left[\frac{1}{2}\int_0^{\pi/2}(1-\cos 2t)\,dt\right]=R^3\sin\varphi\Big|_0^{\pi/2}\cdot\frac{1}{2}\left(t\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{2}}\right)=\frac{R^3\pi}{4}$$

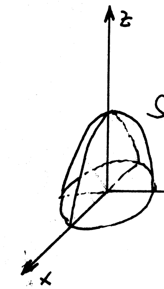
Izračunati površinski integral $\iint\limits_S 3z\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy-ravni.

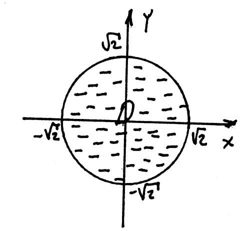
Rj. Neka je D projekcija površi S na xOy ravan. Tada

$$\iint\limits_S f(x,y,z)\,dS=\iint\limits_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

Pronađimo projekciju paraboloida $z=2-(x^2+y^2)$ na xOy ravan.

$z=0 \Rightarrow x^2+y^2=2$ krug sa centrom u tački $(0,0)$ poluprečnika $\sqrt{2}$





$$\frac{\partial z}{\partial x}=-2x$$

$$\frac{\partial z}{\partial y}=-2y$$

$$I=\iint\limits_S 3z\,dS=3\iint\limits_D\left[2-(x^2+y^2)\right]\sqrt{1+4x^2+4y^2}\,dx\,dy$$

Da bi smo riješili ovaj dvostruki integral potrebno je uvesti smjenu promjenjivih.

Uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$

$$D':\begin{cases}0\le\varphi\le 2\pi\\ 0\le r\le\sqrt{2}\\ dx\,dy=r\,dr\,d\varphi\end{cases}$$ ove granice čitamo sa slike

$$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$$

$$1+4x^2+4y^2=1+4(x^2+y^2)=1+4r^2$$

$$I=3\iint\limits_{D'}(2-r^2)\sqrt{1+4r^2}\cdot r\,dr\,d\varphi=3\iint\limits_{D'}2r\sqrt{1+4r^2}\,dr\,d\varphi-3\iint\limits_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi$$

$$6\iint\limits_{D'}r\sqrt{1+4r^2}\,dr\,d\varphi=6\int_0^{2\pi}\left[\int_0^{\sqrt{2}}r\sqrt{1+4r^2}\,dr\right]d\varphi=\begin{vmatrix}1+4r^2=t^2 & r=0\Rightarrow t=1\\ 8r\,dr=2t\,dt & r=\sqrt{2}\Rightarrow t=3\\ r\,dr=\frac{1}{4}t\,dt & \end{vmatrix}=$$

$$=6\int_0^{2\pi}\left[\int_1^3\frac{1}{4}t^2\,dt\right]d\varphi=6\cdot\frac{1}{4}\,\varphi\Big|_0^{2\pi}\cdot\frac{t^3}{3}\Big|_1^3=\frac{3}{2}\cdot\frac{1}{3}\cdot 2\pi\cdot 26=26\pi$$

$$3\iint\limits_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi=3\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\underbrace{r^3}_{r^2\cdot r}\sqrt{1+4r^2}\,dr\right]d\varphi=\begin{vmatrix}1+4r^2=t^2 & r\,dr=\frac{1}{4}t\,dt\\ 4r^2=t^2-1 & r=0\Rightarrow t=1\\ r^2=\frac{1}{4}(t^2-1) & r=\sqrt{2}\Rightarrow t=3\\ 8r\,dr=2t\,dt & \end{vmatrix}=\ldots=\frac{111\pi}{10}$$

Zadaci za vježbu

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

U zadacima 3876—3884 izračunati date integrale.

3876. $\iint\limits_S \left(z+2x+\frac{4}{3}y\right) dq$, pri čemu je S deo ravni $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ koji se nalazi u prvom oktantu.

3877. $\iint\limits_S xyz\, dq$, pri čemu je S deo ravni $x+y+z=1$ koja leži u prvom oktantu.

3878. $\iint\limits_S x\, dq$, pri čemu je S deo sfere $x^2+y^2+z^2=R^2$ koji se nalazi u prvom oktantu.

3879. $\iint\limits_S y\, dq$, po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3880. $\iint\limits_S \sqrt{R^2-x^2-y^2}\, dq$ po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3881. $\iint\limits_S x^2y^2\, dq$ po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3882. $\iint\limits_S \frac{dq}{r^2}$, pri čemu je S deo cilindra $x^2+y^2=R^2$ ograničen ravnima $z=0$ i $z=H$, a r je odstojanje tačke na površini od koordinatnog početka.

3883. $\iint\limits_S \frac{dq}{r^n}$ po sferi $x^2+y^2+z^2=R^2$, pri čemu je r odstojanje tačke na sferi od nepomične tačke $P(0,\ 0,\ c)$, $(c>R)$.

3884. $\iint\limits_S \frac{dq}{r}$, pri čemu je S deo hiperboličnog paraboloida $z=xy$, isečen cilindrom $x^2+y^2=R^2$, a r je odstojanje tačke na površi S od z-ose.

3885*. Naći masu sfere ako je površinska gustina u svakoj njenoj tački brojno jednaka odstojanju te tačke od nekog određenog prečnika sfere.

3886. Naći masu sfere ako je površinska gustina u svakoj njenoj tački brojno jednaka kvadratu odstojanja te tačke od nekog određenog prečnika sfere.

Rješenja

3876. $4\sqrt{61}$. **3877.** $\frac{\sqrt{3}}{120}$. **3878.** $\frac{\pi R^3}{4}$.

3979. 0. **3880.** πR^3. **3881.** $\frac{2\pi R^6}{15}$.

3882. $2\pi \operatorname{arctg}\frac{H}{R}$. **3883.** $\frac{2\pi R}{c(n-2)}\left[\frac{1}{(c-R)^{n-2}}-\frac{1}{(c+R)^{n-2}}\right]$ za $n\neq 2$; $\frac{2\pi R}{c}\ln\frac{c+R}{c-R}$ za $n=2$.

3884. $\pi\left[R\sqrt{R^2+1}+\ln\left(R+\sqrt{R^2+1}\right)\right]$.

3885*. $\pi^2 R^3$. Primeniti sferne koordinate. **3886.** $\frac{8}{3}\pi R^4$.

Površinski integrali II vrste

Obično su oblika: $\iint\limits_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dz\,dx + R(x,y,z)\,dx\,dy$

Uvijek ga svodimo na dvostruki integral.

S je neka data površina. Početni integral se obično podjeli na tri dijela $\iint\limits_S P(x,y,z)\,dy\,dz$, $\iint\limits_S Q(x,y,z)\,dz\,dx$ i $\iint\limits_S R(x,y,z)\,dx\,dy$.

$\vec{n}=(\cos\alpha, \cos\beta, \cos\gamma)$ je vektor normale na površinu S gdje su α, β i γ uglovi koje zaklapa vektor normale sa x, y i z osom.

Kad računamo $\iint\limits_S P(x,y,z)\,dy\,dz$ treba uzeti u obzir predznak broja $\cos\alpha$. Ako je $\cos\alpha<0$ ispred integrala stavljamo − (minus), ako je $\cos\alpha>0$ ispred integrala stavljamo + (plus) i ako je $\cos\alpha=0$ tada $\iint\limits_S P(x,y,z)\,dy\,dz=0$.

Analogno uzimamo vrijednost $\cos\beta$ za $\iint\limits_S Q(x,y,z)\,dz\,dx$ i $\cos\gamma$ za $\iint\limits_S R(x,y,z)\,dx\,dy$. $I=I_1+I_2+I_3$

Integral I_1 rješavamo projekcijom površi S na yOz ravan, integral I_2 projekcijom na xOz ravan i integral I_3 $I_3=\iint\limits_S R(x,y,z)\,dx\,dy$ projekcijom površi S na xOz ravan.

Kod površinskih integrala II vrste mora se označiti koju stranu površi uzimamo. Zavisi od toga sa koje strane vektor normale djeluje (ili sa unutrašnje ili sa spoljašnje oblasti površi).

Kod izbora površi S po kojoj se integrira mora se precizirati da li se uzima vanjska ili unutrašnja strana površi, jer prelaskom na suprotnu stranu integral mijenja predznak.

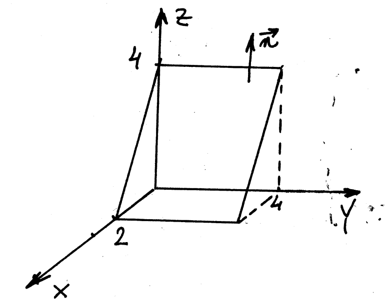
Izračunati $\iint\limits_S z\,dx\,dy + x\,dz\,dx + y\,dy\,dz$ pri čemu je S gornja strana ravni $2x+z=4$, $0<y<4$ u prvom oktantu.

Rj.

$2x+z=4 \quad /:4$

$\frac{x}{2}+\frac{z}{4}=1$ segmentni oblik jednačine ravni

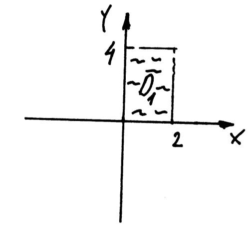
$z=4-2x$



$\vec{n}=(2,0,1)$ vektor normale ravni

$|\vec{n}|=\sqrt{5}$ $\quad \vec{n}_0=\frac{\vec{n}}{|\vec{n}|}=\left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$

$\cos\alpha \quad \cos\beta \quad \cos\gamma$



$I_1=\iint\limits_S z\,dx\,dy$ projeciramo površ na xOy ravan $\quad D_1: \begin{cases} 0\le x\le 2 \\ 0\le y\le 4 \end{cases}$

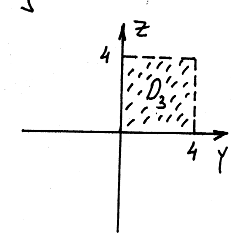
kako je $\cos\gamma>0 \Rightarrow I_1=+\iint\limits_{D_1}(4-2x)\,dx\,dy =$

$=\int_0^4\left[\int_0^2(4-2x)\,dx\right]dy=\int_0^4\left(4x\Big|_0^2-2\cdot\frac{1}{2}x^2\Big|_0^2\right)dy=\int_0^4(8-4)\,dy=4y\Big|_0^4=16$

$I_2=\iint\limits_S x\,dz\,dx$ (gledamo ugao β)

Kako je $\cos\beta=0 \Rightarrow I_2=0$

$I_3=\iint\limits_S y\,dy\,dz$ (gledamo ugao α) $\quad \cos\alpha>0 \Rightarrow I_3=+\iint\limits_{D_3} y\,dy\,dz$.



$D_3: \begin{cases} 0\le y\le 4 \\ 0\le z\le 4 \end{cases}$

$I_3=\int_0^4\left[\int_0^4 y\,dy\right]dz=\int_0^4\frac{1}{2}y^2\Big|_0^4\,dz$

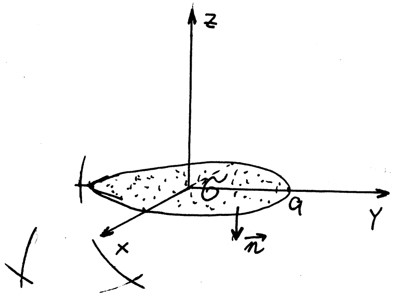
$=\frac{1}{2}\cdot16\cdot z\Big|_0^4=32$

$\iint\limits_S z\,dx\,dy + x\,dz\,dx + y\,dy\,dz = 16+0+32=48$

(#) Izračunati površinski integral drugog tipa (po koordinatama) $I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy$ gdje je

σ - donja strana kruga $x^2+y^2\le a^2$.

Rj. Skicirajmo datu površinu



U našem slučaju ortogonalna projekcija D je jednaka datoj površini σ.
Ugao γ je $\gamma=\pi$ (donja strana kruga) tj. $\cos\pi<0$.

Prisjetimo se, kako se računa površinski integral drugog tipa, npr. $\iint\limits_S R(x,y,z)\,dx\,dy$

poznavano vektor normale $\vec{n}$ na površ S

ako je $\cos\gamma<0$ gdje γ ugao između $\vec{n}$ i z-ose naš integral postaje $\iint\limits_S R(x,y,z)\,dx\,dy=-\iint\limits_D R(x,y,z(x,y))\,dx\,dy$

gdje je D ortogonalna projekcija površi S a $z=z(x,y)$ jednačina površi S

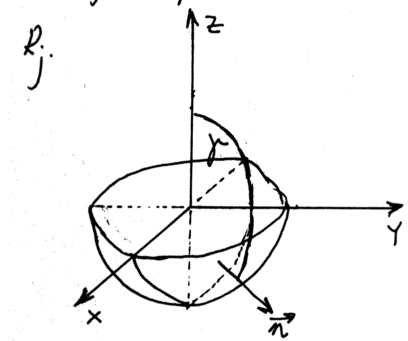
$$I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy=-\iint\limits_{D}\sqrt[4]{x^2+y^2}\,dx\,dy=\left|\begin{array}{l}\text{uvodimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi\end{array}\quad D\xrightarrow{\text{transf.}}D':\begin{cases}0\le r\le a\\ 0\le\varphi\le 2\pi\end{cases}\right|$$

$$=-\iint\limits_{D'}\sqrt[4]{r^2}\,r\,dr\,d\varphi=-\int_0^{2\pi}d\varphi\int_0^a r^{\frac{3}{2}}dr=-\int_0^{2\pi}\frac{2}{5}r^{\frac{5}{2}}\Big|_0^a\,d\varphi=-\frac{2}{5}a^{\frac{5}{2}}\varphi\Big|_0^{2\pi}.$$

$I=-\frac{4}{5}\pi\sqrt{a^5}$ traženo rješenje

(#) Izračunati $\iint\limits_S x^2y^2z\,dx\,dy$ gdje je S-vanjska strana donje polovine sfere $x^2+y^2+z^2=R^2$.

Rj.



Kako imamo $dx\,dy$: zanima nas ugao γ (γ je ugao koji vektor normale $\vec{n}$ na površ zaklapa sa z-osom).

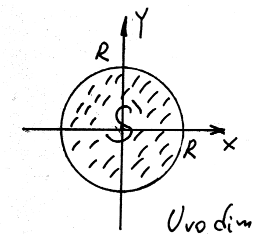
$\gamma>\frac{\pi}{2}\Rightarrow\cos\gamma<0$

$z^2=R^2-x^2-y^2$

$z<0\qquad z=-\sqrt{R^2-x^2-y^2}$

Da smo imali čitavu sferu tada bi integral podjelili na dva dijela za gornji i za donji dio sfere.

Gledamo projekciju površi S na xOy ravan:



$S':\ x^2+y^2\le R^2$

$$\iint\limits_S x^2y^2z\,dx\,dy=-\iint\limits_{S'}x^2y^2\left(-\sqrt{R^2-x^2-y^2}\right)dx\,dy$$

Uvodimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$

$D:\begin{cases}0\le\varphi\le 2\pi\\ 0\le r\le R\\ dx\,dy=r\,d\varphi\,dr\end{cases}\qquad x^2+y^2=r^2$

$$\iint\limits_S x^2y^2z\,dx\,dy=\iint\limits_{S'}x^2y^2\sqrt{R-x^2-y^2}\,dx\,dy=\iint\limits_D r^2\cos^2\varphi\, r^2\sin^2\varphi\sqrt{R^2-r^2}\cdot r\,d\varphi\,dr$$

$$=\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\left[\int_0^R r^5\sqrt{R^2-r^2}\,dr\right]d\varphi\overset{(*)}{=}\frac{8R^7}{105}\cdot\frac{\pi}{4}=\frac{2\pi R^7}{105}$$

(*)

$$\int_0^R r^5\sqrt{R^2-r^2}\,dr=\int_0^R r^4\sqrt{R^2-r^2}\,r\,dr=\left|\begin{array}{ll}R^2-r^2=t^2 & r=0\Rightarrow t=R\\ -2r\,dr=2t\,dt & r=R\Rightarrow t=0\\ r\,dr=-t\,dt & \end{array}\right|=\int_0^R(R^2-t^2)^2\cdot\sqrt{t^2}\,t\,dt=$$

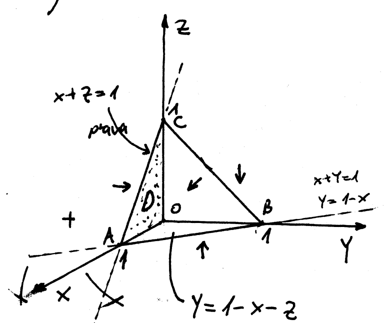
$$=\ldots=\frac{8R^7}{105}$$

$$\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\,d\varphi=\int_0^{2\pi}\frac{1}{4}(2\cos\varphi\sin\varphi)^2d\varphi=\frac{1}{4}\int_0^{2\pi}\sin^2 2\varphi\,d\varphi=\frac{1}{4}\int_0^{2\pi}\frac{1}{2}(1-\cos4\varphi)\,d\varphi=\frac{1}{8}\left(\varphi\Big|_0^{2\pi}-\frac{1}{4}\sin4\varphi\Big|_0^{2\pi}\right)=\frac{\pi}{4}$$

(#) Izračunati površinski integral $K=\oiint\limits_{-W} y\,dx\,dz$ gdje je W - površina tetraedra ograničenog ravnima $x+y+z=1$, $x=0$, $y=0$ i $z=0$.

Rj: integral oblika $\iint\limits_W R(x,y,z)\,dx\,dz$ zovemo površinski integral drugog tipa. Računamo ga tako što napravimo projekciju D površi W na xOz ravan i odredimo predznak broja $\cos\beta$ gdje je β ugao koji zaklapa vektor normale $\vec{n}$ površi W sa Y-osom.

Skicirajmo naš tetraedar



Kako je u zadatku data oblast -W to posmatramo vektore normale koje odgovaraju unutrašnjim površinama tetraedra

$$K=\oiint\limits_{-W} y\,dx\,dz=\iint\limits_{-\Delta AOC} y\,dx\,dz+\iint\limits_{-\Delta AOB} y\,dx\,dz+\iint\limits_{-\Delta BOC} y\,dx\,dz+\iint\limits_{-\Delta ABC} y\,dx\,dz$$

$$\iint\limits_{-\Delta AOC} y\,dx\,dz=+\iint\limits_{D} 0\,dx\,dz=0$$

$$\iint\limits_{-\Delta AOB} y\,dx\,dz=\left|\begin{array}{l}\text{vektor normale } \Delta AOB\\ \text{je okomit na Y-osu}\end{array}\right|=0$$

$$\iint\limits_{-\Delta OBC} y\,dx\,dz=\left|\begin{array}{l}\text{vektor normale } \Delta OBC\\ \text{je okomit na Y-osu}\end{array}\right|=0$$

$$\iint\limits_{-\Delta ABC} y\,dx\,dz=\left|\begin{array}{l}\text{vektor normale } \vec{n} \text{ na}\\ \Delta ABC \text{ sa Y-osom zaklapa}\\ \text{ugao } \beta \text{ koji je između } 90^\circ \text{ i } 180^\circ\\ \text{ZAŠTO? (vidi sliku)}\\ \cos\beta<0\end{array}\right|=-\iint\limits_{D}(1-x-z)\,dx\,dz=$$

$$=-\int_0^1 dx\int_0^{1-x}(1-x-z)\,dz=-\int_0^1\left(z\Big|_0^{1-x}-xz\Big|_0^{1-x}-\frac{1}{2}z^2\Big|_0^{1-x}\right)dx=$$

$$=-\int_0^1\left(1-x-x(1-x)-\frac{1}{2}\overbrace{(1-x)^2}^{1-2x+x^2}\right)dx=-\int_0^1\left(1-x-\cancel{x}+x^2-\frac{1}{2}+\cancel{x}-\frac{1}{2}x^2\right)dx$$

$$=-\int_0^1\left(\frac{1}{2}x^2-x+\frac{1}{2}\right)dx=-\left(\frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^1-\frac{1}{2}x^2\Big|_0^1+\frac{1}{2}x\Big|_0^1\right)=-\left(\frac{1}{6}-\frac{1}{2}+\frac{1}{2}\right)=-\frac{1}{6}$$

traženo rješenje

II način

Možemo upotrebiti formulu Gauss-Ostrogradski

$$\oiint\limits_{S} P(x,y,z)\,dy\,dz+Q(x,y,z)\,dx\,dz+R(x,y,z)\,dx\,dy=\iiint\limits_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

Ω - oblast koju ograničava površ S

U našem slučaju $P(x,y,z)=R(x,y,z)=0$

$$Q(x,y,z)=y\implies\frac{\partial Q}{\partial y}=1$$

$$K=\oiint\limits_{-W} y\,dx\,dz=-\iiint\limits_{\Omega}dx\,dy\,dz=-\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}dz=-\int_0^1 dx\int_0^{1-x}(1-x-y)\,dy$$

$$=\left|\begin{array}{l}\text{primjetimo da smo sličan}\\ \text{integral već imali u prethodnom}\\ \text{slučaju}\end{array}\right|=\ldots=-\frac{1}{6}$$

traženo rješenje

#) Izračunati površinski integral

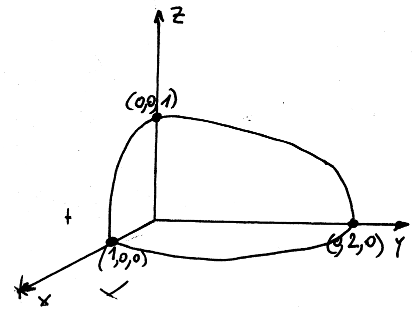
$$J=\iint_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz$$

gdje je T vanjska strana elipsoida $4x^2+y^2+4z^2=4$ koji se nalazi u prvom oktantu.

Rj. skiciramo elipsoid

$4x^2+y^2+4z^2=4 \quad /:4$

$\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{1}=1$



$$J=\iint_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz$$

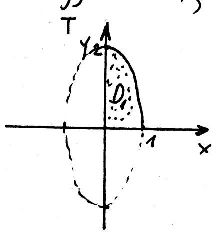
Ovo je površinski integral druge vrste. Prisjetimo se kako se računa npr. $\iint_T P(x,y,z)\,dy\,dz$. Neka je $\vec{n}$ vektor normale površi T koji sa x, y i z osom redom zaklapa uglove α, β i γ, i neka je D ortogonalna projekcija površi T na yOz ravan. Tada

$\iint_T P(x,y,z)\,dy\,dz = \pm\iint_D P(\varphi(y,z),y,z)\,dy\,dz$ gdje je + ako je $\cos\alpha>0$, − (minus) ako je $\cos\alpha<0$ a $x=\varphi(y,z)$ je jednačina koja opisuje površ T.

$$J=\iint_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz = \iint_T 2\,dx\,dy + \iint_T y\,dx\,dz - \iint_T x^2 z\,dy\,dz = J_1+J_2-J_3$$

Izračunajmo redom J_1, J_2 i J_3.

$J_1=\iint_T 2\,dx\,dy$; vektor normale $\vec{n}$ na T sa z osom zaklapa ugao $\gamma\in(0,\frac{\pi}{2})$ tj. $\cos\gamma>0$



$z=0:\quad 4x^2+y^2=4$

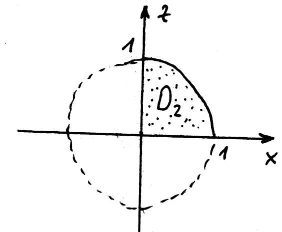
$D_1:\begin{cases}0\le x\le 1\\ 0\le y\le 2\sqrt{1-x^2}\end{cases}$

D_1 je četvrtina elipse

$P_{elipse}=ab\pi$, $\quad J_1=+2\iint_{D_1}dx\,dy=2\cdot\frac{1}{4}P_{elipse}=\frac{1}{2}\cdot 2\pi=\pi$

$J_2=\iint_T y\,dx\,dz$, vektor normale $\vec{n}$ na površ T sa y-osom zaklapa uglove od 0 do $\pi/2$ (I oktant) pa je $\cos\gamma>0$.

Neka je D_2 ortogonalna projekcija površi T na xOz ravan.



$D_2:\quad 4x^2+4z^2=4$

$4x^2+y^2+4z^2=4$

$y^2=4-4x^2-4z^2$

$y=2\sqrt{1-x^2-z^2}$

$D_2:\begin{cases}0\le x\le 1\\ 0\le z\le\sqrt{1-x^2}\end{cases}$

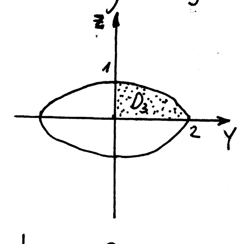
$$J_2=\iint_T y\,dx\,dz=+2\iint_{D_2}\sqrt{1-x^2-z^2}\,dx\,dz=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ z=r\sin\varphi\\ dz\,dx=r\,dr\,d\varphi\\ D_2\longrightarrow D_2'\end{array}\quad D_2'=\begin{cases}0\le r\le 1\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}\right|$$

$$=2\iint_{D_2'}\sqrt{1-r^2\cos^2\varphi-r^2\sin^2\varphi}\,r\,dr\,d\varphi=2\int_0^{\frac{\pi}{2}}d\varphi\int_0^1\sqrt{1-r^2}\,r\,dr=$$

$$=2\int_0^{\frac{\pi}{2}}d\varphi\int_0^1\sqrt{1-r^2}\left(-\frac{1}{2}\right)d(1-r^2)=-\varphi\Big|_0^{\frac{\pi}{2}}\cdot\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1=-\frac{\pi}{2}\cdot\left(0-\frac{2}{3}\right)=\frac{\pi}{3}$$

$J_3=\iint_T x^2 z\,dy\,dz$, vektor normale $\vec{n}$ na površ T sa x-osom zaklapa uglove od 0 do $\frac{\pi}{2}$ pa je $\cos\alpha>0$

Neka je D_3 ortogonalna projekcija površi T na yOz ravan.



$D_3:\quad y^2+4z^2=4 \qquad y^2=4-4z^2$

$\frac{y^2}{4}+\frac{z^2}{1}=1$

$4x^2+y^2+4z^2=4$

$4x^2=4-y^2-4z^2$

$x^2=1-\frac{1}{4}y^2-z^2$

$$J_3=\iint_T x^2 z\,dy\,dz=+\iint_{D_3}\left(1-\frac{1}{4}y^2-z^2\right)z\,dy\,dz=$$

$$\left|D_3:\begin{cases}0\le z\le 1\\ 0\le y\le 2\sqrt{1-z^2}\end{cases}\right|=\int_0^1 z\,dz\int_0^{2\sqrt{1-z^2}}\left(1-\frac{1}{4}y^2-z^2\right)dy=\int_0^1 z\left(y\Big|_0^{2\sqrt{1-z^2}}-\frac{1}{4}\cdot\frac{1}{3}y^3\Big|_0^{2\sqrt{1-z^2}}-z^2y\Big|_0^{2\sqrt{1-z^2}}\right)$$

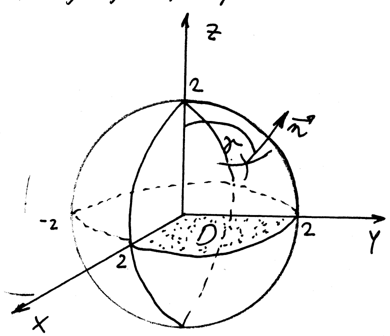
izvučemo $2\sqrt{1-z^2}$

$$=\int_0^1 z\left(2\sqrt{1-z^2}-\frac{2}{3}\sqrt{(1-z^2)^3}-2z^2\sqrt{1-z^2}\right)dz=\frac{4}{3}\int_0^1 z(1-z^2)^{\frac{3}{2}}dz=\frac{2}{3}\cdot\frac{2(1-z^2)^{\frac{5}{2}}}{5}\Big|_1^0=\frac{4}{15}$$

Prema tome $J=\pi+\frac{\pi}{3}-\frac{4}{15}=\frac{4\pi}{3}-\frac{4}{15}$.

Izračunati površinski integral $I=\iint_S xy^3z\,dx\,dy$, ako je S vanjska strana sfere $x^2+y^2+z^2=4$ u prvom oktantu.

Rj: $x^2+y^2+z^2=4$ je sfera sa centrom u koordinatnom početku čiji je poluprečnik dužine 2.



Kad računamo $\iint_S f(x,y,z)\,dx\,dy$ treba uzeti u obzir predznak broja $\cos\gamma$.
Ako je $\cos\gamma<0$ ispred integrala stavljamo minus, ako je $\cos\gamma>0$ ispred integrala stavljamo plus, a ako je $\cos\gamma=0$ tada je integral jednak 0.
γ je ugao koji vektor normale $\vec n$ ($\vec n=(\cos\alpha,\cos\beta,\cos\gamma)$) zaklapa sa z-osom

Vektor normale $\vec n$ je u prvom oktantu $\Rightarrow$ $0<\gamma<\frac{\pi}{2}$

$\Rightarrow$ $\cos\gamma>0$

$x^2+y^2+z^2=4$

$z=\pm\sqrt{4-(x^2+y^2)}$ nama treba +

$$I=\iint_S xy^3z\,dx\,dy=\iint_D xy^3\left(\sqrt{4-(x^2+y^2)}\right)dx\,dy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi \\ y=r\sin\varphi \\ dxdy=r\,dr\,d\varphi \\ x^2+y^2=r^2\end{array}\quad D':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 2\end{cases}\right|$$

$$=\iint_{D'} r\cos\varphi\, r^3\sin^3\varphi\sqrt{4-r^2}\,r\,dr\,d\varphi=\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi\,d\varphi\int_0^2 r^5\sqrt{4-r^2}\,dr=I_1\cdot I_2$$

$$I_1=\int_0^{\frac{\pi}{2}}\cos\varphi\cdot\sin^3\varphi\,d\varphi=\left|\begin{array}{l}\sin\varphi=t\\ \cos\varphi d\varphi=dt\\ \varphi|_0^{\pi/2}\Rightarrow t|_0^1\end{array}\right|=\int_0^1 t^3dt=\frac{1}{4}t^4\Big|_0^1=\frac{1}{4}$$

$$I_2=\int_0^2 r^5\sqrt{4-r^2}\,dr=\int_0^2 r^4\sqrt{4-r^2}\,r\,dr=\left|\begin{array}{l}4-r^2=t^2\\ -2r\,dr=2t\,dt\\ r\,dr=-t\,dt\end{array}\quad r|_0^2\Rightarrow t|_2^0\right|=\int_0^2(4-t^2)^2\cdot t\,t\,dt$$

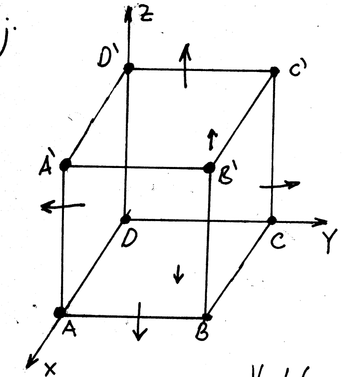
$$=\int_0^2(16-8t^2+t^4)\cdot t^2\,dt=\int_0^2(t^6-8t^4+16t^2)\,dt=\dots=\frac{1024}{105}$$

$$I=\frac{1}{4}\cdot\frac{1024}{105}=\frac{256}{105}$$

traženo rješenje

Izračunati integral $\iint_S x\,dy\,dz+y\,dz\,dx+z\,dx\,dy$ gdje je S vanjska strana kocke koju čine ravni $x=0$, $y=0$, $z=0$, $x=1$, $y=1$, $z=1$.

Rj:



Označimo sa $I_1=\iint_S x\,dy\,dz$

Ovaj integral vadimo po šest površina: ABCD, ABB'A', BCC'B', ADD'A', A'B'C'D' i DCC'D'.

Kako imamo $dydz$ posmatramo ugao α koji zaklapa vektor normale na površ sa x osom

Vektor normala površina ABCD, A'B'C'D', BCC'B' i ADD'A' je okomit na x-osu $\Rightarrow$

$$\Rightarrow \iint_{ABCD} x\,dy\,dz=\iint_{A'B'C'D'} x\,dy\,dz=\iint_{BCC'B'} x\,dy\,dz=\iint_{ADD'A'} x\,dy\,dz=0$$

Kako je $x=0$ za površinu DCC'D' $\Rightarrow$ $\iint_{DCC'D'} x\,dy\,dz=0$

Za I_1 ostaje nam samo površina ABB'A'

$\vec n_0=(1,0,0)$ $\Rightarrow$ $\cos\alpha>0$ $\Rightarrow$ $I_1=+\iint_D dy\,dz$

gdje je D oblast dobijena projekcijom kvadrata ABB'A' na yOz ravan $D:\begin{cases}0\le y\le 1\\ 0\le z\le 1\end{cases}$

$$I_1=\iint_D dy\,dz=\int_0^1\left[\int_0^1 dy\right]dz=z\Big|_0^1\,y\Big|_0^1=1$$

Sad nije teško, analognim zaključivanjem, vidjeti da je $\iint_S y\,dz\,dx=1$ i $\iint_S z\,dx\,dy=1$ redom po površinama BCC'B' i A'B'C'D' dok je po ostalim površinama $=0$ $\Rightarrow$ $\iint_S x\,dy\,dz+y\,dz\,dx+z\,dx\,dy=3$

Zadaci za vježbu

U zadacima 3887—3893 izračunati date površinske integrale.

3887. $\iint\limits_S x\,dy\,dz + y\,dx\,dz + z\,dx\,dy$ po spoljnoj strani kocke obrazovane ravnima $x=0$, $y=0$, $z=0$, $x=1$, $y=1$. $z=1$.

3888. $\iint\limits_S x^2y^2z\,dx\,dy$ po spoljnoj strani donje polovine sfere $x^2+y^2+z^2=R^2$.

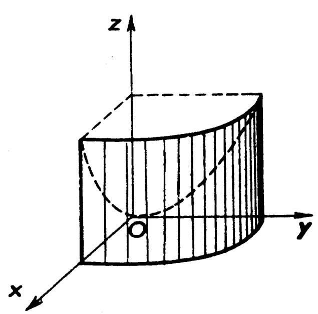
3889. $\iint\limits_S z\,dx\,dy$ po spoljnoj strani elipsoida $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

3890. $\iint\limits_S z^2\,dx\,dy$ po spoljnoj strani elipsoida $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

3891. $\iint\limits_S xz\,dx\,dy + xy\,dy\,dz + yz\,dx\,dz$ po spoljnoj strani piramide obrazovane ravnima $x=0$, $y=0$, $z=0$ i $x+y+z=1$.

3892. $\iint\limits_S yz\,dx\,dy + xz\,dy\,dz + xy\,dx\,dz$ po spoljnoj strani zatvorene površine koja se nalazi u prvom oktantu a sastoji se iz dela cilindra $x^2+y^2=R^2$ i odgovarajućih delova ravni $x=0$, $y=0$, $z=0$ i $z=H$.

3893. $\iint\limits_S y^2z\,dx\,dy + xz\,dy\,dz + x^2y\,dx\,dz$ po spoljnoj strani zatvorene površine koja se nalazi u prvom oktantu a sastoji se iz obrtnog paraboloida $z=x^2+y^2$, cilindra $x^2+y^2=1$ i odgovarajućih delova koordinatnih ravni (sl. 68).



Sl. 68

Rješenja

3887. 3. **3888.** $\dfrac{2\pi R^7}{105}$. **3889.** $\dfrac{4}{3}\pi abc$. **3890.** 0.

3891. $\dfrac{1}{8}$. **3892.** $R^2H\left(\dfrac{2R}{3}+\dfrac{\pi H}{8}\right)$. **3893.** $\dfrac{\pi}{8}$.

Primjena površinskih integrala

I Izračunavanje površine dijela glatke površi, koja pripada prostoru $\mathbb{R}^3$

Neka je površ S zadana jednačinom $z=z(x,y)$ gdje su $(x,y)\in D$. (D - je oblast u ravni xOy u koju se projektuje površ $z=z(x,y)$).

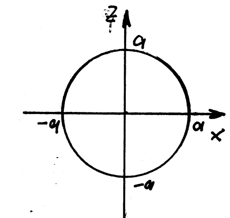
Površina P dijela glatke površi $S\subseteq\mathbb{R}^3$ računa se po formuli $P=\iint_S dS=\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$.

— — —

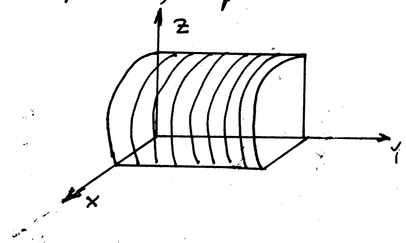
(#) Neka je S površina tijela koje je dobijeno presjekom dva cilindra $S_1=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+z^2=a^2,\ y\in\mathbb{R}\}$ i $S_2=\{(x,y,z)\in\mathbb{R}^3 \mid y^2+z^2=a^2,\ x\in\mathbb{R}\}$. Izračunati površinu dobijenog tijela.

Rj. $P=\iint_S dS$ Skicirajmo S_1 i S_2, pa skicirajmo njihov presjek.

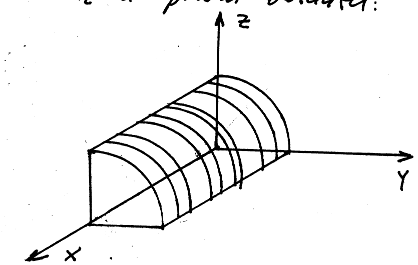
S_1: $x^2+z^2=a^2$ u ravni: xOz



U prostoru, u prvom oktantu:



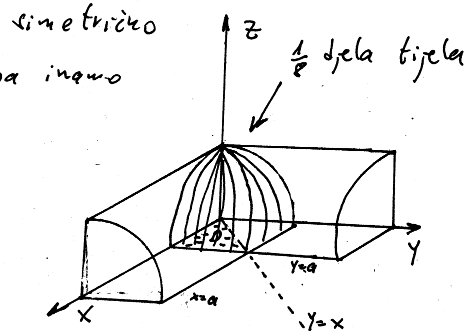
S_2 u prvom oktantu:



Presjek $S_1\cap S_2$ će kao rezultat dati tijelo koje je simetrično u odnosu na sve tri ravni xOy, xOz i yOz.

$\frac{1}{8}$ dijela tijela će se nalaziti u prvom oktantu:

Primjetimo da je i ovo tijelo simetrično u odnosu na pravu $y=x$ pa imamo



$$P=\frac{1}{16}\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

gdje je $D:\begin{cases}0\le x\le a\\ 0\le y\le x\end{cases}$

$z^2=a^2-x^2$ tj. $z=\sqrt{a^2-x^2}$.

$z'_x=\frac{-x}{\sqrt{a^2-x^2}}$, $z'_y=0$ $\quad 1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{a^2-x^2}=\frac{a^2}{a^2-x^2}$

tražena površina

$$P=16a\int_0^a\frac{dx}{\sqrt{a^2-x^2}}\int_0^x dy=16a\int_0^a\frac{x\,dx}{\sqrt{a^2-x^2}}=\left|\begin{matrix}a^2-x^2=t\\ -2x\,dx=dt\\ x\,dx=-\frac{1}{2}dt\\ \dots\end{matrix}\right|=\dots=16a\sqrt{a^2-x^2}\Big|_a^0=16a^2$$

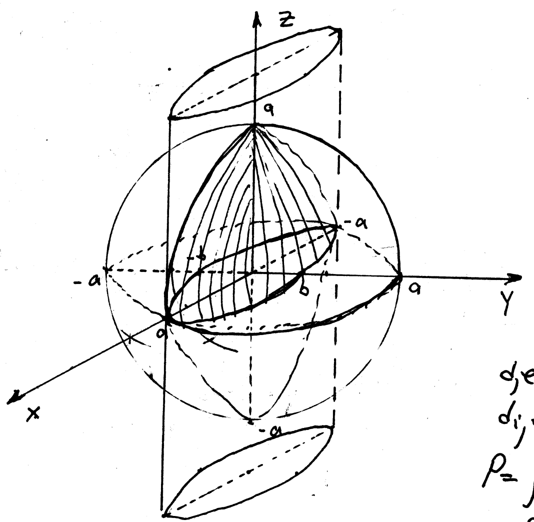
(#) Izračunati površinu djela sfere

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2\}$$

koji se nalazi u unutrašnjosti cilindra

$$S_1=\{(x,y,z)\in\mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z\in\mathbb{R}\},\quad b\le a$$

R: Skiciramo sferu S i cilindar S_1.



Cilindrična površ u presjeku sa sferom, isjeca iz nje simetričnu površ u odnosu na ravan xOy. Ta dva simetrična dijela označimo sa l_1 i l_2. Svaki od ova dva dijela, koordinatne ravni xOz i yOz ih dijele na četiri jednaka dijela.

$P=\iint\limits_S dS$ gdje je S površina djela sfere ograničena cilindrom.

S: $x^2+y^2+z^2=a^2$

$z=\pm\sqrt{a^2-x^2-y^2}$

Zbog navedene simetričnosti posmatraću sferu samo u prvom oktantu

$z=\sqrt{a^2-x^2-y^2}$ $\qquad z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}$, $z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$

$\boxed{\iint\limits_S dS=\iint\limits_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy}$ gdje je D projekcija površi S na xOy ravan

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}=\frac{a^2}{a^2-x^2-y^2}$$

$$P=8\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\\ a\ge0,\ b\ge0}}\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=8a\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ a\ge0,\ b\ge0}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}=8a\iint\limits_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

gdje je $D:\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\\ x\ge0,\ y\ge0\end{cases}$ ili drugačije napisano $D:\begin{cases}0\le x\le a\\ 0\le y\le\frac{b}{a}\sqrt{a^2-x^2}\end{cases}$

$$\frac{y^2}{b^2}\le 1-\frac{x^2}{a^2}$$

$$y^2\le\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$P=8a\int_0^a dx\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\frac{dy}{\sqrt{\underbrace{a^2-x^2}_{\text{ovo je broj za } dy}-y^2}}=8a\int_0^a\left(\arcsin\frac{y}{\sqrt{a^2-x^2}}\Bigg|_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}}\right)dx$$

$$\boxed{\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+c}$$

$$=8a\int_0^a\Big(\arcsin\frac{b}{a}-\underbrace{\arcsin 0}_{=0}\Big)dx=$$

$$=8a\arcsin\frac{b}{a}\int_0^a dx=8a^2\arcsin\frac{b}{a}$$ tražena površina

Stoksova formula

Dat je krivoliniski integral $\int\limits_C P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz$

gde je c kontura u prostoru.

Stoksova formula glasi:

$$\int\limits_c P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=\iint\limits_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

površinski integral prve vrste

$$\int\limits_c P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=\iint\limits_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

gde je S površina u prostoru ograničena konturom c a $\vec{n}=(\cos\alpha\ \cos\beta,\ \cos\gamma)$ jedinični vektor normale na površinu S.

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma$$

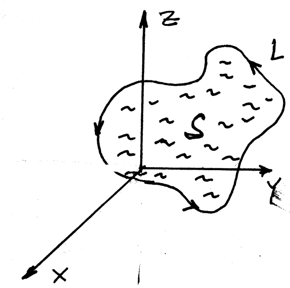
Vidimo da Stoksova formula povezuje krivoliniski integral druge vrste sa površinskim integralom prve i druge vrste.

Ranije smo spomenuli Greenovu formulu koja povezuje krivoliniski integral druge vrste sa dvostrukim integralom. Formula Gauss-Ostrogradski povezuje površinski integral druge vrste sa trostrukim integralom.

Integral $I=\int\limits_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$

uzet po nekoj zatvorenoj konturi L, pretvoriti pomoću formule Stoksa u površinski integral, nad površinom koju zatvara spomenuta kontura.

R.



$$\int\limits_L P\,dx+Q\,dy+R\,dz=\iint\limits_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$R(x,y,z)=x^2+y^2$

$P(x,y,z)=y^2+z^2$

$Q(x,y,z)=x^2+z^2$

$\frac{\partial R}{\partial y}=2y \qquad \frac{\partial Q}{\partial z}=2z$

$\frac{\partial R}{\partial x}=2x \qquad \frac{\partial P}{\partial z}=2z$

$\frac{\partial Q}{\partial x}=2x \qquad \frac{\partial P}{\partial y}=2y$

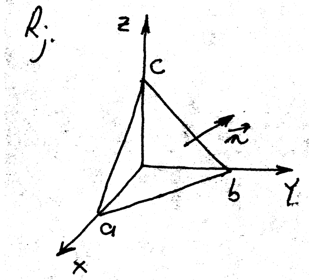
$$I=\iint\limits_S (2y-2z)\,dy\,dz-(2x-2z)\,dx\,dz+(2x-2y)\,dx\,dy=$$

$$=2\iint\limits_S (y-z)\,dy\,dz+(z-x)\,dx\,dz+(x-y)\,dx\,dy$$

(#) Izračunati krivoliniski integral $-\int_C y^2dx+z^2dy+x^2dz$

pri čemu je c kontura ΔABC gde su tačke $A(a,0,0)$, $B(0,b,0)$ i $C(0,0,c)$, $a,b,c>0$.

Rj.



$$-\int_C y^2dx+z^2dy+x^2dz \overset{\text{Stoksova formula}}{=\!=\!=} -\iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$P=y^2$, $Q=z^2$, $R=x^2$

$\frac{\partial Q}{\partial x}=0$, $\frac{\partial P}{\partial y}=2y$, $\frac{\partial R}{\partial y}=0$, $\frac{\partial Q}{\partial z}=2z$

$\frac{\partial R}{\partial x}=2x$ $\quad \frac{\partial P}{\partial z}=0$

$$\begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = -2z\,dydz-2x\,dzdx-2y\,dxdy$$

$$-\int_C y^2dx+z^2dy+x^2dz = 2\iint_S (z\,dydz+x\,dzdx+y\,dxdy)$$

S oblast ograničena ΔABC

Izračunajmo $\iint_S z\,dydz$. Površinu S proicirajmo na yOz ravan:

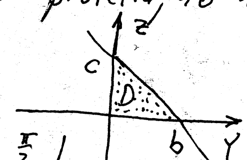
$\frac{y}{b}+\frac{z}{c}=1$

$cy+bz=bc$

$bz=bc-cy$

$z=c-\frac{c}{b}y=\frac{c}{b}(b-y)$

Ugao koji zaklapa vektor normale $\vec{n}$ na površinu S je između 0 i $\frac{\pi}{2}$.



$\frac{y}{b}+\frac{z}{c}=1$

$\vec{n}=(\frac{1}{a},\frac{1}{b},\frac{1}{c})$

$\cos\alpha\geq 0$

$D: \begin{cases} 0\leq y\leq b \\ 0\leq z\leq \frac{c}{b}(b-y) \end{cases}$

$$\iint_S z\,dydz = +\iint_D z\,dydz = \int_0^b \Big[\int_0^{\frac{c}{b}(b-y)} z\,dz\Big]dy = \int_0^b \frac{1}{2}z^2\Big|_0^{\frac{c}{b}(b-y)}dy = \frac{1}{2}\Big(\frac{c}{b}\Big)^2\int_0^b (b-y)^2dy$$

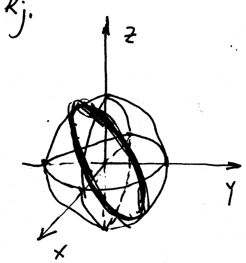
$$= \left| \begin{matrix} b-y=t & -dy=dt \\ y=0 \Rightarrow t=b & dy=-dt \\ y=b \Rightarrow t=0 & \end{matrix} \right| = \frac{1}{2}\frac{c^2}{b^2}\int_0^b t^2dt = \frac{1}{2}\cdot\frac{c^2}{b^2}\cdot\frac{t^3}{3}\Big|_0^b = \frac{1}{2}\cdot\frac{bc^2}{3}$$

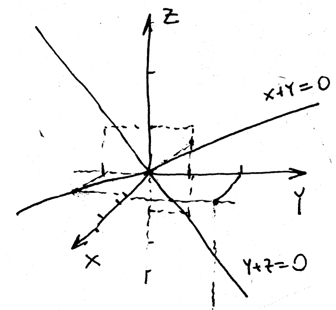
Analogno izračunamo $\iint_S x\,dzdx=\frac{1}{2}\frac{a^2c}{3}$; $\iint_S y\,dxdy=\frac{1}{2}\cdot\frac{ab^2}{3}$ $\Rightarrow$ $I=\frac{ab^2+bc^2+ca^2}{3}$

(#) Izračunati krivoliniski integral $\int_C ydx+zdy+xdz$

ako je c krug dobijen presekom sfere $x^2+y^2+z^2=a^2$ i ravni $x+y+z=0$.

Rj.





$$\int_C y\,dx+z\,dy+x\,dz \overset{\text{Stoksova formula}}{=\!=\!=} \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$\frac{\partial Q}{\partial x}=0$ $\quad \frac{\partial P}{\partial y}=1$ $\quad P=y$, $Q=z$, $R=x$

S je površina ograničena krugom

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \Big(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\Big)\cos\alpha-\Big(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\Big)\cos\beta+\Big(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\Big)\cos\gamma$$

$\frac{\partial R}{\partial y}=0$ $\quad \frac{\partial Q}{\partial z}=1$ $\quad \frac{\partial R}{\partial x}=1$ $\quad \frac{\partial P}{\partial z}=0$

$$\int_C y\,dx+z\,dy+x\,dz = \iint_S(-\cos\alpha-\cos\beta-\cos\gamma)\,dS$$

gde je $\vec{n}_0=(\cos\alpha,\cos\beta,\cos\gamma)$ vektor (jedinični) normale na površinu S

$x+y+z=0$

$\vec{n}=(1,1,1)$ vektor normale na ravan $x+y+z=0$ (a time i na našu površinu S)

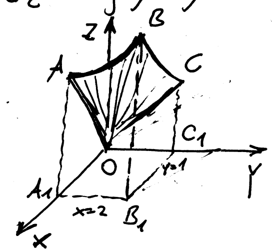
$|\vec{n}|=\sqrt{1+1+1}=\sqrt{3}$ $\quad \vec{n}_0=\Big(\underbrace{\frac{1}{\sqrt{3}}}_{\cos\alpha},\underbrace{\frac{1}{\sqrt{3}}}_{\cos\beta},\underbrace{\frac{1}{\sqrt{3}}}_{\cos\gamma}\Big)$

$$\iint_S(-\cos\alpha-\cos\beta-\cos\gamma)\,dS=\iint_S\Big(-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}\Big)dS=-\frac{3}{\sqrt{3}}\iint_S dS$$

$\iint_S dS$ je površina oblasti S (S je krug poluprečnika a $\quad P_{kruga}=a^2\pi$)

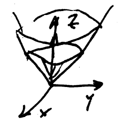
$$\int_C y\,dx+z\,dy+x\,dz=-\frac{3}{\sqrt{3}}a^2\pi=-\sqrt{3}\,a^2\pi$$

(#) Uz pomoć formule Stoksa, izračunati krivolinijski integral $K=\oint\limits_{\ell} e^x dx + z(x^2+y^2)^{\frac{3}{2}} dy + yz^3 dz$ gdje je



ℓ - zakrivljena linija OCBAO (vidi sliku) dobijena presjekom površina

$z=\sqrt{x^2+y^2}$, $x=0$, $x=2$, $y=0$, $y=1$.



Rj: $z=\sqrt{x^2+y^2}$ je čunj iznad xOy ravni

$x=0$, $x=2$ su ravni paralelne sa yOz ravni

$y=0$, $y=1$ su ravni paralelne sa xOz ravni

Stoksova formula glasi

$$\int\limits_c P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=\iint\limits_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

$P=e^x$, $\frac{\partial P}{\partial y}=0$, $\frac{\partial P}{\partial z}=0$

$Q=z(x^2+y^2)^{\frac{3}{2}}$, $\frac{\partial Q}{\partial x}=z\cdot\frac{3}{2}(x^2+y^2)^{\frac{1}{2}}\cdot 2x=3xz\sqrt{x^2+y^2}$, $\frac{\partial Q}{\partial z}=(x^2+y^2)^{\frac{3}{2}}$

$R=yz^3$, $\frac{\partial R}{\partial x}=0$, $\frac{\partial R}{\partial y}=z^3$

$$K=\oint\limits_{\ell} e^x dx+z(x^2+y^2)^{\frac{3}{2}}dy+yz^3dz=\left|\text{formula Stoksa}\right|=$$

$$=\iint\limits_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & z(x^2+y^2)^{\frac{3}{2}} & yz^3 \end{vmatrix}=\iint\limits_S \underbrace{(z^3-(x^2+y^2)^{\frac{3}{2}})}_{=0}\,dy\,dz-(0-0)\,dx\,dz$$

$z^3 = (\sqrt{x^2+y^2})^3$

$$+(3xz\sqrt{x^2+y^2}-0)\,dx\,dy=\iint\limits_S 3xz\sqrt{x^2+y^2}\,dx\,dy$$ površinski integral II vrste

Tj. dobili smo $K=\iint\limits_S 3xz\sqrt{x^2+y^2}\,dx\,dy$

Kako naša data kriva pravi površinu S: $z=\sqrt{x^2+y^2}$ u prvom oktantu imamo

$$K=\iint\limits_S 3x(x^2+y^2)\,dx\,dy$$

Prisjetimo se kako se računa površinski integral II vrste npr. $I=\iint\limits_S R(x,y,z)\,dx\,dy$. Neka je $\vec{n}$ vektor normale na površ S, i neka je γ ugao koji $\vec{n}$ gradi sa z-osom, i neka je D projekcija površi S na xOy ravan. Tada

$$I=\iint\limits_S R(x,y,z)\,dx\,dy=\pm\iint\limits_D R(x,y,z(x,y))\,dx\,dy$$ gdje predznak ispred integrala zavisi od $\cos\gamma$ (za $\cos\gamma>0$ $+$, $\cos\gamma<0$ $-$).

Mi posmatramo vanjsku stranu površi, iz čega možemo zaključiti (sa slike) da je $\gamma\in(\frac{\pi}{2},\pi)$ pa je $\cos\gamma<0$.

Projekcija D površi S je data u sklopu zadatka (vidi sliku) ($\square A_1B_1C_1O$)

$D: \begin{cases} 0\le x\le 2 \\ 0\le y\le 1 \end{cases}$ $\quad K=\iint\limits_S 3x(x^2+y^2)\,dx\,dy=-\iint\limits_D 3x(x^2+y^2)\,dx\,dy=$

$$=-3\int_0^1 dy\int_0^2 (x^3+xy^2)\,dx=-3\int_0^1\left(\frac{1}{4}x^4\Big|_0^2+\frac{1}{2}x^2y^2\Big|_0^2\right)dy=$$

$$=-3\int_0^1 (4+2y^2)\,dy=-3\left(4y\Big|_0^1+\frac{2}{3}y^3\Big|_0^1\right)=-12-2=-14$$ traženo rješenje

Zadaci za vježbu

3894. Integral $\int\limits_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$, uzet po nekoj zatvorenoj konturi L, primenom Štoksove formule transformisati u integral po površini „razapetoj“ nad tom konturom.

3895. Izračunati integral $\int\limits_L x^2y^3\,dx+dy+z\,dz$ po krugu $x^2+y^2=R^2$, $z=0$, na dva načina: a) neposredno, i b) koristeći Štoksovu formulu, uzimajući za površinu S polusferu $z=+\sqrt{R^2-x^2-y^2}$. (Integracija po krugu u ravni xOy računa se u pozitivnom smeru obilaženja).

Rješenja

3894. $2\iint\limits_S (x-y)\,dx\,dy+(y-z)\,dy\,dz+(z-x)\,dx\,dz.$

3895. $-\frac{\pi R^6}{8}.$

Formula Gauss-Ostrogradski

Ova formula daje vezu između površinskog integrala druge vrste i trostrukog integrala.

$$\iint P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy =$$

$$= \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S (S je zatvorena površina).

(1.) Izračunati $\iint\limits_S xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx$ gdje je S bilo koja zatvorena površ.

Rj.

$$\iint\limits_S yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iint\limits_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy \xrightarrow{\text{Formula Gauss-Ostr.}} \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S, $\frac{\partial P}{\partial x} = 0$, $\frac{\partial Q}{\partial y} = 0$, $\frac{\partial R}{\partial z} = 0$

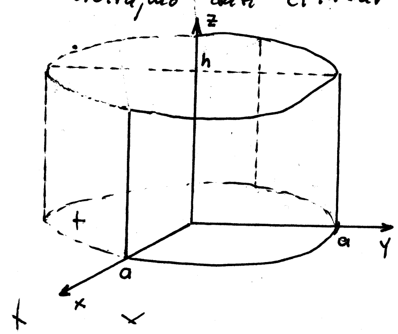
$$\iint\limits_S yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iiint\limits_{\Omega} 0\,dx\,dy\,dz =$$

$$\Omega : \begin{cases} a \le x \le b \\ c \le y \le d \\ e \le z \le f \end{cases} \qquad = \int_a^b dx \int_c^d dy \int_e^f 0\,dz = 0$$

(#) Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral $I = \oiint\limits_S 4x^3\,dy\,dz + 4y^3\,dx\,dz - 6z^4\,dx\,dy$

gdje je S vanjska strana cilindra $x^2+y^2=a^2$ koji se nalazi između ravni $z=0$ i $z=h$.

Rj. Skicirajmo dati cilindar



Prisjetimo se formule Gauss-Ostrogradski

$$\oiint\limits_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy =$$

$$= \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

Ω - unutrašnjost oblasti S

$P(x,y,z) = 4x^3 \qquad \frac{\partial P}{\partial x} = 12x^2$

$Q(x,y,z) = 4y^3 \qquad \frac{\partial Q}{\partial y} = 12y^2$

$R(x,y,z) = 6z^4 \qquad \frac{\partial R}{\partial z} = 24z^3$

$$\oiint\limits_S 4x^3\,dy\,dz + 4y^3\,dx\,dz - 6z^4\,dx\,dy = 12 \iiint\limits_{\Omega} (x^2+y^2-2z^3)\,dx\,dy\,dz =$$

$$= \left| \begin{array}{ll} \text{uvedimo cilindrične koordinate} & \\ x = r\cos\varphi & \Omega \xrightarrow{\text{transformiše}} \Omega' \\ y = r\sin\varphi & 0 \le r \le a \\ z = z & 0 \le \varphi \le 2\pi \\ dx\,dy\,dz = r\,dr\,d\varphi\,dz & 0 \le z \le h \\ x^2+y^2 = \ldots = r^2 & \end{array} \right| = 12 \iiint\limits_{\Omega'} (r^2 - 2z^3)\, r\,dr\,d\varphi\,dz =$$

$$= 12 \int_0^{2\pi} d\varphi \int_0^a dr \int_0^h (r^3 - 2rz^3)\,dz = 12 \int_0^{2\pi} d\varphi \int_0^a \left(r^3 z \Big|_0^h - 2r \cdot \frac{1}{4} z^4 \Big|_0^h \right) dr$$

$$= 12 \int_0^{2\pi} d\varphi \int_0^a \left(r^3 h - \frac{1}{2} r h^4 \right) dr = 12\, \varphi \Big|_0^{2\pi} \left(h \frac{1}{4} r^4 \Big|_0^a - \frac{1}{2} h^4 \cdot \frac{1}{2} r^2 \Big|_0^a \right) =$$

$$= 24\pi \cdot \frac{1}{4} h \left(a^4 - h^3 a^2 \right) = 6\pi h a^2 (a^2 - h^3)$$ traženo rješenje

(#) Površinski integral po zatvorenoj površini pretvoriti uz pomoć formule Ostrogradskog u trostruki integral po zapremini tijela, koje je ograničeno spomenutom površinom

$$\iint_S \sqrt{x^2+y^2+z^2}\,\left[\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)\right]dS$$

gdje je $\vec{n}$ vanjska normala na površinu S.

Rj: $\cos(\vec{n},x)$ je kosinus ugla između normale i x-ose. $\cos(\vec{n},y)$ i $\cos(\vec{n},z)$ je kosinus ugla između normale na površinu S i y-ose i z-ose redom.

Uvedimo oznake $\cos(\vec{n},x)=\cos\alpha$, $\cos(\vec{n},y)=\cos\beta$ i $\cos(\vec{n},z)=\cos\gamma$.

Prema formuli Stoksa znamo da je
$$dy\,dz = dS\cos\alpha$$
$$dz\,dx = dS\cos\beta$$
$$dx\,dy = dS\cos\gamma$$

$$I=\iint_S \sqrt{x^2+y^2+z^2}\,(\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)]\,dS =$$

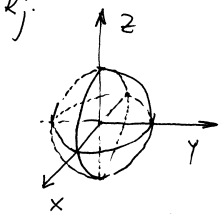
$$=\iint_S \sqrt{x^2+y^2+z^2}\,(dy\,dz+dz\,dx+dx\,dy)$$

$$\iint_S P\,dy\,dz+Q\,dx\,dz+R\,dx\,dy=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$\frac{\partial P}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}} \qquad I=\iiint_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$$

(#) Izračunati $\iint_S x^3dy\,dz+y^3dz\,dx+z^3dx\,dy$ gdje je S - vanjski dio sfere $x^2+y^2+z^2=R^2$.

Rj:



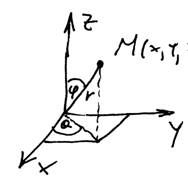
$$I=\iint_S x^3dy\,dz+y^3dz\,dx+z^3dx\,dy \underset{\text{Gauss-Ostrogradski}}{\overset{\text{Formula}}{=}}$$

$$=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$P=x^3,\quad \frac{\partial P}{\partial x}=3x^2,\quad Q=y^3,\quad \frac{\partial Q}{\partial y}=3y^2,\quad R=z^3,\quad \frac{\partial R}{\partial z}=3z^2$$

$$I=\iiint_\Omega (3x^2+3y^2+3z^2)\,dx\,dy\,dz \qquad \Omega: x^2+y^2+z^2\le R^2$$

Uvedimo sferne koordinate:



$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$

$$\Omega':\begin{cases} 0\le r\le R\\ 0\le\theta\le 2\pi\\ 0\le\varphi\le\pi\\ dx\,dy\,dz=r^2\sin\varphi\,d\varphi\,d\theta\,dr\end{cases}$$

$$x^2+y^2+z^2=r^2$$

$$I=3\iiint_{\Omega'} r^2r^2\sin\varphi\,d\varphi\,d\theta\,dr=3\int_0^R r^4dr\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi=$$

$$=3\cdot\frac{1}{5}r^5\Big|_0^R\cdot\theta\Big|_0^{2\pi}\cdot(-\cos\varphi)\Big|_0^{\pi}=\frac{3}{5}\cdot R^5\cdot 2\pi\cdot 2=\frac{12}{5}R^5\pi$$

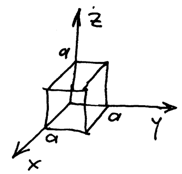
(#) Izračunati $\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$ gdje je

S – vanjska strana kocke $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$.

Rj.:

$$\iint\limits_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy \overset{\text{formula Gauss-Ostr.}}{=} \iiint\limits_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$$\frac{\partial P}{\partial x} = 2x, \quad \frac{\partial Q}{\partial y} = 2y, \quad \frac{\partial R}{\partial z} = 2z$$



$$\Omega : \begin{cases} 0 \le x \le a \\ 0 \le y \le a \\ 0 \le z \le a \end{cases}$$

Prema tome:

$$\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy =$$

$$= \iiint\limits_\Omega (2x + 2y + 2z)\,dx\,dy\,dz = 2\int_0^a dx \int_0^a dy \int_0^a (x+y+z)\,dz =$$

$$= 2\int_0^a dx \int_0^a \left(xz\Big|_0^a + yz\Big|_0^a + \frac{1}{2}z^2\Big|_0^a \right) dy = 2\int_0^a dx \int_0^a \left(ax + ay + \frac{1}{2}a^2\right) dy =$$

$$2a\int_0^a \left(xy\Big|_0^a + \frac{1}{2}y^2\Big|_0^a + \frac{1}{2}ay\Big|_0^a \right) dx = 2a\int_0^a \left(ax + \frac{1}{2}a^2 + \frac{1}{2}a^2\right) dx = 2a^2\int_0^a (x+a)\,dx =$$

$$= 2a^2\left(\frac{1}{2}a^2 + a^2\right) = 3a^4$$

Zadaci za vježbu

3896. Površinski integral $\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$, uzet po zatvorenoj površini S, primenom formule Ostrogradskog, transformisati u trojni integral po zapremini ograničenoj tom površinom (Integral se računa po spoljnoj strani površine S).

3897. Površinski integral $\iint\limits_S x^2 + y^2 + z^2 \{\cos(N,x) + \cos(N,y) + \cos(N,z)\} d\sigma$ po zatvorenoj površini S, primenom formule Ostrogradskog transformisati u trojni integral po zapremini ograničenoj tom površinom, pri čemu je N spoljna normala površine S.

3898. Izračunati integral u prethodnom zadatku ako je S sfera poluprečnika R sa centrom u koordinatnom početku.

3899. Izračunati integral

$$\iint\limits_S [x^3\cos(N,\ x) + y^3\cos(N,\ y) + z^3\cos(N,\ z)]\,d\sigma,$$

u kojem je S — sfera poluprečnika R sa centrom u koordinatnom početku, a N — spoljna normala.

3900. Izračunati integral u zadacima 3891—3863 primenom formule Ostrogradskog.

Rješenja

3896. $2\iiint\limits_\Omega (x+y+z)\,dx\,dy\,dz.$

3897. $\iiint\limits_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz.$ **3898.** 0. **3899.** $\frac{12}{5}\pi R^5.$

Diferenciranje pod znakom integrala

Neka je dat integral koji zavisi od parametra α:

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$$

Ako su $f(x,\alpha)$, $f'(x,\alpha)$ neprekidne f-je, ako postoje $b'(\alpha)$ i $a'(\alpha)$ tada

$$I'(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f'_\alpha(x,\alpha)\,dx + b'(\alpha)\, f(b(\alpha),\alpha) - a'(\alpha)\cdot f(a(\alpha),\alpha)$$

Ako granice a i b ne zavise od α tada

$$I'(\alpha) = \int_a^b f'_\alpha(x,\alpha)\,dx$$

\# Polazeći od integrala $\int_0^b \frac{dx}{1+\alpha x}$ izračunati

$$\int_0^b \frac{x\,dx}{(1+\alpha x)^2} \quad ; \quad \int_0^b \frac{x^2\,dx}{(1+\alpha x)^3}.$$

Rj.

$$I(\alpha) = \int_0^b \frac{dx}{1+\alpha x} = \left| \begin{matrix} 1+\alpha x = t & x=0 \Rightarrow t=1 \\ \alpha\,dx = dt & x=b \Rightarrow t=1+\alpha b \\ dx = \frac{1}{\alpha}dt & \end{matrix} \right| =$$

$$= \frac{1}{\alpha}\int_1^{1+\alpha b} \frac{dt}{t} = \frac{1}{\alpha} \ln|t| \Big|_1^{1+\alpha b} = \frac{1}{\alpha}\ln|1+\alpha b|$$

$$f'_\alpha(x,\alpha) = \left(\frac{1}{1+\alpha x}\right)'_\alpha = (-1)(1+\alpha x)^{-2}\cdot x = \frac{-x}{(1+\alpha x)^2}$$

$$I'(\alpha) = \int_a^b f'_\alpha(x,\alpha)\,dx \Rightarrow \quad I'(\alpha) = \int_0^b \frac{-x}{(1+\alpha x)^2}\,dx \Rightarrow$$

$$\Rightarrow \int_0^b \frac{x\,dx}{(1+\alpha x)^2} = -I'(\alpha)$$

Kako je $I(\alpha) = \frac{1}{\alpha}\ln|1+\alpha b|$ to je $I'_\alpha = -\frac{1}{\alpha^2}\ln|1+\alpha b| + \frac{1}{\alpha}\cdot\frac{1}{1+\alpha b}\cdot b$

Prema tome $\displaystyle\int_0^b \frac{x\,dx}{(1+\alpha x)^2} = \frac{1}{\alpha^2}\ln|1+\alpha b| - \frac{b}{\alpha(1+\alpha b)}$

Slično bi imali $\displaystyle I''(\alpha) = \int_0^b \left(\frac{-x}{(1+\alpha x)^2}\right)'_\alpha dx = \int_0^b \frac{2x^2}{(1+\alpha x)^3}\,dx \Rightarrow$

$$\Rightarrow \int_0^b \frac{x^2}{(1+\alpha x)^3}\,dx = \frac{1}{2} I''(\alpha), \quad I''_\alpha = (I'_\alpha)' = \frac{2}{\alpha^3}\ln|1+\alpha b| + \left(-\frac{1}{\alpha^2}\right)\frac{b}{1+\alpha b}$$

$$-\frac{b}{\alpha^2(1+\alpha b)} + \frac{1}{\alpha}\cdot\frac{-b}{(1+\alpha b)^2}\cdot b \quad \Rightarrow \int_0^b \frac{x^2}{(1+\alpha x)^3}\,dx = \frac{1}{\alpha^3}\ln|1+\alpha b| - \frac{b}{\alpha^2(1+\alpha b)}$$

$$-\frac{b^2}{\alpha(1+\alpha b)^2} \quad \text{traženo rješenje}$$

(#) Izračunati pomoću diferenciranja po parametru integral

$$I(\alpha)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\alpha^2\cos^2 x)\,dx,\quad \alpha>0.$$

Rj. Ako je data f-ja dvije promjenjive $f(x,\alpha)$, ako su $f(x,\alpha)$ i $f'_\alpha(x,\alpha)$ neprekidne f-je tada za integral $I(\alpha)=\int_a^b f(x,\alpha)\,dx$ vrijedi $I'_\alpha(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx$.

f'_α — predstavlja izvod f-je f po promjenjivoj α

$$I(\alpha)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\alpha^2\cos^2 x)\,dx$$

$$f(x,\alpha)=\ln(\sin^2 x+\alpha^2\cos^2 x)$$

$$f'_\alpha=\frac{1}{\sin^2 x+\alpha^2\cos^2 x}\cdot 2\alpha\cos^2 x=\frac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}$$

$$I'_\alpha(\alpha)=\int_0^{\pi/2} f'_\alpha\,dx=\int_0^{\pi/2}\frac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}\,dx \;/:\cos^2 x\; =2\alpha\int_0^{\pi/2}\frac{dx}{\operatorname{tg}^2 x+\alpha^2}$$

$$=\left|\begin{array}{ll} \operatorname{tg} x=t & x=\operatorname{arctg} t \\ x=0 \Rightarrow t=0 & dx=\frac{dt}{1+t^2} \\ x=\frac{\pi}{2}\Rightarrow t=\infty & \end{array}\right| = 2\alpha\int_0^\infty \frac{dt}{(t^2+\alpha^2)(t^2+1)}$$

$$\frac{1}{(x^2+\alpha^2)(x^2+1)}=\frac{Ax+B}{x^2+\alpha^2}+\frac{Cx+D}{x^2+1}\quad /\cdot(x^2+\alpha^2)(x^2+1)$$

$$1=A(x^3+x)+B(x^2+1)+C(x^3+\alpha^2 x)+D(x^2+\alpha^2)$$

$A+C=0$ (1)

$B+D=0$ (2)

$A+\alpha^2 C=0$ (3)

$B+\alpha^2 D=1$ (4)

(1)-(4): $C-\alpha^2 C=0 \Rightarrow C=0 \Rightarrow A=0$

(2)-(4): $D-\alpha^2 D=-1$ $\quad(\alpha^2-1)D=1$

$\alpha^2 D-D=1$ $\quad D=\frac{1}{\alpha^2-1} \Rightarrow B=\frac{-1}{\alpha^2-1}$

$$I'_\alpha(\alpha)=2\alpha\int_0^\infty\frac{dx}{(x^2+\alpha^2)(x^2+1)}=\frac{-2\alpha}{\alpha^2-1}\int_0^\infty\frac{dx}{x^2+\alpha^2}+\frac{2\alpha}{\alpha^2-1}\int_0^\infty\frac{dx}{x^2+1}=$$

$$-\frac{2\alpha}{\alpha^2-1}\cdot\frac{1}{\alpha}\operatorname{arctg}\frac{x}{\alpha}\Big|_0^\infty+\frac{2\alpha}{\alpha^2-1}\operatorname{arctg} x\Big|_0^\infty=$$

$$=-\frac{2}{\alpha^2-1}\left(\frac{\pi}{2}-0\right)+\frac{2\alpha}{\alpha^2-1}\left(\frac{\pi}{2}-0\right)=$$

$$=-\frac{\pi}{\alpha^2-1}+\frac{\pi}{\cancel{2}}\cdot\frac{\cancel{2}\alpha}{\alpha^2-1}=\frac{\pi(\alpha-1)}{\underbrace{\alpha^2-1}_{(\alpha-1)(\alpha+1)}}=\frac{\pi}{\alpha+1}$$

$$I'_\alpha(\alpha)=\frac{\pi}{\alpha+1}\Rightarrow I(\alpha)=\pi\ln|\alpha+1|+C=\left|\text{kako je } \alpha>0\right|$$

$$=\pi\ln(\alpha+1)+C\quad\ldots(*)$$

$$I(\alpha)=\int_0^{\pi/2}\ln(\sin^2 x+\alpha^2\cos^2 x)\,dx\Rightarrow I(1)=\int_0^{\pi/2}\ln(1)\,dx=0\quad\ldots(**)$$

$$I(1)\overset{(*)}{=}\pi\ln 2+C\overset{(**)}{=}0\quad\Rightarrow\quad C=-\pi\ln 2$$

$$\int_0^{\pi/2}\ln(\sin^2 x+\alpha^2\cos^2 x)\,dx=\pi\ln(\alpha+1)-\pi\ln 2=\pi\ln\frac{\alpha+1}{2}$$

traženo rješenje

Izračunati $I(\alpha)=\int_0^\infty \frac{1-e^{-\alpha x}}{xe^{x}}dx$ ako je $\alpha>-1$.

Rj: $I'(\alpha)=\int_a^b f'_\alpha(\alpha,x)\,dx$

$f(\alpha,x)=\frac{1-e^{-\alpha x}}{x e^{x}}$, $\quad f'_\alpha=\frac{-e^{-\alpha x}\cdot(-x)}{xe^{x}}$

$$I'(\alpha)=\int_0^\infty \frac{x e^{-\alpha x}}{xe^{x}}dx=\int_0^\infty e^{-\alpha x-x}dx=\int_0^\infty e^{-(\alpha+1)x}dx=\left|\begin{array}{l} -(\alpha+1)x=s\\ -(\alpha+1)dx=ds\\ dx=-\frac{ds}{\alpha+1}\\ x=0\Rightarrow s=0\\ x=\infty\Rightarrow s=-\infty\end{array}\right|=-\frac{1}{\alpha+1}\int_0^{-\infty}e^{s}ds=\frac{-1}{\alpha+1}e^{s}\Big|_0^{-\infty}=0-\frac{(-1)}{\alpha+1}=\frac{1}{\alpha+1}$$

$$I'_\alpha=\frac{1}{\alpha+1}\Rightarrow I(\alpha)=\int\frac{1}{\alpha+1}d\alpha=\ln|\alpha+1|+C$$

Kako je $I(0)=\int_0^\infty\frac{1-e^{0}}{xe^{x}}dx=0$ to je $I(0)=\ln 1+C=0$

$\Rightarrow C=0$ $\qquad I(\alpha)=\ln|\alpha+1|$ traženo rješenje

za vježbu

Izračunati $I(\alpha)=\int_0^\infty\frac{1-e^{-\alpha x^2}}{xe^{x^2}}dx$, ako je $\alpha>-1$.

za vježbu

Izračunati $\int_0^{\pi/2}\frac{\operatorname{arc\,tg}(\alpha\,\operatorname{tg}x)}{\operatorname{tg}x}dx$

rješenje: $I(\alpha)=\frac{\pi}{2}\ln|1+\alpha|$.

Izračunati $\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}dx$.

Rj: $I'(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx$, $\qquad I(\alpha)=\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}dx$

$f(\alpha,x)=e^{-x}\frac{\sin\alpha x}{x}$, $\quad f'_\alpha=\frac{e^{-x}}{x}\cdot x\cos\alpha x=e^{-x}\cos\alpha x$

$$I'(\alpha)=\int_0^\infty e^{-x}\cos\alpha x\,dx=\lim_{R\to\infty}\int_0^R e^{-x}\cos\alpha x\,dx=$$

$$=\left|\begin{array}{ll} u=e^{-x} & dv=\cos\alpha x\,dx\\ du=-e^{-x}dx & v=\frac{1}{\alpha}\sin\alpha x\end{array}\right|=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-x}\sin\alpha x\Big|_0^R+\frac{1}{\alpha}\int_0^R e^{-x}\sin\alpha x\,dx\right)$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R+\frac{1}{\alpha}\int_0^R e^{-x}\sin\alpha x\,dx\right)=\left|\begin{array}{ll} u=e^{-x} & dv=\sin\alpha x\,dx\\ du=-e^{-x}dx & v=-\frac{1}{\alpha}\cos\alpha x\end{array}\right|=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R+\frac{1}{\alpha}\left(-\frac{1}{\alpha}\underbrace{e^{-x}\cos\alpha x\Big|_0^R}_{(e^{-R}\cos\alpha R-e^{0}\cdot 1)}-\frac{1}{\alpha}\int_0^R e^{-x}\cos\alpha x\,dx\right)\right)=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R-\frac{1}{\alpha^2}e^{-R}\cos\alpha R+\frac{1}{\alpha^2}-\frac{1}{\alpha^2}\int_0^R e^{-x}\cos\alpha x\,dx\right)=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\underbrace{\sin\alpha R}_{\text{ovo je između }-1\text{ i }1}-\frac{1}{\alpha^2}e^{-R}\underbrace{\cos\alpha R}_{\text{uzima vrijednosti između }-1\text{ i }1}+\frac{1}{\alpha^2}\right)-\frac{1}{\alpha^2}\underbrace{\lim_{R\to\infty}\int_0^R e^{-x}\cos\alpha x\,dx}_{I'(\alpha)}$$

Sad imamo

$$\left(1+\frac{1}{\alpha^2}\right)\int_0^\infty e^{-x}\cos\alpha x\,dx=\frac{1}{\alpha^2}\Rightarrow\int_0^\infty e^{-x}\cos\alpha x\,dx=\frac{\frac{1}{\alpha^2}}{\frac{\alpha^2+1}{\alpha^2}}=\frac{1}{\alpha^2+1}$$

Kako je $I'(\alpha)=\frac{1}{\alpha^2+1}$ to je $I(\alpha)=\int\frac{1}{\alpha^2+1}d\alpha=\operatorname{arc\,tg}\alpha+C$

$I(0)=0=\operatorname{arc\,tg}0+C\Rightarrow C=0$

Prema tome $\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}dx=\operatorname{arc\,tg}\alpha$ traženo rješenje

Zadaci za vježbu

3730. Naći oblast definisanosti funkcije $f(x)=\int\limits_0^1 \frac{dz}{\sqrt{x^2+z^2}}$.

3731. Naći krivinu krive $y=\int\limits_{\pi}^{2\pi} \frac{\sin \alpha x}{\alpha}\,d\alpha$ u tački čija je apscisa $x=1$.

3732. Polazeći od jednakosti $\int\limits_0^b \frac{dx}{1+ax}=\frac{1}{a}\ln(1+ab)$ izvesti diferenciranjem po parametru, sledeću formulu:

$$\int\limits_0^b \frac{x\,dx}{(1+ax)^2}=\frac{1}{a^2}\ln(1+ab)-\frac{b}{a(1+ab)}.$$

3733. Polazeći od jednakosti $\int\limits_0^b \frac{dx}{x^2+a^2}=\frac{1}{a}\operatorname{arctg}\frac{b}{a}$, izračunati integral $\int\limits_0^b \frac{dx}{(x^2+y^2)^3}$.

3734. Polazeći od jednakosti $\int\limits_0^\infty \frac{dx}{a^2+x^2}=\frac{\pi}{2a}$. izračunati $\int\limits_0^\infty \frac{dx}{(x^2+a^2)^n}$ (n je ceo pozitivan broj).

3735. Izračunati vrednost integrala $\int\limits_0^\infty e^{-ax}x^{n-1}\,dx$ (n je ceo pozitivan broj) za $a>0$, našavši prethodno vrednost $\int\limits_0^\infty e^{-ax}\,dx$.

3736*. Polazeći od jednakosti (vidi zad. 2318)

$$\int\limits_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi}{2|ab|},\quad \text{naći}\quad \int\limits_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}.$$

Rješenja

3730. Definisana je za sve vrednosti $x\neq 0$. **3731.** 3π.

3733. $\frac{b}{8a^4}\left\{\frac{5a^2+3b^2}{(a^2+b^2)^2}+\frac{3}{ab}\operatorname{arctg}\frac{b}{a}\right\}$. **3734.** $\frac{1\cdot3\cdot5\ldots(2n-3)}{2\cdot4\cdot6\ldots(2n-2)}\frac{\pi}{2a^{2n-1}}$ $(n>1)$.

3735. $\frac{(n-1)!}{a^n}$. **3736*.** $\frac{\pi(a^2+b^2)}{4|ab|^3}$. Diferencirati po a ili b i sabrati rezultate.

U zadacima 3737 — 3749 izračunati vrednosti datih integrala metodom diferenciranja po parametru.

3737. $\int\limits_0^\infty \frac{1-e^{-ax}}{xe^x}\,dx \quad (a>-1)$.

3738. $\int\limits_0^\infty \frac{1-e^{-ax^2}}{xe^{x^2}}\,dx \quad (a>-1)$.

3739. $\int\limits_0^1 \frac{\operatorname{arctg} ax}{x\sqrt{1-x^2}}\,dx$. **3740.** $\int\limits_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}\,dx \quad (a^2<1)$.

3741. $\int\limits_0^\infty \frac{\operatorname{arctg} ax}{x(1+x^2)}\,dx$. **3742.** $\int\limits_0^1 \frac{\ln(1-a^2x^2)}{\sqrt{1-x^2}}\,dx \quad (a^2<1)$.

3743. $\int\limits_0^\pi \frac{\ln(1+a\cos x)}{\cos x}\,dx \quad (a^2<1)$.

3744. $\int\limits_0^{\frac{\pi}{2}} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x} \quad (a^2<1)$.

3745. $\int\limits_0^\infty \frac{1-e^{-ax^2}}{x^2}\,dx \quad (a>0)$, znajući da je

$$\int\limits_0^\infty e^{-ax^2}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a>0) \text{ (vidi zadatak 2439).}$$

3746*. $\int\limits_0^\infty \frac{e^{-ax^2}-e^{-bx^2}}{x^2}\,dx \quad (a>0,\ b>0)$.

3747*. $\int\limits_0^\infty e^{-ax}\frac{\sin bx-\sin cx}{x}\,dx \quad (a>0)$.

3748. $\int\limits_0^\infty e^{-ax}\frac{\cos bx-\cos cx}{x}\,dx \quad (a>0)$.

3749*. $\int\limits_0^{\frac{\pi}{2}} \ln(a^2\cos^2 x+b^2\sin^2 x)\,dx$.

Rješenja

3737. $\ln(1+a)$. **3738.** $\frac{1}{2}\ln(1+a)$.

3739. $\frac{\pi}{2}\ln\left(a+\sqrt{1+a^2}\right)$.

3740. $\pi\left(\sqrt{1-a^2}-1\right)$.

3741. $\frac{\pi}{2}\ln(1+a)$, ako je $a>0$;
$-\frac{\pi}{2}\ln(1-a)$, ako je $a<0$.

3742. $\pi\ln\frac{1+\sqrt{1-a^2}}{2}$.

3743. $\pi\arcsin a$. **3744.** $\pi\arcsin a$.

3745. $\sqrt{\pi a}$.

3746*. $\sqrt{\pi}\left(\sqrt{b}-\sqrt{a}\right)$.

Naći izvode po a ili po b.

3747*. $\operatorname{arctg}\frac{b}{a}-\operatorname{arctg}\frac{c}{a}=\operatorname{arctg}\frac{a(b-c)}{a^2+bc}$.

Diferencirati po b ili po c.

3748. $\frac{1}{2}\ln\frac{a^2+b^2}{a^2+c^2}$.

3749*. $\pi\ln\frac{a+b}{2}$. Diferencirati po a ili po b.

Zadaci za vježbu

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

3750. Izračunavši integral $\int_0^{\frac{\pi}{2}} \frac{\operatorname{arctg}(a \operatorname{tg} x)}{\operatorname{tg} x} dx$, naći $\int_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} x} dx$.

3751. Koristeći jednakost $\int_0^1 x^n dx = \frac{1}{n+1}$, izračunati integral

$$\int_0^1 \frac{x^\beta - x^\alpha}{\ln x} dx \quad (\alpha > -1, \ \beta > -1).$$

3752. Koristeći jednakost $2a \int_0^\infty e^{-a^2x^2} dx = \sqrt{\pi}$ (vidi zadatak 2439), izračunati integral $\int_0^\infty \left(e^{-\frac{a^2}{x^2}} - e^{-\frac{b^2}{x^2}}\right) dx$.

3753. Iz relacije $\int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$ (Puasonov integral) izvesti jednakost

$$\frac{1}{\sqrt{x}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2x} dz \ (x > 0$$

i iskoristiti je za izračunavanje integrala (integral difrakcije ili Frenelov integral):

$$\text{a)} \int_0^\infty \frac{\cos x \, dx}{\sqrt{x}}; \quad \text{b)} \int_0^\infty \frac{\sin x \, dx}{\sqrt{x}}.$$

Rješenja

3750. $\frac{\pi}{2} \ln(1+a)$, ako je $a>0$; $-\frac{\pi}{2} \ln(1-a)$, ako je $a<0$; $\int_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} x} dx = \frac{\pi}{2} \ln 2$.

3751*. $\ln \frac{1+\beta}{1+\alpha}$. Integrirati po parametru n u granicama od α do β.

3752. $\sqrt{\pi}\,(b-a)$. **3753.** $\int_0^\infty \frac{\cos x \, dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x \, dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}$.

Vektorska teorija polja

Skalarno polje je f-ja $u = f(T) = f(x,y,z)$ u oblasti prostora ili na površi (na primjer, temperatura u svakoj tački prostora, nadmorska visina tačke i dr.) Skalarno polje se predstavlja nivoskim površima tj. površima s jednačinom $u = c \cdot f(T) = c \cdot f(x,y,z)$ (gdje je c-konstanta) i u ima neprekidne parcijalne izvode koji se ne anuliraju istovremeno).

Na primjer $u = x^2 + y^2 + z^2$ je skalarno polje.

Ranije smo spomenuli da je gradijent f-je $u = f(x,y,z)$, date u nekoj oblasti prostora, vektor čije su projekcije na ose Dekartovog koordinatnog sistema $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ i $\frac{\partial u}{\partial z}$. Označava se simbolom

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

Izvod u pravcu gradijenta u datoj tački dostiže najveću vrijednost jednaku $|\operatorname{grad} u| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}$ tj. pravac gradijenta je pravac najbržeg rasta f-je.

Vektorsko polje je oblast prostora u čijoj je svakoj tački definisan vektor.

$\vec{v} = (V_x, V_y, V_z) = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$ gdje su V_x, V_y, V_z skalarna polja.

Na primjer $\vec{v} = (y^2 + z^2)\vec{i} + x^2 \vec{j} + xyz^2 \vec{k}$ je vektorsko polje.

Nabla operator (∇ operator ili Hamiltonov operator) je diferencijalni operator oblika $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ gdje su $\vec{i}, \vec{j}, \vec{k}$ jedinični ortogonalni vektori.

Ako je $u = f(x,y,z)$ skalarna f-ja biće

$$\nabla \cdot f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \operatorname{grad} f$$

Ako je $\vec{v} = (V_x, V_y, V_z)$ vektorska f-ja onda je $\nabla \vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Važne osobine vektorskog polja su divergencija i rotor vektorskog polja

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \quad \text{(skalarni proizvod } \nabla \text{ i } \vec{v}\text{)}$$

$$\operatorname{rot} \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \quad \text{(vektorski proizvod } \nabla \text{ i } \vec{v}\text{)}$$

Ako je $\operatorname{div} \vec{v} = 0$ tada kažemo da je $\vec{v}$ solenoidno polje.

Ako je $\operatorname{rot} \vec{v} = \vec{0}$ tada kažemo da je $\vec{v}$ potencijalno polje.

F-ju u za koju vrijedi da je $\vec{v} = \operatorname{grad} u$ zovemo potencijalom polja $\vec{v}$.

relacija $u(x,y,z) = C$ gdje je C konstanta, predstavlja površ koju zovemo ekviskalarna površ (nivo površ) skalarnog polja

Naći veličinu i pravac gradijenta skalarnog polja: a) $u = x^2+y^2+z^2$ u tački $T(2,-2,1)$
b) $u = xyz$ u tački $T(1,2,3)$.

Rj. a)

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\operatorname{grad} u = (2x, 2y, 2z) \Rightarrow \operatorname{grad} u(T) = (4,-4,2)$$

$$|\operatorname{grad} u| = \sqrt{16+16+4} = 6 \quad \text{veličina gradijenta}$$

$$\frac{\operatorname{grad} u(T)}{|\operatorname{grad} u(T)|} = \left(\frac{4}{6}, -\frac{4}{6}, \frac{2}{6}\right) = \left(\underbrace{\frac{2}{3}}_{\cos\alpha}, \underbrace{-\frac{2}{3}}_{\cos\beta}, \underbrace{\frac{1}{3}}_{\cos\gamma}\right) \quad \text{jedinični vektor pravca gradijenta}$$

$$\alpha = \arccos\frac{2}{3} \qquad \gamma = \arccos\frac{1}{3}$$
$$\beta = \arccos\left(-\frac{2}{3}\right)$$

b) $|\operatorname{grad} u(T)| = 7 \qquad \dfrac{\operatorname{grad} u(T)}{|\operatorname{grad} u(T)|} = \left(\dfrac{6}{7}, \dfrac{3}{7}, \dfrac{2}{7}\right)$

Dato je skalarno polje $u = x^3+y^3+z^3-3xyz$. U kojim tačkama je a) $\operatorname{grad} u = \vec{0}$
b) $\vec{k}\cdot \operatorname{grad} u = 0$.

Rj.
a) $\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$

$$\operatorname{grad} u = (3x^2-3yz, 3y^2-3xz, 3z^2-3xy)$$

$$\operatorname{grad} u = \vec{0} \Rightarrow \begin{array}{ll} 3x^2-3yz=0 & x^2-yz=0 \quad (I) \\ 3y^2-3xz=0 & y^2-xz=0 \quad (II) \\ 3z^2-3xy=0 & z^2-xy=0 \quad (III) \end{array}$$

Trivijalno rješenje sistema je $x=0, y=0, z=0$.
Ako pomnožimo (I) sa x ($x \neq 0$), (II) sa y ($y \neq 0$) i (III) sa z ($z \neq 0$) dobijamo

$$\begin{array}{lll} x^3-xyz=0 & xyz=x^3 & x^3=y^3=z^3 \\ y^3-xyz=0 & xyz=y^3 & x=y=z \\ z^3-xyz=0 & xyz=z^3 & \end{array}$$

Ako ovu zadnju jednakost napišemo u obliku $\dfrac{x-0}{1} = \dfrac{y-0}{1} = \dfrac{z-0}{1}$ (prava u prostoru) vidimo da je $\operatorname{grad} u = \vec{0}$ za sve tačke ove prave.

b) $\operatorname{grad} u = (3x^2-3yz)\vec{i} + (3y^2-3xz)\vec{j} + (3z^2-3xy)\vec{k}$

$$\vec{k}\cdot\operatorname{grad} u = 3z^2-3xy = 0$$

$\vec{k}\cdot\operatorname{grad} u = 0$ je za sve tačke krive $z^2-xy=0$

Odrediti ugao kojeg zatvaraju gradijenti polja

$z=\sqrt{x^2+y^2}$ i $u=x-3y+\sqrt{3xy}$ u tački $A(3,4)$.

Rj. Gradijent f-je $z=f(x,y)$ se računa po formuli

$$\operatorname{grad} z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right).$$

$$\operatorname{grad} z=\left(\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x,\ \frac{1}{2\sqrt{x^2+y^2}}\cdot 2y\right)=\left(\frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}}\right)$$

$u=x-3y+\sqrt{3xy}$, $\quad \frac{\partial u}{\partial x}=1+\frac{1}{2\sqrt{3xy}}\cdot 3y=1+\frac{3y}{2\sqrt{3xy}}$

$$\frac{\partial u}{\partial y}=-3+\frac{3x}{2\sqrt{3xy}}$$

$$\operatorname{grad} u=\left(1+\frac{3y}{2\sqrt{3xy}},\ -3+\frac{3x}{2\sqrt{3xy}}\right).$$

$A(3,4)$, $\quad \operatorname{grad} z(A)=\left(\frac{3}{\sqrt{9+16}},\ \frac{4}{\sqrt{9+16}}\right)=\left(\frac{3}{5},\frac{4}{5}\right)=\frac{3}{5}\vec{i}+\frac{4}{5}\vec{j}$,

$$\operatorname{grad} u(A)=\left(1+\frac{12}{2\sqrt{36}},\ -3+\frac{9}{2\sqrt{36}}\right)=\left(1+1,\ -3+\frac{\cancel{9}^{3}}{\cancel{12}_{4}}\right)=$$

$$=\left(2,-\frac{9}{4}\right)=2\vec{i}-\frac{9}{4}\vec{j}$$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cdot\cos\varphi \quad\Rightarrow\quad \cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

U našem slučaju $\vec{a}=\left(\frac{3}{5},\frac{4}{5}\right)$, $\vec{b}=\left(2,-\frac{9}{4}\right)$

$$\vec{a}\cdot\vec{b}=\frac{3}{5}\cdot 2+\frac{4}{5}\cdot\left(-\frac{9}{4}\right)=\frac{6\cdot 4}{5\cdot 4}-\frac{36}{20}=\frac{24-36}{20}=\frac{-12}{20}=\frac{-3}{5}$$

$$|\vec{a}|=\sqrt{\frac{9}{25}+\frac{16}{25}}=1,\quad |\vec{b}|=\sqrt{4+\frac{81}{16}}=\sqrt{\frac{64+81}{16}}=\frac{\sqrt{145}}{4}$$

$$\cos\varphi=\frac{-\frac{3}{5}}{1\cdot\frac{\sqrt{145}}{4}}=\frac{-12}{\sqrt{145}}\quad\Rightarrow\quad \varphi=\arccos\left(\frac{-12}{\sqrt{145}}\right)$$ ugao kojeg zatvaraju gradijenti polja

Odrediti divergenciju i rotor vektorskih polja

a) $\vec{v}=(y^2+z^2)\vec{i}+(z^2+x^2)\vec{j}+(x^2+y^2)\vec{k}$

b) $\vec{v}=x^2yz\,\vec{i}+xy^2z\,\vec{j}+xyz^2\,\vec{k}$

Rj. a) $\operatorname{div}\vec{v}=\nabla\cdot\vec{v}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$, $\quad (\vec{v}=(V_x,V_y,V_z))$

$V_x=y^2+z^2$ $\quad \frac{\partial V_x}{\partial x}=0$

$V_y=z^2+x^2$ $\quad \frac{\partial V_z}{\partial z}=0$ $\quad \operatorname{div}\vec{v}=0+0+0=0$ divergencija vektorskog polja

$V_z=x^2+y^2$ $\quad \frac{\partial V_y}{\partial y}=0$

Kako je $\operatorname{div}\vec{v}=0$ to je polje $\vec{v}$ solenoidno

$$\operatorname{rot}\vec{v}=\nabla\times\vec{v}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(V_x,V_y,V_z)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}$$

$\frac{\partial V_z}{\partial y}=2y$ $\quad \frac{\partial V_y}{\partial z}=2z$

$\frac{\partial V_z}{\partial x}=2x$ $\quad \frac{\partial V_x}{\partial z}=2z$ $\quad \operatorname{rot}\vec{v}=(2y-2z)\vec{i}-(2x-2z)\vec{j}$

$\frac{\partial V_y}{\partial x}=2x$ $\quad \frac{\partial V_x}{\partial y}=2y$ $\quad +(2x-2y)\vec{k}=$

$=(2y-2z,\ 2z-2x,\ 2x-2y)$ rotor vektorskog polja

Kako je $\operatorname{rot}\vec{v}\neq 0$ to polje nije potencijalno polje.

b) URADITI ZA VJEŽBU

Rj. $\operatorname{div}\vec{v}=6xyz$

$\operatorname{rot}\vec{v}=(yx^2-yz^2)\vec{j}+(zy^2-zx^2)\vec{k}+(xz^2-xy^2)\vec{i}$.

Izračunati ∇u ako je $u=f(r)$, $\vec{a}=(x,y,z)$ je vektor položaja tačke $M(x,y,z)$ i $r=|\vec{a}|$.

Rj. Da li je u vektorska ili skalarna f-ja?

$$r=|\vec{a}|=\sqrt{x^2+y^2+z^2}$$

$u=f(\sqrt{x^2+y^2+z^2})$ je skalarna f-ja

$$\nabla u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}, \quad u=f(r)$$

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}=f'_r\cdot\left(\sqrt{x^2+y^2+z^2}\right)'_x=f'_r\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}=f'_r\cdot\frac{x}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y}=f'_r\cdot\left(\sqrt{x^2+y^2+z^2}\right)'_y=f'_r\cdot\frac{2y}{2\sqrt{x^2+y^2+z^2}}=f'_r\cdot\frac{y}{r}$$

$$\frac{\partial u}{\partial z}=\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z}=f'_r\cdot\left(\sqrt{x^2+y^2+z^2}\right)'_z=f'_r\cdot\frac{2z}{2\sqrt{x^2+y^2+z^2}}=f'_r\cdot\frac{z}{r}$$

$$\nabla u=\left(f'_r\cdot\frac{x}{\sqrt{x^2+y^2+z^2}},\ f'_r\cdot\frac{y}{\sqrt{x^2+y^2+z^2}},\ f'_r\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=\frac{f'_r}{r}\left(x\vec{i}+y\vec{j}+z\vec{k}\right)=f'_r\cdot\frac{\vec{a}}{r}$$

Iskoristiti prethodni zadatak i izračunati $\nabla\frac{1}{r}$.

Rj. Ako stavimo $f(r)=\frac{1}{r}$ u prethodni zadatak dobijamo:

$$f'_r=\left(\frac{1}{r}\right)'_r=\frac{-1}{r^2}$$

$$\nabla u=\nabla\frac{1}{r}=-\frac{1}{r^2}\cdot\frac{\vec{a}}{r}=\frac{-1}{r^3}\vec{a}$$

Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v}=2x(y^2+z^2)\vec{i}+2y(x^2+z^2)\vec{j}+2z(x^2+y^2)\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $\mathrm{rot}\,\vec{v}=\vec{0}$.
Rotor vektorskog polja $\mathrm{rot}\,\vec{v}$ se računa . .

$$\mathrm{rot}\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$V_x=2x(y^2+z^2)$ $\quad V_y=2y(x^2+z^2)$ $\quad V_z=2z(x^2+y^2)$

$\frac{\partial V_x}{\partial y}=4xy$ $\quad \frac{\partial V_y}{\partial x}=4xy$ $\quad \frac{\partial V_z}{\partial x}=4xz$

$\frac{\partial V_x}{\partial z}=4xz$ $\quad \frac{\partial V_y}{\partial z}=4yz$ $\quad \frac{\partial V_z}{\partial y}=4yz$

$$\mathrm{rot}\,\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}=\vec{i}(4yz-4yz)-\vec{j}(4xz-4xz)+\vec{k}(4xy-4xy)$$

$=(0,0,0)=\vec{0}$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v}=\mathrm{grad}\,u$.

$$\mathrm{grad}\,u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=2x(y^2+z^2)\\ \frac{\partial u}{\partial y}&=2y(x^2+z^2)\\ \frac{\partial u}{\partial z}&=2z(x^2+y^2)\end{aligned}\right\}\ldots(1)$$

$u=u(x,y,z)$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$u=\int 2x(y^2+z^2)\,dx+\varphi(y,z)$$

$$u=x^2(y^2+z^2)+\varphi(y,z)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial y}&=2yx^2+\varphi'_y\\ \frac{\partial u}{\partial z}&=2x^2z+\varphi'_z\end{aligned}\right\}\ldots(2)$$

(1) i (2) $\Rightarrow$ $\varphi'_y=2yz^2$, $\varphi'_z=2zy^2$ ...(*)

Odredimo f-ju φ $\quad \varphi=\int 2yz^2\,dy+\psi(z)$

$\varphi=y^2z^2+\psi(z)$

$\varphi'=2y^2z+\psi'$ ...(**)

(*) i (**) $\Rightarrow$ $\psi'_z=0$ $\Rightarrow$ $\psi(z)=C$

$\Rightarrow$ $\varphi=y^2z^2+C$ $\Rightarrow$ $u=x^2y^2+x^2z^2+y^2z^2+C$

Potencijal vektorskog polja je $u=x^2y^2+x^2z^2+y^2z^2+C$

Odrediti konstante a, b i c tako da vektorsko polje $\vec{v}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$ bude potencijalno i naći njegov potencijal.

Rj. Ako je $rot\,\vec{v}=\vec{0}$ tada je vektorsko polje $\vec{v}$ potencijalno.

$$rot\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}\qquad \begin{aligned}V_x&=x+2y+az\\ V_y&=bx-3y-z\\ V_z&=4x+cy+2z\end{aligned}$$

$$\frac{\partial V_z}{\partial y}=c\quad \frac{\partial V_y}{\partial z}=-1\quad \frac{\partial V_y}{\partial x}=b$$

$$\frac{\partial V_z}{\partial x}=4\quad \frac{\partial V_x}{\partial z}=a\quad \frac{\partial V_x}{\partial y}=2$$

$$rot\,\vec{v}=(c+1)\vec{i}-(4-a)\vec{j}+(b-2)\vec{k}=(c+1,\ a-4,\ b-2)$$

Za vrijednosti $a=4$, $b=2$ i $c=-1$ vektorsko polje $\vec{v}$ je potencijalno polje.

$\vec{v}=(x+2y+4z,\ 2x-3y-z,\ 4x-y+2z)$

Potencijal polja $\vec{v}$ je f-ja u koja zavisi od 3 promjenjive $u=u(x,y,z)$ i za koju vrijedi $\vec{v}=grad\,u$.

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

Nađimo f-ju u.

$$\frac{\partial u}{\partial x}=x+2y+4z\qquad \frac{\partial u}{\partial y}=2x-3y-z\qquad \frac{\partial u}{\partial z}=4x-y+2z\quad \dots(*)$$

$$u=\int(x+2y+4z)\,dx+\varphi(y,z)\ \Rightarrow\ \frac{\partial u}{\partial y}=2x+\varphi'_y$$

$$u=\frac{1}{2}x^2+(2y+4z)x+\varphi(y,z)\qquad \frac{\partial u}{\partial z}=4x+\varphi'_z\ ,\ \overset{(*)}{\Rightarrow}$$

$\overset{(*)}{\Rightarrow}\ \varphi'_y=-3y-z\ ;\ \varphi'_z=-y+2z\ \dots(**)$ Odredimo f-ju φ.

$$\varphi=\int(-3y-z)\,dy+\psi(z)=-\frac{3}{2}y^2-yz+\psi(z)$$

$$\varphi'_z=-y+\psi'_z\ \overset{(**)}{\Rightarrow}\ \psi'_z=2z\ \Rightarrow\ \psi(z)=\int 2z\,dz=z^2+c$$

$$\varphi(y,z)=-\frac{3}{2}y^2-yz+z^2+c\ \Rightarrow\ u=\frac{1}{2}x^2+2xy+4xz-\frac{3}{2}y^2-yz+z^2+c$$

Dato je vektorsko polje $\vec{A}=(e^x z-2xy,\ 1-x^2,\ e^x+z)$. Pokazati da je polje $\vec{A}$ potencijalno i odrediti mu potencijal. Izračunati integral $\int_L \vec{A}\cdot d\vec{r}$ gdje je L duž PQ, $P(0,1,-1)$, $Q(2,3,0)$ orijentisana od tačke P prema tački Q.

Rj. Ako je rotor vektorskog polja $\vec{A}$ jednak $\vec{0}$ ($rot\,\vec{A}=\vec{0}$), tada za $\vec{A}$ kažemo da je potencijalno polje.

$$rot\,\vec{A}=\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^x z-2xy & 1-x^2 & e^x+z\end{vmatrix}$$

$$=(0-0)\vec{i}-(e^x-e^x)\vec{j}+(-2x+2x)\vec{k}=(0,0,0)\ \Rightarrow$$ $\vec{A}$ je potencijalno polje

F-ju $u=u(x,y,z)$ za koju vrijedi da je $\vec{A}=grad\,u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$ zovemo potencijal polja $\vec{A}$. $\vec{A}=(e^x z-2xy,\ 1-x^2,\ e^x+z)$

$$u=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$u=\int(e^x z-2xy)\,dx+\varphi(y,z)$$

$$u=e^x z-x^2y+\varphi(y,z)$$

$$\frac{\partial u}{\partial y}=-x^2+\varphi'_y\qquad \frac{\partial u}{\partial y}=1-x^2\qquad \varphi'_y=1$$

$$\varphi(y,z)=y+\psi(z)\quad \dots(1)$$

$$\frac{\partial u}{\partial z}=e^x+\varphi'_z\qquad \frac{\partial u}{\partial z}=e^x+z$$

$$\varphi'_z=z\ \Rightarrow\ \varphi(y,z)=\frac{z^2}{2}+\psi(y)\ \dots(2)$$

$$(1)\ i\ (2)\ \Rightarrow\ \varphi(y,z)=y+\frac{z^2}{2}$$

Potencijal vektorskog polja $\vec{A}$ je $u=e^x z-x^2y+y+\frac{z^2}{2}+c$

$\int_L \vec{A}\cdot d\vec{r}$ zovemo cirkulacija vektorskog polja $\vec{A}$ duž krive L

$C=\int_L \vec{A}\,d\vec{r}=\int V_x dx+V_y dy+V_z dz$ gdje je $\vec{A}=(V_x,V_y,V_z)$, $d\vec{r}=(dx,dy,dz)$.

$$C=\int_L (e^x z-2xy)\,dx+(1-x^2)\,dy+(e^x+z)\,dz$$

ovo je krivolinijski integral druge vrste po krivoj datoj u prostoru

Prisjetimo se, ako je c kriva u ravni opisana parametarskim jednačinama $x=\eta(t)$, $y=\mu(t)$ gdje je $t_1\le t\le t_2$ tada krivolinijski integral se računa

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\Big(P(\eta(t),\mu(t))\,\eta'(t)+Q(\eta(t),\mu(t))\,\mu'(t)\Big)\,dt$$

Postavimo pravu kroz dvije date tačke $P(0,1,-1)$ i $Q(2,3,0)$.

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke

$P(\overset{x_1}{0},\overset{y_1}{1},\overset{z_1}{-1})$
$Q(\overset{x_2}{2},\overset{y_2}{3},\overset{z_2}{0})$

$$\frac{x-0}{2}=\frac{y-1}{2}=\frac{z+1}{1}\quad(=t)$$

$$L:\begin{cases}x=2t & dx=2\,dt\\ y=2t+1 & dy=2\,dt\\ z=t-1 & dz=dt\\ 0\le t\le 1\end{cases}$$

$$C=\int_0^1\Big(2\cdot\big(e^{2t}(t-1)-2\cdot 2t\cdot(2t+1)\big)+(1-4t^2)\cdot 2+\big(e^{2t}+(t-1)\big)\Big)\,dt$$

$$=\int_0^1\big(2e^{2t}t-2e^{2t}-16t^2-8t+2-8t^2+e^{2t}+t-1\big)\,dt$$

$$=\int_0^1 2e^{2t}t\,dt-\int_0^1 e^{2t}dt-24\int_0^1 t^2dt+\int_0^1(-7t+1)\,dt=\ldots=-\frac{19}{2}$$

$$2\int_0^1 e^{2t}t\,dt=\left|\begin{matrix}u=t & dv=e^{2t}dt\\ du=dt & v=\frac{1}{2}e^{2t}\end{matrix}\right|=2\cdot\frac{1}{2}te^{2t}\Big|_0^1-2\cdot\frac{1}{2}\int_0^1 e^{2t}dt=$$

$$=e^2-\frac{1}{2}e^{2t}\Big|_0^1=e^2-\frac{1}{2}e^2+\frac{1}{2}e^0=\frac{1}{2}e^2+\frac{1}{2}$$

Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v}=2x(y^2+z^2)\,\vec{i}+2y(x^2+z^2)\,\vec{j}+2z(x^2+y^2)\,\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $rot\,\vec{v}=\vec{0}$.
Rotor vektorskog polja $rot\,\vec{v}$ se računa

$$rot\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$V_x=2x(y^2+z^2)$, $\quad \frac{\partial V_x}{\partial y}=4xy$, $\quad \frac{\partial V_x}{\partial z}=4xz$

$V_y=2y(x^2+z^2)$, $\quad \frac{\partial V_y}{\partial x}=4xy$, $\quad \frac{\partial V_y}{\partial z}=4yz$

$V_z=2z(x^2+y^2)$, $\quad \frac{\partial V_z}{\partial x}=4xz$, $\quad \frac{\partial V_z}{\partial y}=4yz$

$$rot\,\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}=\vec{i}(4yz-4yz)-\vec{j}(4xz-4xz)+\vec{k}(4xy-4xy)$$
$$=(0,0,0)=\vec{0}$$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v}=grad\,u$.

$$grad\,u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=2x(y^2+z^2)\\ \frac{\partial u}{\partial y}&=2y(x^2+z^2)\\ \frac{\partial u}{\partial z}&=2z(x^2+y^2)\end{aligned}\right\}\ldots(1)$$

$u=u(x,y,z)$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$u=\int 2x(y^2+z^2)\,dx+\varphi(y,z)$$

$$u=x^2(y^2+z^2)+\varphi(y,z)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial y}&=2yx^2+\varphi'_y\\ \frac{\partial u}{\partial z}&=2x^2z+\varphi'_z\end{aligned}\right\}\ldots(2)$$

(1) i (2) $\Rightarrow$ $\varphi'_y=2yz^2$, $\varphi'_z=2zy^2$ ... (*)

Odredimo f-ju φ $\quad\varphi=\int 2yz^2dy+\psi(z)$

$\varphi=y^2z^2+\psi(z)$

$\varphi'=2y^2z+\psi'$ ... (**)

(*) i (**) $\Rightarrow$ $\psi'_z=0$ $\Rightarrow$ $\psi(z)=c$

$\Rightarrow$ $\varphi=y^2z^2+c$ $\Rightarrow$ $u=x^2y^2+x^2z^2+y^2z^2+c$

Potencijal vektorskog polja je $u=x^2y^2+x^2z^2+y^2z^2+C$

Zadaci za vježbu

Vektorsko polje, divergencija i rotor

4401. Naći vektorske linije homogenog polja $A(P) = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ (a, b i c su konstante).

4402. Naći vektorske linije ravnog polja $A(P) = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}$, ($\omega$ je konstanta).

4403. Naći vektorske linije polja $A(P) = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j} + h\boldsymbol{k}$ (ω i h su konstante).

4404. Naći vektorske linije polja:

1) $A(P) = (y+z)\boldsymbol{i} - x\boldsymbol{j} - x\boldsymbol{k}$;

2) $A(P) = (z-y)\boldsymbol{i} + (x-z)\boldsymbol{j} + (y-x)\boldsymbol{k}$;

3) $A(P) = x(y^2-z^2)\boldsymbol{i} - y(z^2+x^2)\boldsymbol{j} + z(x^2+y^2)\boldsymbol{k}$.

U zadacima 4405 — 4408 izračunati divergenciju i rotor datih vektorskih polja.

4405. $A(P) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

4406. $A(P) = (y^2+z^2)\boldsymbol{i} + (z^2+x^2)\boldsymbol{j} + (x^2+y^2)\boldsymbol{k}$.

4407. $A(P) = x^2yz\boldsymbol{i} + xy^2z\boldsymbol{j} + xyz^2\boldsymbol{k}$.

4408. $A(P) = \operatorname{grad}(x^2+y^2+z^2)$.

4409. Sila $F\boldsymbol{i}$ konstantnog intenziteta F obrazuje vektorsko polje; izračunati divergenciju i rotor toga polja.

Rješenja

4401. Prave paralelne vektoru $A\{a, b, c,\}$: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.

4402. Krugovi sa centrom u kodrdinatnom početku.

4403. Zavojnice sa visinom hoda $\frac{2\pi h}{\omega}$, koje leže na cilindrima čije se ose poklapaju sa z-osom: $x = R\cos(\omega t+\alpha)$, $y = R\sin(\omega t+\alpha)$, $z = ht + z_0$, pri čemu su R, α i z_0 proizvoljne konstante.

4404. 1) Krugovi $x^2+y^2+z^2=R^2$, $y-z+C=0$, po kojima ravni paralelne simetralnoj ravni $y-z=0$ presecaju sfere sa zajedničkim centrom u koordinatnom početku (R i C su proizvoljne konstante.

2) Krugovi $x^2+y^2+z^2=R^2$, $x+y+z=C$ po kojima ravni, koje od koordinatnih osa odsecaju odsečke iste dužine i znaka, presecaju sfere sa zajedničkim controm u koordinatnom početku.

3) Krive po kojima se presecaju sfere $x^2+y^2+z^2=R^2$ i hiperbolični paraboloidi $zy = Cx$.

4405. div $A = 3$, rot $A = 0$.

4406. div $A = 0$, rot $A = 2[(y-z)\boldsymbol{i} + (z-x)\boldsymbol{j} + (x-y)\boldsymbol{k}]$.

4407. div $A = 6xyz$, rot $A = x(z^2-y^2)\boldsymbol{i} + y(x^2-z^2)\boldsymbol{j} + z(y^2-x^2)\boldsymbol{k}$.

4408. div $A = 6$, rot $A = 0$.

4409. div $A = 0$, rot $A = 0$.

4410. Ravno vektorsko polje definisano je silom obrnuto proporcionalnom kvadratu odstojanja njene napadne tačke od koordinatnog početka i usmerenom prema koordinatnom početku (npr. ravno električno polje obrazovano naelektrisanom materijalnom tačkom); naći divergenciju i rotor polja.

4411. Naći divergenciju i rotor prostranog polja ako je sila polja podčinjena istim uslovima kao i u zadatku 4410.

4412. Vektorsko polje je definisano silom obrnuto proporcionalnom odstupanju njene napadne tačke od z-ose, normalnom na tu osu i usmerenom prema njoj; izračunati divergenciju i rotor toga polja.

4413. Vektorsko polje je definisano silom obrnuto proporcionalnom odstojanju njene napadne tačke od ravni xOy i usmerenom prema koordinatnom početku; izračunati divergenciju tog polja.

4414. Izračunati div $(a\boldsymbol{r})$ ako je a konstantan skalar.

4415. Dokazati relaciju

$$\operatorname{div}(\varphi A) = \varphi \operatorname{div} A + (A \operatorname{grad} \varphi),$$

u kojoj je $\varphi = \varphi(x, y, z)$ skalarna funkcija.

4416. Izračunati div $\boldsymbol{b}(\boldsymbol{r}\cdot\boldsymbol{a})$ i div $\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{a})$ ako su $\boldsymbol{a}$ i $\boldsymbol{b}$ konstantni vektori.

4417. Izračunati div $(\boldsymbol{a}\times\boldsymbol{r})$ ako je $\boldsymbol{r}$ konstantan vektor.

4418. Ne prelazeći na koordinate izračunati divergenciju vektorskog polja:

1) $A(P) = \boldsymbol{r}(\boldsymbol{a}\boldsymbol{r}) - 2\boldsymbol{a}r^2$, 2) $A(P) = \frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}$,

3) $\operatorname{grad}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}$

Rješenja

4410. div $A = \frac{k}{r^2}$, gde je k koeficijent proporcionalnosti, a r—odstojanje napadne tačke sile od koordinatnog početka; rot $A = 0$.

4411. div $A = 0$, rot $A = 0$.

4412. div $A = 0$, rot $A = 0$. U tačkama z-ose polje nije definisano.

4413. div $A = -\frac{k}{z\sqrt{x^2+y^2+z^2}}$, gde je k koeficijent proporcionalnosti. U tačkama ravni Oxy polje nije definisano.

4414. $3a$. **4416.** div $\boldsymbol{b}(\boldsymbol{r}\boldsymbol{a}) = (\boldsymbol{a}\boldsymbol{b})$, div $\boldsymbol{r}(\boldsymbol{r}\boldsymbol{a}) = 4(\boldsymbol{r}\boldsymbol{a})$.

4417. 0. **4418.** 1) 0. 2) 0. 3) 0.

4419. Izračunati divergenciju vektorskog polja

$$A(P)=f(|r|)\frac{r}{|r|}.$$

Dokazati da je divergencija ovog polja jednaka nuli samo onda kad je $f(|r|)=\frac{C}{r^2}$ ako je polje prostorno, i $f(|r|)=\frac{C}{|r|}$ ako je polje ravno, pri čemu je C proizvoljna skalarna konstanta.

4420. Dokazati da je

$$\operatorname{rot}[A_1(P)+A_2(P)]=\operatorname{rot}A_1(P)+\operatorname{rot}A_2(P).$$

4421. Izračunati $\operatorname{rot}[\varphi A(P)]$, ako je $\varphi=\varphi(x, y, z)$ skalarna funkcija.

4422. Izračunati $\operatorname{rot} r a$ ako je r intenzitet vektora položaja tačke, a a je konstantan vektor.

4423. Izračunati $\operatorname{rot}(a \times r)$ ako je a konstantan vektor.

4424. Kruto telo obrće se konstantnom ugaonom brzinom ω oko ose: naći divergenciju i rotor polja linearnih brzina.

4425. Dokazati relaciju

$$n(\operatorname{grad}(A n)-\operatorname{rot}(A \times n))=\operatorname{div} A,$$

ako je n jedinični konstantan vektor.

Diferencijalne operacije vektorske analize (grad, div, rot) zgodno je obeležavati pomoću simboličnog vektora ∇ (Hamiltonov „nabla“ operator):

$$\nabla=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k.$$

Primenu ovog operatora na ovu ili onu (skalarnu ili vektorsku veličinu) treba shvatiti ovako: po pravilima vektorske algebre treba pomnožiti vektor ∇ datom veličinom, a zatim množenje simbola $\frac{\partial}{\partial x}$ i tsl. veličinom S shvatiti kao izračunavanje odgovarajućeg izvoda. Tada je $\operatorname{grad} u=\nabla u$; $\operatorname{div} A=\nabla A$; $\operatorname{rot} A=\nabla \times A$.

Pomoću Hamiltonova operatora mogu se predstaviti i diferencijalne operacije drugog reda: $\operatorname{div}\operatorname{grad} u=\nabla\nabla u$; $\operatorname{rot}\operatorname{grad} u=\nabla\times\nabla u$; $\operatorname{grad}\operatorname{div} A=\nabla(\nabla A)$; $\operatorname{div}\operatorname{rot} A=\nabla(\nabla\times A)$; $\operatorname{rot}\operatorname{rot} A=\nabla\times(\nabla\times A)$.

4426. Dokazati da je $r\cdot\nabla r^n=n r^n$, pri čemu je r vektor položaja tačke.

4427. Dokazati relacije:

1) $\operatorname{rot}\operatorname{grad} u=0$; 2) $\operatorname{div}\operatorname{rot} A=0$.

4428. Dokazati da je

$$\operatorname{div}\operatorname{grad} u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}.$$

(Ovaj se izraz naziva Laplasovim operatorom i obično se obeležava sa Δu. Pomoću Hamiltonova operatora ova se veličina može pisati u obliku $\Delta u=(\nabla\nabla)u=\nabla^2 u$).

4429. Dokazati da je

$$\operatorname{rot}\operatorname{rot} A(P)=\operatorname{grad}\operatorname{div} A(P)-\Delta A(P),$$

pri čemu je

$$\Delta A(P)=\Delta A_x i+\Delta A_y j+\Delta A_z k.$$

4430. Vektorsko polje definisano je konstantnim vektorom A; uveriti se da to polje ima potencijal i naći taj potencijal.

4431. Vektorsko polje definisano je silom proporcionalnom odstojanju napadne tačke od koordinatnog početka i usmerenom prema koordinatnom početku; pokazati da je to polje konzervativno i naći njegov potencijal.

4432. Sile polja su obrnuto proporcionalne odstojanju njihovih napadnih tačaka od ravni Oxy i usmerene su prema koordinatnom početku; hoće li polje biti konzervativno?

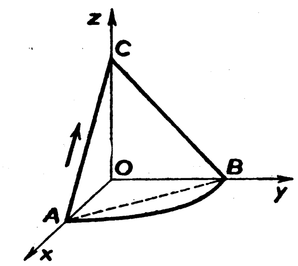
4433. Sile polja su obrnuto proporcionalne kvadratu odstojanja njihovih napadnih tačaka od z-ose i usmerene prema koordinatnom početku; hoće li polje biti konzervativno?

4434. Vektorsko polje definisano je silom obrnuto proporcionalnom odstojanju njene napadne tačke od z-ose, normalnom na tu osu i usmerenom ka njoj; pokazati da je to polje konzervativno i naći njegov potencijal.

4435. Linearne brzine tačaka krutog tela koje se obrće oko neke ose obrazuju vektorsko polje; je li to polje potencijalno?

4436. Sile polja definisane su ovako: $A(P)=f(r)\frac{r}{r}$ (tzv. centralno polje; $r=\sqrt{x^2+y^2+z^2}$); pokazati da je potencijal polja: $u(x, y, z)=\int_a^r f(r)\,dr$ i odavde kao specijalan slučaj izvesti potencijal polja sila privlačenja koje potiču od tačkaste mase, i potencijal polja u zadatku 4431.

4437. Naći rad sila polja $A(p)=xy\,i+yz\,j+xz\,k$ pri pomeranju tačke po zatvorenoj krivoj koja se sastoji iz odsečka prave $x+z=1$, $y=0$, četvrtine kružne linije $x^2+y^2=1$, $z=0$, i odsečka prave $y+z=1$, $x=0$ (sl. 78), — u smeru naznačenom na slici. Koliki će biti taj rad ako se luk BA zameni izlomljenom linijom BOA ili pravolinijskim odsečkom BA?



Sl. 78

Rješenja

4419. $\operatorname{div} A=\frac{2f(r)}{r}+f'(r)$, ako je polje prostorno, i $\operatorname{div} A=\frac{f(r)}{r}+f'(r)$ ako je polje ravno.

4421. $\varphi\operatorname{rot} A+(\operatorname{grad}\varphi\times A)$. **4422.** $\frac{r\times A}{r}$.

4423. $2a$. **4424.** ωn_0, gde je n_0 jedinični vektor paralelan osi obrtanja.

4430. $u=A r+C$. **4431.** $u=-\frac{1}{2}k(x^2+y^2+z^2)+C$. **4432.** Neće. **4433.** Neće.

4434. $u=-\frac{1}{2}\ln x^2+y^2)+C$. **4435.** Nema.

4437. $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{2}$. **4438.** $k\delta\ln\frac{\sqrt{(l-x)^2+y^2}+l-x}{\sqrt{(l+x)^2+y^2}-l-x}$.

Cirkulacija i fluks vektorskog polja

Neka je $\vec{v}=(v_x, v_y, v_z)$ dato vektorsko polje.
Cirkulacija vektorskog polja $\vec{v}$ duž krive c je integral

$$C=\int_c \vec{v}\, d\vec{r} = \int_c v_x\,dx+v_y\,dy+v_z\,dz \qquad \text{gdje je } \vec{r}=(x,y,z)$$
$$d\vec{r}=(dx, dy, dz)$$

Ako je c zatvorena kontura možemo koristiti formulu Stoksa u vektorskom obliku

$$C=\oint_c \vec{v}\,d\vec{r} = \iint_S \vec{n}\cdot \operatorname{rot}\vec{v}\,dS=\iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} dS$$

Fluks (tok, proticanje) vektorskog polja (kroz površ S) je površinski integral

$$\Phi=\iint_S \vec{v}\cdot\vec{n}\,dS=\iint_S (v_x\cos\alpha+v_y\cos\beta+v_z\cos\gamma)\,dS$$

$$=\iint_S v_x\,dy\,dz+v_y\,dx\,dz+v_z\,dx\,dy$$

Ako je S zatvorena površ, fluks polja se može računati pomoću formule Gauss-Ostrogradski

$$\Phi=\oiint_S \vec{v}\cdot\vec{n}\,dS=\iiint_\Omega \operatorname{div}\vec{v}\,dx\,dy\,dz=\iiint_\Omega\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)dx\,dy\,dz$$

gdje je Ω oblast u prostoru koja je ograničena površinom S.

Izračunati cirkulaciju polja $\vec{v}=x\vec{i}+y\vec{j}+(x+y-1)\vec{k}$ duž odsječka prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

Rj. Cirkulacija vektorskog polja $\vec{v}=(v_x,v_y,v_z)$ duž krive c je integral

$$C=\int_c v_x\,dx+v_y\,dy+v_z\,dz.$$

U našem slučaju $\vec{v}=(x, y, x+y-1)$, dok je c dio prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.
Imamo krivolinijski integral druge vrste

$$C=\int_c x\,dx+y\,dy+(x+y-1)\,dz$$

$A(1,1,1)$
$B(2,3,4)$

Kako glasi jednačina prave kroz dvije tačke u prostoru?

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3} \qquad (=t)$$

Napišimo pravu u parametarskom obliku:

$x=t+1$
$y=2t+1$
$z=3t+1$

Dio prave između tačke $A(1,1,1)$ i $B(2,3,4)$ je ~~za~~ $t\in[0,1]$.

$dx=dt,\quad dy=2\,dt,\quad dz=3\,dt$

$$C=\int_0^1 (t+1)\,dt+(2t+1)\,2\,dt+(3t+1)\,3\,dt=\int_0^1(t+1+4t+2+9t+3)\,dt$$

$$=\int_0^1(14t+6)\,dt=14\cdot\frac{1}{2}t^2\Big|_0^1+6t\Big|_0^1=7+6=13$$ vrijednost cirkulacije polja

Izračunati tok (fluks) vektora $\vec{v} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ kroz sferu $x^2+y^2+z^2=R^2$.

Rj. $\vec{v} = (v_x, v_y, v_z) = (x^3, y^3, z^3)$

Tok vektorskog polja (kroz površ S) je površinski integral

$$\phi = \iint_S v_x\,dy\,dz + v_y\,dx\,dz + v_z\,dx\,dy$$

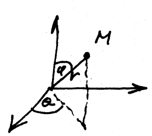
Kako je data zatvorena površina S to možemo upotrebiti formulu Gauss-Ostrogradski

$$\iint_S v_x\,dy\,dz + v_y\,dx\,dz + v_z\,dx\,dy = \iiint_\Omega \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\,dy\,dz$$

$\frac{\partial v_x}{\partial x} = 3x^2$, $\frac{\partial v_y}{\partial y} = 3y^2$, $\frac{\partial v_z}{\partial z} = 3z^2$, Ω oblast ograničena sferom $x^2+y^2+z^2=R^2$

$$\phi = \iiint_\Omega 3(x^2+y^2+z^2)\,dx\,dy\,dz \overset{(\Delta)}{=}$$

uvedimo sferne koordinate



$$\Omega' : \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi$

$x^2+y^2+z^2 = r^2\left[\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi\right] = r^2$

$$\overset{(\Delta)}{=} 3\iiint_{\Omega'} r^2\, r^2 \sin\varphi\,dr\,d\varphi\,d\theta =$$

$$= 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,d\varphi \int_0^R r^4\,dr = 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, \frac{1}{5} r^5 \Big|_0^R d\varphi = 3\,\frac{R^5}{5} \int_0^{2\pi} (-\cos\varphi)\Big|_0^{\pi} d\theta$$

$$= \frac{6R^5}{5}\pi\, \theta\Big|_0^{2\pi} = \frac{12R^5}{5}\pi$$ traženi tok vektora kroz sferu

Izračunati cirkulaciju vektorskog polja $\vec{v} = -y\vec{i} + x\vec{j} + a\vec{k}$ (a = konstanta) duž kruga $(x-2)^2+y^2=1$, $z=0$.

Rj. $\vec{v} = -y\vec{i} + x\vec{j} + a\vec{k}$

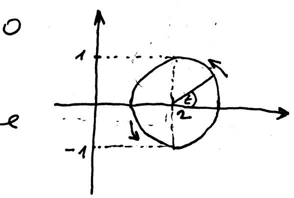
$c: (x-2)^2 + y^2 = 1,\ z=0$

$$C = \int_c \vec{v}\,d\vec{r} = \int_c v_x\,dx + v_y\,dy + v_z\,dz$$

cirkulacija polja $\vec{v}$

Imamo krivolinijski integral

$$C = \int_c -y\,dx + x\,dy + a\,dz \quad \text{gdje je } c: \begin{cases} (x-2)^2+y^2=1 \\ z=0 \end{cases}$$



Parametrizirajmo kružnicu tj. uvedimo smjene

$$\begin{cases} x-2 = \cos t \\ y = \sin t \\ z = 0 \end{cases} \quad 0 \le t \le 2\pi \qquad \begin{matrix} dx = -\sin t\,dt \\ dy = \cos t\,dt \\ dz = 0 \end{matrix} \qquad x = 2+\cos t$$

$$C = \int_0^{2\pi} (-\sin t)(-\sin t)\,dt + (2+\cos t)\cos t\,dt + 0 =$$

$$\int_0^{2\pi} (\sin^2 t + 2\cos t + \cos^2 t)\,dt = \int_0^{2\pi} (1 + 2\cos t)\,dt = (t + 2\sin t)\Big|_0^{2\pi} = 2\pi$$

II način: pomoću Stoksove formule

$$C = \int_c \vec{v}\,d\vec{r} = \iint_S \vec{n}\cdot \mathrm{rot}\,\vec{v}\,dS = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} dS$$

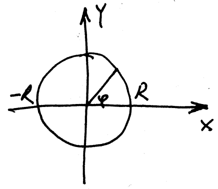
$$\mathrm{rot}\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & a \end{vmatrix} = 0\vec{i} + 0\vec{j} + 2\vec{k} = (0,0,2)$$

$$C = \iint_S \vec{n}\cdot \mathrm{rot}\,\vec{v}\,dS = \iint_S 2\cos\gamma\,dS = 2\underbrace{\iint_S dx\,dy}_{\text{površina kruga}} = 2\cdot 1^2\cdot\pi = 2\pi$$

Iz formule Stoksa znamo da je $\cos\gamma\,dS = dx\,dy$

Izračunati cirkulaciju vektorskog polja $\vec{v} = x^2y^3\vec{i} + \vec{j} + z\vec{k}$ duž kružnice c koja je data kao presjek kružnice $x^2+y^2=R^2$ i xOy ravni.

Rj. $c: \begin{cases} x^2+y^2=R^2 \\ z=0 \end{cases}$



$C = \oint_c \vec{v}\, d\vec{r} = \oint_c v_x\,dx + v_y\,dy + v_z\,dz$

cirkulacija polja $\vec{v}$

I način

Parametrizirajmo kružnicu $\begin{cases} x=R\cos t \\ y=R\sin t \\ z=0 \end{cases}$ ... ZAVRŠITI ZA VJEŽBU

II način Pomoću formule Stoksa:

$$C = \oint_c \vec{v}\,d\vec{r} = \iint_S \vec{n}\cdot \operatorname{rot}\vec{v}\,dS \qquad \operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = (0,0,-3x^2y^2)$$

$$C = \iint_S (\cos\alpha, \cos\beta, \cos\gamma)\cdot(0,0,-3x^2y^2)\,dS = \iint_S -3x^2y^2\cos\gamma\,dS =$$

$$= -3\iint_S x^2y^2\,dx\,dy \quad \text{gdje je sad } S: \{x^2+y^2=R^2$$

Uvodimo polarne koordinate $\begin{matrix} x=r\cos\varphi \\ y=r\sin\varphi \end{matrix} \Rightarrow S': \begin{cases} 0\le r\le R \\ 0\le\varphi\le 2\pi \end{cases}$ $dx\,dy = r\,dr\,d\varphi$

$$C = -3\iint_{S'} r^2\cos^2\varphi\, r^2\sin^2\varphi\cdot r\,dr\,d\varphi = -3\int_0^R r^5\left[\int_0^{2\pi}\frac{1}{4}\cdot\underbrace{4\cos^2\varphi\sin^2\varphi}_{(\sin 2\varphi)^2}\,d\varphi\right]dr$$

$$= -3\int_0^R r^5\left[\int_0^{2\pi}\frac{1}{4}\sin^2 2\varphi\,d\varphi\right]dr = -\frac{3}{4}\int_0^R r^5\left[\int_0^{2\pi}\frac{1-\cos 4\varphi}{2}\,d\varphi\right]dr$$

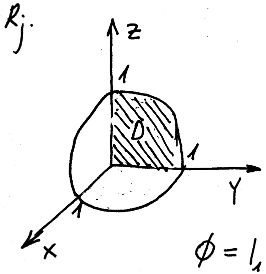
$$= -\frac{\cancel{3}^{1}}{4}\left[\frac{1}{2}\varphi\Big|_0^{2\pi} - \frac{1}{4}\sin 4\varphi\Big|_0^{2\pi}\right]\cdot\frac{1}{\cancel{6}_{2}}r^6\Big|_0^R = -\frac{1}{8}R^6\cdot\pi$$

$1 = \cos^2 2\varphi + \sin^2 2\varphi$

$\cos 4\varphi = \cos^2 2\varphi - \sin^2 2\varphi$

Naći fluks polja $\vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ kroz dio sfere $x^2+y^2+z^2=1$ u I oktantu.

Rj.



I način

$$\Phi = \iint_S \vec{v}\cdot\vec{n}\,dS = \iint_S (v_x\cos\alpha + v_y\cos\beta + v_z\cos\gamma)\,dS$$

$$= \iint_S v_x\,dy\,dz + v_y\,dx\,dz + v_z\,dx\,dy$$

$$\Phi = I_1+I_2+I_3 = \iint_S xy\,dy\,dz + \iint_S yz\,dx\,dz + \iint_S zx\,dx\,dy$$

Zbog simetrije $I_1=I_2=I_3$ pa je $\Phi = 3I_1$. Računamo samo I_1

$$I_1 = \iint_S xy\,dy\,dz = +\iint_D \sqrt{1-(y^2+z^2)}\,y\,dy\,dz \quad \text{gdje je } D: y^2+z^2\le 1,\ x\ge 0,\ y\ge 0$$

$x^2 = 1-(y^2+z^2)$

$x = \pm\sqrt{1-(y^2+z^2)}$

Vektor normale zaklapa ugao $\alpha\in(0,\frac{\pi}{2})$ sa x-osom. $\cos\alpha > 0$ (u I oktantu).

uzimamo + jer smo u prvom oktantu

Uvodimo polarne koordinate $\begin{matrix} y=r\cos\varphi \\ z=r\sin\varphi \end{matrix}$

$D': \begin{cases} 0\le\varphi\le\frac{\pi}{2} \\ 0\le r\le 1 \\ dy\,dz = r\,d\varphi\,dr \end{cases}$ $y^2+z^2=r^2$

$$I_1 = \iint_{D'} r\cos\varphi\sqrt{1-r^2}\cdot r\,d\varphi\,dr = \int_0^1 r^2\sqrt{1-r^2}\left[\int_0^{\frac{\pi}{2}}\cos\varphi\,d\varphi\right]dr = \int_0^1 r^2\sqrt{1-r^2}\cdot 1\,dr$$

$$= \left|\begin{matrix} r=\sin t \\ dr=\cos t\,dt \end{matrix}\right| = \int_0^1 \sin^2 t\sqrt{1-\sin^2 t}\,\cos t\,dt = \int_0 \sin^2 t\cos^2 t\,dt = \ldots = \frac{3\pi}{16}$$

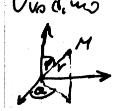
u prethodnom zadatku smo imali slično

II način: Kako je S zatvorena površ možemo primjeniti formulu Gauss-Ostrogradski.

$$\Phi = \iint_S \vec{v}\cdot\vec{n}\,dS = \iiint_\Omega \operatorname{div}\vec{v}\,dx\,dy\,dz = \iiint_\Omega\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dx\,dy\,dz$$

U našem slučaju $\Phi = \iiint_\Omega (x+y+z)\,dx\,dy\,dz$ gdje je $\Omega: \begin{cases} x^2+y^2+z^2\le 1 \\ x\ge 0,\ y\ge 0 \\ z\ge 0 \end{cases}$

Uvodimo sferne koordinate



$\begin{matrix} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{matrix} \Rightarrow \Omega': \begin{cases} 0\le r\le 1 \\ 0\le\varphi\le\pi/2 \\ 0\le\theta\le\pi/2 \end{cases}$ $dx\,dy\,dz = r^2\sin\varphi\,d\varphi\,dr\,d\theta$

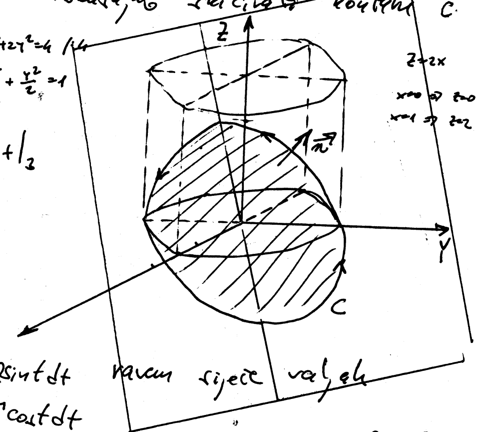
$$\Phi = \iiint_{\Omega'} (r\sin\varphi\cos\theta + \ldots)\,r^2\sin\varphi \ldots = \ldots = \frac{3\pi}{16}$$

Izračunati cirkulaciju vektorskog polja $\vec{v}=(1, xy^2, yz^2)$ duž konture $x^2+2y^2=4$, $z=2x$.

R: Cirkulacija vektorskog polja $\vec{v}$ $(\vec{v}=(v_x, v_y, v_z))$ duž krive c je integral

$$C=\int_c \vec{v}\, d\vec{r} = \int_c v_x\, dx + v_y\, dy + v_z\, dz$$

Pokušajmo skicirati konturu c.



U našem slučaju

$$C=\int_c dx + xy^2\, dy + yz^2\, dz = I_1+I_2+I_3$$

parametriziramo konturu c

kako je $\left(\frac{x}{2}\right)^2+\left(\frac{y}{\sqrt{2}}\right)^2=1$ uvedimo smjene

$$\left.\begin{aligned}\frac{x}{2}&=\cos t\\ \frac{y}{\sqrt{2}}&=\sin t\\ z&=z\end{aligned}\right\} \Rightarrow \begin{aligned}x&=2\cos t\\ y&=\sqrt{2}\sin t\\ z&=4\cos t\\ 0&\le t\le 2\pi\end{aligned} \qquad \begin{aligned}dx&=-2\sin t\, dt\\ dy&=\sqrt{2}\cos t\, dt\\ dz&=-4\sin t\, dt\end{aligned}$$

$\sin^2x+\cos^2x=1$
$\cos^2x-\sin^2x=\cos 2x$
$2\sin^2x=1-\cos 2x$
$2\cos^2x=1+\cos 2x$

$$C=\int_0^{2\pi}\left(-2\sin t + 2\cos t\cdot 2\sin^2 t\cdot \sqrt{2}\cos t + \sqrt{2}\sin t\cdot 16\cdot \cos^2 t(-4\sin t)\right)dt$$

pojednostavimo računanje ovog integrala

$$I_1=\int_c dx = \int_0^{2\pi} -2\sin t\, dt = 2\cos t\Big|_0^{2\pi} = 2(1-1)=0$$

$4\cos^2t\sin^2t=(2\cos t\sin t)^2$

$$I_2=\int_c xy^2\, dy = \int_0^{2\pi} 2\cos t\cdot 2\sin^2 t\cdot\sqrt{2}\cos t\, dt = 4\sqrt{2}\int_0^{2\pi}\cos^2 t\sin^2 t\, dt = \sqrt{2}\int_0^{2\pi}\sin^2 2t\, dt$$

$$=\frac{\sqrt{2}}{2}\int_0^{2\pi}(1-\cos 2t)\,dt = \frac{\sqrt{2}}{2}\left(t\Big|_0^{2\pi} - \frac{1}{2}\sin t\Big|_0^{2\pi}\right) = \frac{\sqrt{2}}{2}(2\pi-0) = \pi\sqrt{2}$$

$$I_3=\int_c yz^2\, dz = \int_0^{2\pi}\sqrt{2}\sin t\cdot 16\cos^2 t\cdot(-4)\sin t\, dt = \overset{(-16)\cdot 4}{-64\sqrt{2}}\int_0^{2\pi}\sin^2 t\cos^2 t = -16\, I_2 = -16\pi\sqrt{2}$$

$$C=\pi\sqrt{2}-16\pi\sqrt{2}=-15\pi\sqrt{2}$$

II način

pomoću Stokesove formule

površinski integral ↓

$$C=\int_c \vec{v}\, d\vec{r} = \iint_S \vec{n}\,\mathrm{rot}\,\vec{v}\, dS = \iint_S \begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix} dS$$

gdje je S površina koju zatvara kontura c, $\vec{n}=(\cos\alpha, \cos\beta, \cos\gamma)$ jedinični vektor normale na S

$$\mathrm{rot}\,\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 1 & xy^2 & yz^2\end{vmatrix} = (z^2-0)\vec{i}-(0-0)\vec{j}+(y^2-0)\vec{k} = (z^2, 0, y^2)$$

$$C=\iint_S (z^2\cos\alpha + y^2\cos\gamma)\, dS$$

posmatramo ravan $z=2x$ i vektor normale na ovu ravan, zato što je naša elipsa unutar ove ravni

$2x-z=0 \qquad \vec{n}=(2,0,-1)$

$|\vec{n}|=\sqrt{4+1}=\sqrt{5} \qquad \vec{n}_0=\left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right)$

projekcija površi S na xOy ravan je elipsa $x^2+2y^2=4$

prema tome

$$C=\iint_S (z^2\cos\alpha + y^2\cos\gamma)\, dS = \iint_{D'} z^2\, dy\, dz - \iint_{D''} y^2\, dx\, dy$$

$x^2+2y^2=4$
$z=2x$
$\left(\frac{z}{2}\right)^2+2y^2=4$ /·4
$z^2+8y^2=16$ /:16

Projekcija površi S na yOz ravan je elipsa $D': \left\{\frac{z^2}{16}+\frac{y^2}{2}=1\right.$ $\quad \frac{z^2}{16}+\frac{y^2}{2}=1$

$$\iint_{D'} z^2\, dy\, dz = \left|\begin{matrix}\frac{z}{4}=r\cos\varphi & z=4r\cos\varphi & dy\,dz=4\sqrt{2}\,r\,dr\,d\varphi\\ \frac{y}{\sqrt{2}}=r\sin\varphi & y=\sqrt{2}\,r\sin\varphi & \\ & 0\le\varphi\le 2\pi & \\ & 0\le r\le 1 & \end{matrix}\right| = \iint_{D_1'} 16r^2\cos^2\varphi\, 4\sqrt{2}\, r\, dr\, d\varphi =$$

$$=64\sqrt{2}\int_0^1 r^3\, dr\int_0^{2\pi}\cos^2\varphi\, d\varphi = 64\sqrt{2}\int_0^1 r^3\, dr\int_0^{2\pi}\frac{1}{2}(1+\cos 2\varphi)\, d\varphi = 32\sqrt{2}\cdot\frac{r^4}{4}\Big|_0^1\left(2\pi-\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}\right) = 16\sqrt{2}\pi$$

Projekcija površi S na xOy ravan je elipsa $D'': \left\{\frac{x^2}{4}+\frac{y^2}{2}=1\right.$

$$\iint_{D''} y^2\, dx\, dy = \left|\begin{matrix}\frac{x}{2}=r\cos\varphi & x=2r\cos\varphi & dx\,dy=2\sqrt{2}\,r\,dr\,d\varphi\\ \frac{y}{\sqrt{2}}=r\sin\varphi & y=\sqrt{2}\,r\sin\varphi & 0\le\varphi\le 2\pi\\ & & 0\le r\le 1\end{matrix}\right| = \int_0^1\int_0^{2\pi} 2r^2\sin^2\varphi\cdot 2\sqrt{2}\, r\, dr\, d\varphi =$$

$$=4\sqrt{2}\int_0^1 r^3\, dr\int_0^{2\pi}\sin^2\varphi\, d\varphi = \ldots = \pi\sqrt{2}$$

$$C=16\pi\sqrt{2}-\pi\sqrt{2}=15\pi\sqrt{2}$$

Zadaci za vježbu

Protok (fluks) i cirkulacija (u ravni)

4450. Izračunati protok i cirkulaciju konstantnog vektora A duž proizvoljne zatvorene krive L.

4451. Izračunati protok i cirkulaciju vektora $A(P)=a\,\boldsymbol{r}$, pri čemu je a — konstantan skalar, a $\boldsymbol{r}$ — vektor položaja tačke P, — duž proizvoljne zatvorene krive L.

4452. Izračunati protok i cirkulaciju vektora $A(P)=x\boldsymbol{i}-y\boldsymbol{j}$ duž proizvoljne zatvorene krive L.

4453. Izračunati protok i cirkulaciju vektora $A(P)=(x^3-y)\boldsymbol{i}+(y^3+x)\boldsymbol{j}$ duž kružnice poluprečnika R sa centrom u koordinatnom početku.

4454. Potencijal polja brzinâ čestica tečnosti je $u=\ln r\,(r=\sqrt{x^2+y^2})$; odrediti količinu tečnosti koja ističe u jedinici vremena kroz zatvorenu konturu opisanu oko kordinatnog početka (protok), i količinu tečnosti koja protiče u jedinici vremena duž te konture (cirkulacija). Koliki će biti rezultat ako centar leži van konture?

4455. Potencijal polja brzinâ čestica tečnosti je $u=\varphi$, pri čemu je $\varphi=\operatorname{arctg}\dfrac{y}{x}$; odrediti protok i cirkulaciju vektora brzina duž zatvorene konture L.

4456. Potencijal polja brzinâ čestica tečnosti je $u(x, y)=x(x^2-3y^2)$; izračunati količinu tečnosti koja protekne u jedinici vremena kroz pravolinijski odsečak koji spaja koordinatni početak sa tačkom (1,1).

Rješenja

4450. I protok i cirkulacija su jednaki nuli.

4451. Vrednost protoka je $2aS$, gde je S površina oblasti ograničene konturom L cirkulacija je jednaka nuli.

4452. I protok i cirkulacija su jednaki nuli.

4453. Vrednost protoka je $\dfrac{2}{3}\pi R^4$, a cirkulacija je $2\pi R^2$.

4454. U slučaju kad koordinatni početak leži unutar konture protok ima vrednost 2π, protivnom slučaju njegova je vrednost nula; cirkulacija je u oba slučaja jednaka nuli.

4455. Ako koordinatni početak leži unutar konture cirkulacija je 2π, a ako leži van onture vrednost cirkulacije je 0; protok je u oba slučaja jednak nuli.

Zadaci za vježbu

Protok i cirkulacija (u prostoru)

4457. Dokazati da je početak vektora položaja $\boldsymbol{r}$ kroz svaku zatvorenu površinu jednak trostrukoj zapremini tela ograničenog tom površinom.

4458. Izračunati protok vektora položaja kroz bočnu površinu kružnog cilindra (poluprečnik osnove je R, visina H), ako osa cilindra prolazi kroz koordinatni početak.

4459. Koristeći rezultate zadataka 4457 i 4458 utvrditi koliki je protok vektora položaja kroz obe osnove cilindra prethodnog zadatka.

4460. Izračunati protok vektora položaja kroz bočnu površinu kružnog konusa čija osnova leži u ravni xOy, a osa mu se poklapa sa z-osom. (Visina konusa je $=1$, a poluprečnik osnove je $=2$).

4461. Naći protok vektora $A(P)=xy\boldsymbol{i}+yz\boldsymbol{j}+xz\boldsymbol{k}$ kroz onaj deo površine sfere $x^2+y^2+z^2=1$ koji leži u prvom oktantu.

4462*. Naći protok vektora $A(P)=yz\boldsymbol{i}+xz\boldsymbol{j}+xy\boldsymbol{k}$ kroz bočnu površinu piramide sa vrhom u tački $S(0, 0, 2)$, čija je osnova trougao sa temenima $0(0, 0, 0)$, $A(2, 0, 0)$ i $B(0, 1, 0)$.

4463. Izračunati cirkulaciju vektora položaja jednog zavoja AB zavojnice $x=a\cos t$, $y=a\sin t$, $z=bt$, ako su A i B tačke koje odgovaraju vrednostima 0 i 2π parametra t.

4464. Kruto telo se obrće konstantnom ugaonom brzinom ω oko z-ose; izračunati cirkulaciju polja linearnih brzina duž kružne linije poluprečnika R, čiji centar leži na osi obrtanja a ravan joj je normalna na tu osu, — u smeru u kom se vrši obrtanje.

4465*. Izračunati protok rotora vektorskog polja $A(P)=y\boldsymbol{i}+z\boldsymbol{j}+x\boldsymbol{k}$ kroz površinu obrtnog paraboloida $z=2(1-x^2-y^2)$ koju od njega odseca ravan $z=0$

Rješenja

4456. 2. **4458.** $2\pi R^2H$. **4459.** πR^2H.

4460. 4π. Izračunati protok kroz osnovu konusa i iskoristiti rezultat zadatka 4457.

4461 $\dfrac{3\pi}{16}$.

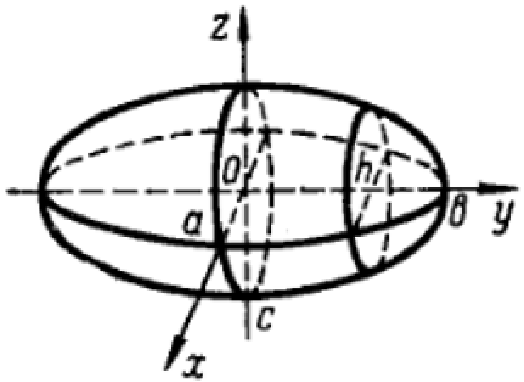
4462*. $\dfrac{1}{6}$. Primeniti formulu Ostrogradskog i izračunati protok kroz osnovu piramide

4463. $2\pi^2b^2$. **4464.** $2\pi\omega R^2$.

4465. $-\pi$. Primeniti Škotsovu formulu uzimajući za konturu L krivu po kojoj ravan Oxy preseca paraboloid.

1. Naći protok (fluks) vektorskog polja $\boldsymbol{\vec{p}} = \boldsymbol{x\vec{i}} - \boldsymbol{y}^2\boldsymbol{\vec{j}} + \left(\boldsymbol{x}^2 + \boldsymbol{y}^2 - 1\right)\boldsymbol{\vec{k}}$ kroz elipsoidu $\frac{\boldsymbol{x}^2}{\boldsymbol{a}^2} + \frac{\boldsymbol{y}^2}{\boldsymbol{b}^2} + \frac{\boldsymbol{z}^2}{\boldsymbol{c}^2} = 1$.

Rješenje:



Slika 1: elipsoid

Kako je S zatvorena površ možemo primijeniti formulu Gauss - Ostrogradski

$$\Phi = \iint_S \vec{p}\vec{n}\, ds = \iiint_\Omega div\,\vec{p}\, dxdydz = \iiint_\Omega \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dxdydz$$

Imamo:

$$\vec{p} = \left(v_x, v_y, v_z\right) = \left(x, -y^2, x^2 + y^2 - 1\right)$$

$$\frac{\partial v_x}{\partial x} = 1,\ \frac{\partial v_y}{\partial y} = -2y,\ \frac{\partial v_z}{\partial z} = 0$$

Oblast Ω je ograničena elipsoidom (vidi sliku 1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\Phi = \iiint_\Omega \left(1 - 2y\right) dxdydz = (*)$$

Uvedimo sferne koordinate:

$$x = ra\,sin\varphi\,cos\theta$$
$$y = rb\,sin\varphi\,sin\theta$$
$$z = rc\,cos\varphi$$
$$dx\,dy\,dz = r^2 sin\varphi abc\,dr\,d\varphi\,d\theta$$

$$\Omega' = \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\begin{aligned}
(*) &= \iiint_{\Omega'} (1 - 2\mathrm{rb}\ \sin\varphi\ \sin\theta)\ \mathrm{r}^2 \sin\varphi \mathrm{abc}\ \mathrm{dr}\ \mathrm{d}\varphi\ \mathrm{d}\theta \\
&= \mathrm{abc}\int_0^1 \mathrm{r}^2\ \mathrm{dr}\int_0^\pi \sin\varphi\ \mathrm{d}\varphi \int_0^{2\pi} \left(1 - 2\mathrm{rb}\ \sin\varphi\ \sin\theta\right)\ \mathrm{d}\theta \\
&= \mathrm{abc}\int_0^1 \mathrm{r}^2\ \mathrm{dr}\int_0^\pi (2\pi - 2\mathrm{rb}\ \sin\varphi(-\cos 2\pi + \cos 0)\sin\varphi\ \mathrm{d}\varphi \\
&= \mathrm{abc}\int_0^1 \mathrm{r}^2\ \mathrm{dr}\int_0^\pi (2\pi - 2\mathrm{rb}\ \sin\varphi\left(-1 + 1\right))\sin\varphi\ \mathrm{d}\varphi \\
&= \mathrm{abc}\int_0^1 \mathrm{r}^2\ \mathrm{dr}\int_0^\pi 2\pi\ \sin\varphi\ \mathrm{d}\varphi \\
&= 2\pi\mathrm{abc}\int_0^1 \mathrm{r}^2\ \mathrm{dr}\int_0^\pi \sin\varphi\ \mathrm{d}\varphi \\
&= 2\pi\mathrm{abc}\int_0^1 (-\cos\pi + \cos 0)\mathrm{r}^2\ \mathrm{dr} \\
&= 2\pi\mathrm{abc}\int_0^1 (-(-1) + 1)\mathrm{r}^2\ \mathrm{dr} \\
&= 4\pi\mathrm{abc}\int_0^1 \mathrm{r}^2\ \mathrm{dr} \\
&= 4\pi\mathrm{abc}\frac{1}{3}(1 - 0) = \frac{4}{3}\boldsymbol{\pi}\mathbf{abc}
\end{aligned}$$

Prema tome

$$\mathbf{\Phi} = \frac{4}{3}\boldsymbol{\pi abc}\,.$$

Pismeni dio ispita iz **Matematike II**, 03.02.2011.

GRUPA A

1. Naći površinu figure koja je ograničena linijama $y=-x^2, x-y-2=0$.
2. Naći ekstreme funkcije $z=x^3+3xy^2-15x-12y$.
3. Naći zapreminu tijela ograničenog ravnima $x=1, x=3, y=1, y=5,\ 2x-y+z-1=0,\ z=0$.
4. Izračunati krivolinijski integral $I=\int_c z\sqrt{x^2+y^2+2z^2}\,ds$, ako je c kriva

$$x=\frac{r\sqrt{2}}{2}\cos t, y=\frac{r\sqrt{2}}{2}\cos t, z=r\sin t, t\in[0,\pi].$$

GRUPA B

1. Izračunati površinu rotacionog tijela koje se dobije rotacijom parabole $y^2=4x$ od tačke $x=0$ do tačke $x=2$.
2. Naći uslovne ekstreme funkcije $z=6+4x+3y$ uz uslov $x^2+y^2=1$.
3. Naći zapreminu tijela ograničenog ravnima $x=-1,\ x=2,\ y=-2,\ y=2,\ 4x-3y+z-2=0,\ z=0$.
4. Izračunati krivolinijski integral $I=\int_c y\sqrt{x^2+4y^2+z^2}\,ds$, ako je c kriva

$$x=\frac{a\sqrt{6}}{3}\sin t, y=a\cos t, z=\frac{a\sqrt{3}}{3}\sin t, t\in\left[0,\frac{\pi}{2}\right].$$

Pismeni dio ispita iz **Matematike II**, 18.02.2011

GRUPA A

1. Izračunati integrale: $I_1=\int_1^3 x^3\sqrt{x^2-1}\,dx, I_2=\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sin x+\cos x}$.
2. Izmjeniti poredak integracije u integralu $I=\int_1^2 dx\int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy$.
3. Izračunati površinski integral $P=\iint_S (z^2+1)\,dS$, S je dio sfere $x^2+y^2+z^2=4$ u prvom oktantu.
4. Izračunati integral $I(\alpha)=\int_0^{\infty}\frac{1-e^{-\alpha x^2}}{xe^{x^2}}dx$ pomoću diferenciranja po parametru ako je $\alpha>-1$.

GRUPA B

1. Izračunati integrale: $I_1=\int_0^4\frac{dx}{1+\sqrt{2x+1}}, I_2=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\sin x+\cos x}dx.$
2. Izmjeniti poredak integracije u integralu $I=\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x,y)\,dy$.
3. Izračunati površinski integral $\iint_{(S)}\sqrt{-x^2+4}\,dS$, gdje je (S) omotač površi

$$\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}, 0\le z\le 3.$$

4. Izračunati pomoću diferenciranja po parametru integral

$$I(\alpha)=\int_0^{\frac{\pi}{2}}\ln\left(\sin^2 x+\alpha^2\cos^2 x\right)dx,\ \alpha>0.$$

Pismeni dio ispita iz **Matematike II**, 23.06.2011.

GRUPA A

1. Izračunati dužinu luka krive $y=\ln\frac{e^x-1}{e^x+1}$ od tačke sa apscisom $x=1$ do tačke sa apscisom $x=2$.
2. Izračunati pomoću dvostrukog integrala zapreminu tijela kojeg ograničavaju površi $x^2+y^2-2az=0, \left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right),\ a>0$.
3. Izračunati pomoću Greenove formule krivolinijski integral $I=\oint_c\sqrt{x^2+y^2}dx+y\left[xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right]dy$, ako je c kontura koja ograničava oblast $y^2\le 2x-2,\ x\le 2,\ y\ge 0$.
4. Izračunati površinski integral $I=\iint_S\left(x+y+z^2\right)dS$, ako je S polulopta $x^2+y^2+z^2=9, z\ge 0$.

GRUPA B

1. Izračunati dužinu luka krive $y=a\ln\frac{a^2}{a^2-x^2}$ $(a>0)$ od tačke $A(0,0)$ do tačke $B\left(\frac{a}{2}, a\ln\frac{4}{3}\right)$.
2. Izračunati pomoću dvostrukog integrala zapreminu tijela kojeg ograničavaju površi $x^2+y^2=4\ (x\ge 0),\ x^2-y^2=1\ (x\ge 1), z=4-x^2\ (z\ge 0)$ i ravan $z=0$.

3. Izračunati krivolinijski integral $\oint\limits_c xds$, ako je c lemniskata

$\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$, $a>0$.

4. Izračunati površinski integral $I=\iint\limits_S (x-y+z)dS$, ako je S polulopta

$x^2+y^2+z^2=4, z\geq 0$.

Pismeni dio ispita iz **Matematike II**, 08.07.2011.

Grupa A

1. Odrediti jednačinu tangentne ravni na površ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, koja je normalna na pravoj $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

2. Izračunati integral po glatkom luku koji spaja tačke A i B

$\int\limits_{\widehat{AB}}\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz$, $A(1,1,1), B(1,2,3), \widehat{AB}\subset\{(x,y,z): x>0, y>0, z>0\}$

3. Izračunati zapreminu onog dijela lopte $x^2+y^2+z^2=1$ koji se nalazi unutar cilindra $x^2+(y-1)^2=1$.

4. Dato je vektorsko polje $\vec{A}=(e^x z-2xy, 1-x^2, e^x+z)$. Pokazati da je polje A potencijalno i odrediti mu potencijal. Izračunati integral $\int\limits_L \vec{A}\cdot d\vec{r}$, gdje je L duž PQ, P (0, 1, -1), Q (2, 3, 0), orijentisana od P prema Q.

Grupa B

1. Dokazati da proizvoljna tangentna ravan površi S: $xyz=a^3\,(a>0, \text{konstanta})$ obrazuje sa koordinatnim ravnima tetraedar stalne zapremine $\left(V=\frac{9}{2}a^3\right)$.

2. Izračunati integral po glatkom luku koji spaja tačke A i B

$\int\limits_{\widehat{AB}}\frac{zxdy+xydz-yzdx}{(x-yz)^2}$ $A(7,2,3),\ B(5,3,1),\ \left(z\neq\frac{x}{y}\right)$.

3. Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2=y, x^2+y^2=2y, z=y^2, z=0$.

4. Dato je vektorsko polje $\vec{A}=(2x(y^2+z^2)+yz, 2y(z^2+x^2)+xz, 2z(x^2+y^2)+xy)$. Pokazati da je polje A potencijalno i odrediti mu potencijal. Izračunati fluks vektorskog polja $\vec{A}$ kroz spoljnu stranu polusfere $\Gamma: x^2+y^2+z^2-2z=0, y\geq 0$

Pismeni dio ispita iz **Matematike II,** 15.09.2011.

GRUPA A

1. Izračunati površinu figure koju određuju prava $2x+3\sqrt{3}y-12=0$ i dio elipse $4x^2+9y^2=36$ u prvom kvadrantu.

2. Promijeniti poredak integracije i izračunati dvostruki integral $I=\int\limits_0^2 ydy\int\limits_{\frac{1}{2}}^{2-\frac{y^2}{8}}\frac{dx}{\sqrt{x^5}}$.

3. Izračunati površinu dijela površi $x^2+y^2+z-1=0$ koji se nalazi iznad ravni $z=0$.

4. Dati su krivolinijski integrali $I_1=\int\limits_{c_1}\frac{xdy-ydx}{x^2+y^2}$, $I_2=\int\limits_{c_2}\frac{xdy-ydx}{x^2+y^2}$, gdje je c_1 duž $\overline{AB}$, $A(1,2), B(-1,4)$, orijentisana od tačke A prema tački B, a c_2 je parabola koja prolazi kroz tačke $A(1,2), B(-1,4)$ i $C\left(\frac{-1}{2},\frac{11}{4}\right)$. Dokazati da je $I_1=I_2$ i izračunati taj broj.

GRUPA B

1. Izračunati površinu krivolinijskog četverougla omeđenog parabolama $y=x^2,\ y=\frac{x^2}{3},\ y^2=2x,\ y^2=3x.$

2. Promijeniti poredak integracije i izračunati dvostruki integral $I=\int\limits_0^a ydy\int\limits_0^{a-\sqrt{a^2-y^2}}\frac{x\ln(x+a)dx}{(x-a)^2}.$

3. Izračunati površinu dijela sfere $x^2+y^2+z^2=a^2$ koji se nalazi u unutrašnjosti cilindra $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0<b<a.$

4. Izračunati krivolinijski integral $I=\oint\limits_c \frac{1}{x}\operatorname{arctg}\frac{y}{x}dx+y^3e^{-y}dy$, ako je c pozitivno orjentisana kontura oblasti određene isječkom kružnog prstena $1\leq x^2+y^2\leq 4,\ 0\leq y\leq x.$

Pismeni dio ispita iz **Matematike II,** 23.09.2011.

GRUPA A

1. Izračunati površinu figure koju određuju prava $2x+3\sqrt{3}y-12=0$ i dio elipse $4x^2+9y^2=36$ u prvom kvadrantu.

2. Promijeniti poredak integracije i izračunati dvostruki integral $I=\int\limits_0^2 ydy\int\limits_{\frac{1}{2}}^{2-\frac{y^2}{8}}\frac{dx}{\sqrt{x^5}}$.

3. Izračunati površinu dijela površi $x^2+y^2+z-1=0$ koji se nalazi iznad ravni $z=0$.

4. Dati su krivolinijski integrali $I_1=\int\limits_{c_1}\frac{xdy-ydx}{x^2+y^2}$, $I_2=\int\limits_{c_2}\frac{xdy-ydx}{x^2+y^2}$, gdje je c_1 duž $\overline{AB}$, $A(1,2), B(-1,4)$, orjentisana od tačke A prema tački B, a c_2 je parabola koja prolazi kroz tačke $A(1,2), B(-1,4)$ i $C\left(\frac{-1}{2},\frac{11}{4}\right)$. Dokazati da je $I_1=I_2$ i izračunati taj broj.

GRUPA B

1. Izračunati površinu krivolinijskog četverougla omeđenog parabolama $y=x^2,\ y=\frac{x^2}{3},\ y^2=2x,\ y^2=3x.$
2. Promijeniti poredak integracije i izračunati dvostruki integral $I=\int\limits_0^a ydy\int\limits_0^{a-\sqrt{a^2-y^2}}\frac{x\ln(x+a)dx}{(x-a)^2}.$
3. Izračunati površinu dijela sfere $x^2+y^2+z^2=a^2$ koji se nalazi u unutrašnjosti cilindra $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0<b<a.$
4. Izračunati krivolinijski integral $I=\oint\limits_c\frac{1}{x}\operatorname{arctg}\frac{y}{x}dx+y^3e^{-y}dy,$ ako je c pozitivno orjentisana kontura oblasti određene isječkom kružnog prstena $1\le x^2+y^2\le 4,\ 0\le y\le x.$

Pismeni dio ispita iz **Matematike II,** oktobar 2011.

1. Odrediti jednačinu tangentne ravni na površ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, koja je normalna na pravoj $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.
2. Promijeniti poredak integracije i izračunati dvostruki integral $I=\int\limits_0^a ydy\int\limits_0^{a-\sqrt{a^2-y^2}}\frac{x\ln(x+a)dx}{(x-a)^2}.$
3. Izračunati krivolinijski integral $\oint\limits_c xds,$ ako je c lemniskata $(x^2+y^2)^2=a^2(x^2-y^2),\ a>0.$
4. Dato je vektorsko polje $\vec{A}=(2x(y^2+z^2)+yz, 2y(z^2+x^2)+xz, 2z(x^2+y^2)+xy)$. Pokazati da je polje A potencijalno i odrediti mu potencijal. Izračunati fluks vektorskog polja $\vec{A}$ kroz spoljnu stranu polusfere $\Gamma: x^2+y^2+z^2-2z=0, y\ge 0$.