GRAĐEVINSKI FAKULTET
SVEUČILIŠTA U RIJECI

SKRIPTA RIJEŠENIH ZADATAKA IZ OTPORNOSTI MATERIJALA

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1. Zadatak

Izračunati naprezanja i nacrtati dijagram naprezanja na konzolnom nosaču ako je zadano :

$F_1 = F_2 = 2 \cdot 10^4 N$

$F_3 = 4 \cdot 10^4 N$

$E = 2{,}2 \cdot 10^{11} Pa$

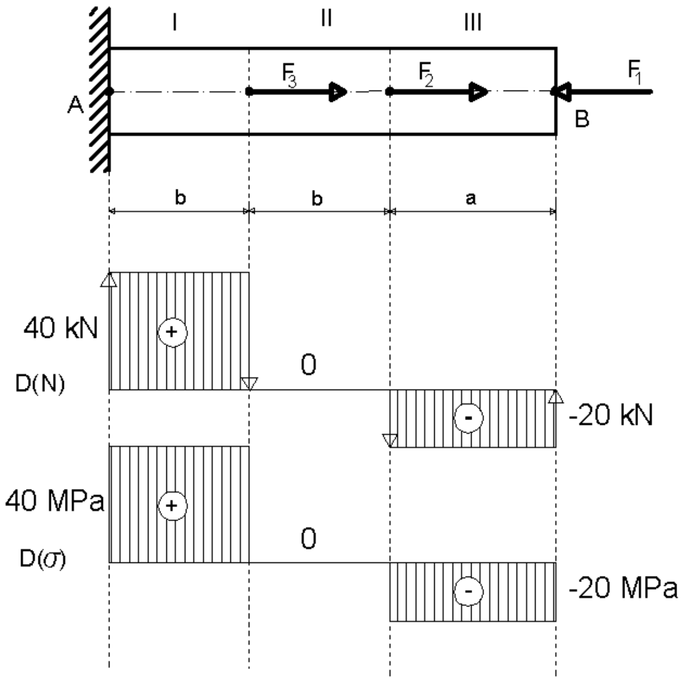
$A = 10 cm^2 = 10^{-3} m^2$

$a = 2m$

$b = 1m$

$1Pa = 1\frac{N}{m^2}$

$1MPa = 1\frac{N}{mm^2}$



Vrijednosti uzdužnih sila na pojedinim segmentima :

$N_{III} = -F_1 = -2 \cdot 10^4 N$

$N_{II} = -F_1 + F_2 = 0$

$N_I = F_3 = 4 \cdot 10^4 N$

Vrijednosti naprezanja na pojedinim segmentima :

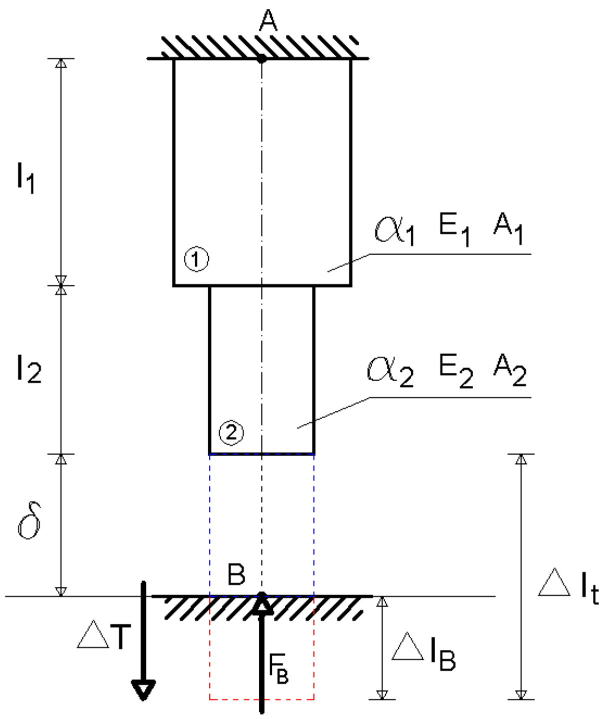
$$\sigma_{III} = \frac{N_{III}}{A} = -20 \cdot 10^6 Pa = -20 MPa$$

$$\sigma_I = \frac{N_{II}}{A} = 0$$

$$\sigma_I = \frac{N_I}{A} = 40 \cdot 10^6 Pa = 40 MPa$$

2. Zadatak

Odrediti naprezanja u štapu pri promjeni temperature za +ΔT :



Rješenje :

Ukoliko nema vanjskog otpora izduženju štapa, nema niti naprezanja u štapu. To znači da se štap AC može slobodno izdužiti pod utjecajem temperature za veličinu δ. Sve dok je $\Delta l_t \leq \delta$ u štapu su naprezanja jednaka nuli.
Ukoliko je tendencija štapa da se izduži za $\Delta l_t > \delta$, aktivira se u štapu sila koja opet " vraća " štap na realnu dužinu AB, kao što se vidi na slici. Može se zaključiti i da će sila u štapu biti tlačna , odnosno s predznakom – (minus).

δ – realno izdužrnje
Δl_B – nerealno izduženje

$$\Delta l_t = \alpha_1 l_1 \Delta T + \alpha_2 l_2 \Delta T$$
$$\Delta l_t \leq \delta \rightarrow \varepsilon > 0, \sigma = 0$$
$$\Delta l_t > \delta \rightarrow \varepsilon > 0, \sigma \neq 0$$

$$\delta = \Delta l_t - \Delta l_B \ \ldots (1)$$
$$\Delta l_t = (\alpha_1 l_1 + \alpha_2 l_2)\Delta T \ \ldots (2)$$
$$\Delta l_B = \frac{F_B l_1}{E_1 A_1} + \frac{F_B l_2}{E_2 A_2} \ \ldots (3)$$

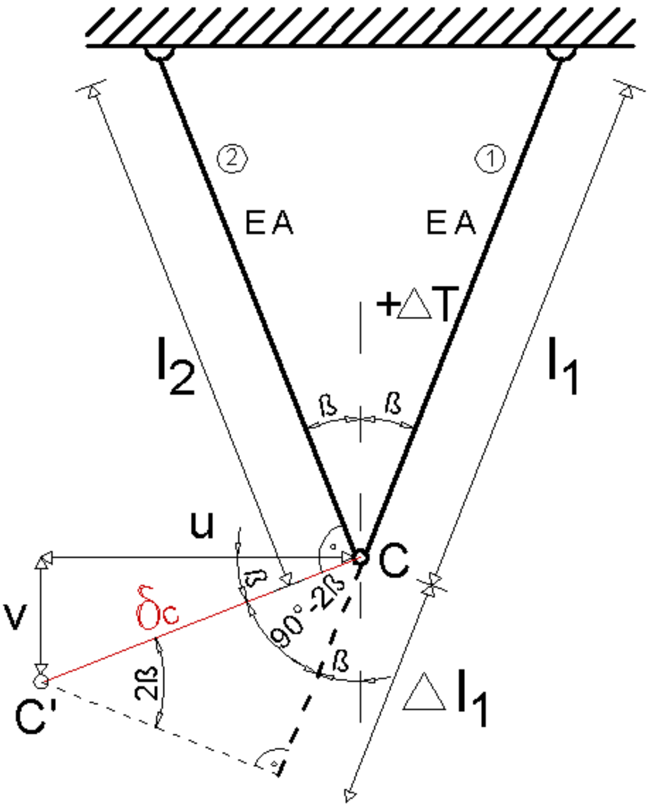
$$(2);(3) \rightarrow (1) \Rightarrow F_B = \frac{[(\alpha_1 l_1 + \alpha_2 l_2)\Delta T - \delta]E_1 A_1}{l_1\left(1 + \frac{l_2}{l_1}\frac{E_1 A_1}{E_2 A_2}\right)}$$

$$\sigma_{x1} = -\frac{F_B}{A_1}$$

$$\sigma_{x2} = -\frac{F_B}{A_2}$$

3. Zadatak

Odrediti pomak točke C ukoliko se temperatura štapa 1 poveća za ΔT :



Rješenje :

$$\varepsilon_t = \left|\frac{\Delta l_t}{l}\right| \Rightarrow \Delta l_t = \varepsilon_t l = \alpha \Delta T l$$

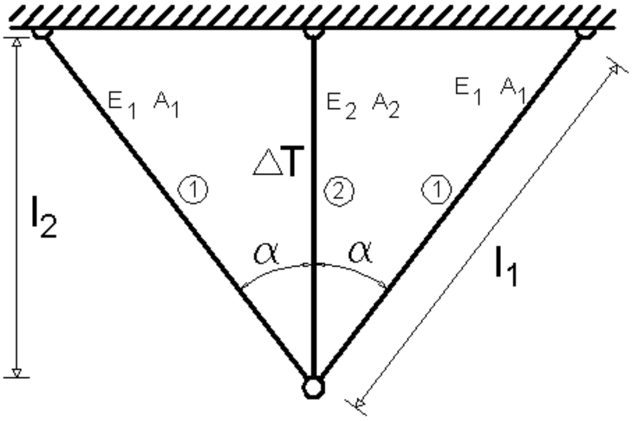
$$\sin 2\beta = \frac{\Delta l_1}{\delta_C}\,;\, \Delta l_2 = 0$$

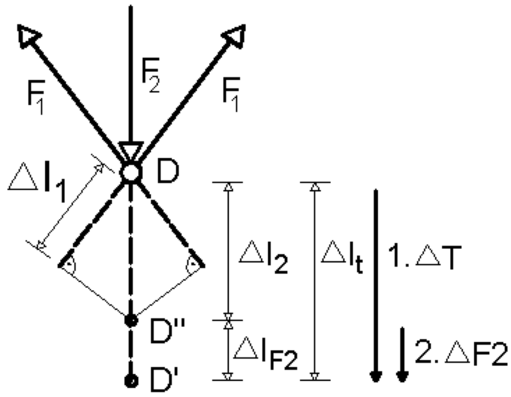
$$\delta_C = \frac{\Delta l_1}{\sin 2\beta} = \frac{\alpha \Delta T l}{\sin 2\beta}$$

$$u = \delta_C \cos\beta$$
$$v = \delta_C \sin\beta$$

4. Zadatak

Odrediti naprezanja u štapovima ukoliko se temperatura štapa 2 poveća za ΔT :





Rješenje

Iz uvjeta ravnoteže sila :

$$\underline{\Sigma Y = 0} \rightarrow 2F_1\cos\alpha - F_2 = 0 \ldots (1)$$

Iz plana pomaka :

$$\cos\alpha = \frac{l_2}{l_1} = \frac{\Delta l_1}{\Delta l_2} \rightarrow \Delta l_1 = \Delta l_2 \cos\alpha \ldots (2)$$

Iz Hookovog zakona :

$$\Delta l_1 = \frac{F_1 l_1}{E_1 A_1} \ldots (3)$$

$$\Delta l_2 = \frac{F_2 l_2}{E_2 A_2} + \alpha_2 \cdot \Delta T \cdot l_2 \ldots (4)$$

Iz plana pomaka i opterećenja (ovdje promjena temperature) možemo zaključiti da se štap 2 nastoji izdužiti, no njegovo potpuno izduženje sprječavaju druga dva štapa spojena u čvoru D, koji nisu direktno opterećeni i nemaju tendenciju izduženja. Štap 2 se izduži za Δl_t pri čemu čvor D zauzme položaj D'. Tada se javi otpor štapova 1 koji u štapu 2 aktivira tlačnu silu F_2 koja čvor D vraća iz D' u D''.

Budući da su sva tri štapa međusobno spojena u čvoru D, izduženjem štapa 2 prisilno se izdužuju i štapovi s brojem 1. Kao posljedica izduženja u njima se javlja vlačna sila F_1(vlačna sila uzrokuje izduženje ako nema nikakvog drugog opterećenja).

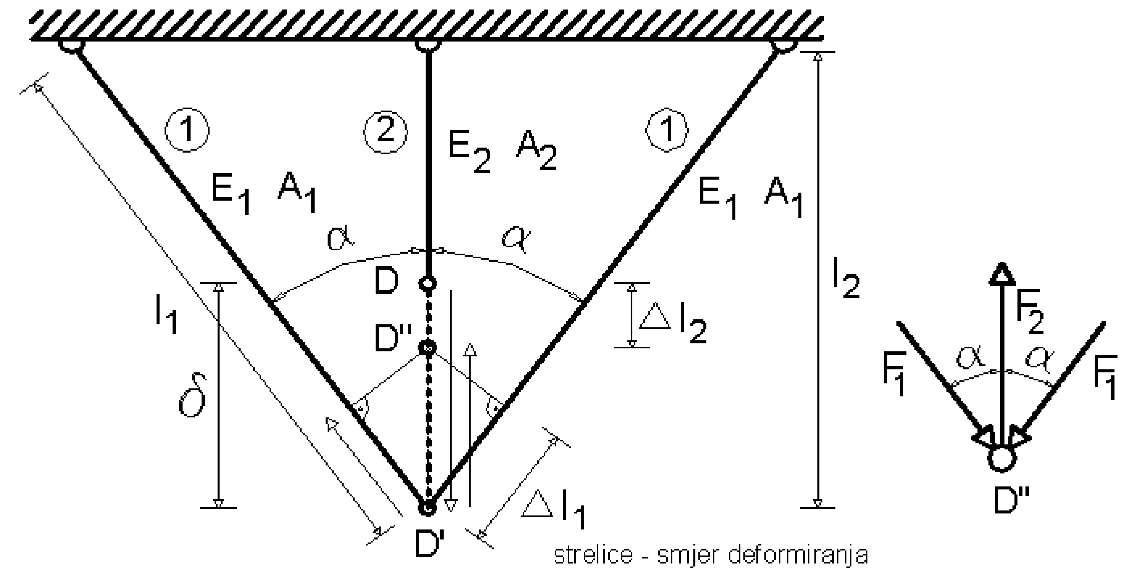
$$(1);(3);(4) \rightarrow (2) \Rightarrow F_1 = \frac{\alpha_2 \Delta T E_1 A_1 \cos^2\alpha}{1 - 2\dfrac{E_1 A_1}{E_2 A_2}\cos^3\alpha}$$

$$(1) \rightarrow F_2 = 2F_1\cos\alpha \Rightarrow F_2 = \frac{2\alpha_2 \Delta T E_1 A_1 \cos^3\alpha}{1 - 2\dfrac{E_1 A_1}{E_2 A_2}\cos^3\alpha}$$

$$\sigma_1 = \frac{F_1}{A_1}; \sigma_2 = -\frac{F_2}{A_2}$$

5. Zadatak

Odrediti naprezanja u štapovima ako je srednji štap kraći za δ od predviđene duljine l.



Rješenje

Iz uvjeta ravnoteže sila :

$$\underline{\Sigma Y = 0} \rightarrow F_2 - 2F_1 \cos\alpha = 0 \rightarrow F_2 = 2F_1 \cos\alpha \ \ldots (1)$$

Iz plana pomaka :

$$\Delta l_1 = \frac{F_1 l_1}{E_1 A_1}$$

$$\Delta l_2 = \frac{F_2 l_2}{E_2 A_2}$$

$$\cos\alpha = \frac{l_2}{l_1} = \frac{\Delta l_1}{\delta - \Delta l_2}$$

$$\delta = \Delta l_2 + \frac{\Delta l_1}{\cos\alpha} = \frac{F_2 l_2}{E_2 A_2} + \frac{F_1 l_1}{E_1 A_1}\frac{1}{\cos\alpha} \ \ldots (2)$$

$$(1) \rightarrow (2) \Rightarrow \delta = 2\frac{F_1 l_2}{E_2 A_2}\cos\alpha + \frac{F_1 l_1}{E_1 A_1}\frac{1}{\cos\alpha} = F_1\left(2\frac{l_2}{E_2 A_2}\cos\alpha + \frac{l_1}{E_1 A_1}\frac{1}{\cos\alpha}\right)$$

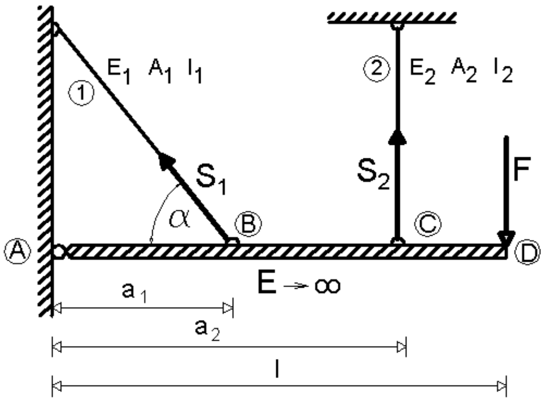
$$F_1 = \frac{E_1 A_1 E_2 A_2 \delta \cos^2\alpha}{l_2\left(2E_1 A_1 \cos^3\alpha + E_2 A_2\right)}$$

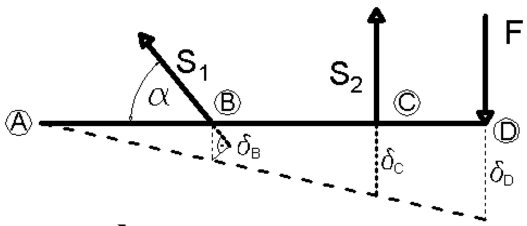
$$(1) \Rightarrow F_2 = \frac{2E_1 A_1 E_2 A_2 \delta \cos^3\alpha}{l_2\left(2E_1 A_1 \cos^3\alpha + E_2 A_2\right)}$$

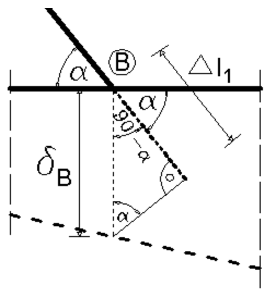
$$\sigma_1 = -\frac{F_1}{A_1} \qquad \sigma_2 = -\frac{F_2}{A_2}$$

6. Zadatak

Odrediti naprezanja u štapovima 1 i 2 te vertikalni pomak točke D.







Rješenje

Sustav je jedanput statički neodređen (ima jednu prekomjernu veličinu), odnosno nepoznate su 4 veličine (R_H , R_V , S_1 , S_2).
Uz tri uvjeta ravnoteže postavljamo dodatnu jednadžbu na deformiranom sustavu na principu sličnosti trokuta.
Iz uvjeta ravnoteže sila :

$$\underline{\Sigma M_A = 0} \rightarrow Fl - S_2 a_2 - S_1 a_1 \sin\alpha = 0 \ldots (1)$$

$$\underline{\Sigma X = 0} \rightarrow R_H - S_1 \cos\alpha = 0 \rightarrow R_H = S_1 \cos\alpha \ldots (2)$$

$$\underline{\Sigma Y = 0} \rightarrow S_2 - F + S_1 \sin\alpha + R_V = 0 \ldots (3)$$

Iz plana pomaka :

$$\delta_C \equiv \Delta l_2 ; \delta_B = \frac{\Delta l_1}{\sin\alpha}$$

Iz sličnosti trokuta :

$$\frac{\delta_C}{a_2} = \frac{\delta_B}{a_1} \ldots (4)$$

Iz Hookovog zakona :

$$\Delta l_1 = \frac{S_1 l_1}{E_1 A_1} ; \Delta l_2 = \frac{S_2 l_2}{E_2 A_2} \quad \rightarrow (4)$$

$$S_1 = S_2 \frac{E_1 A_1}{E_2 A_2} \frac{l_2}{l_1} \frac{a_1}{a_2} \sin\alpha$$

$$S_1 \rightarrow (1) \Rightarrow S_2 = \frac{Fl}{a_2 \left[1 + \frac{E_1 A_1}{E_2 A_2} \frac{l_2}{l_1} \frac{a_1^2}{a_2^2} \sin^2\alpha \right]}$$

$$\sigma_1 = \frac{S_1}{A_1}$$

$$\sigma_2 = \frac{S_2}{A_2}$$

$$\frac{\Delta l_2}{a_2} = \frac{\delta_D}{l} \Rightarrow \delta_D = \Delta l_2 \frac{l}{a_2} = \frac{S_2 l_2 l}{E_2 A_2 a_2}$$

7. Zadatak

Dimenzionirati štapove AB i DG kružnog poprečnog presjeka napravljene od čelika, te odrediti njihova produljenja.

$\sigma_{dop} = 140 Mpa$

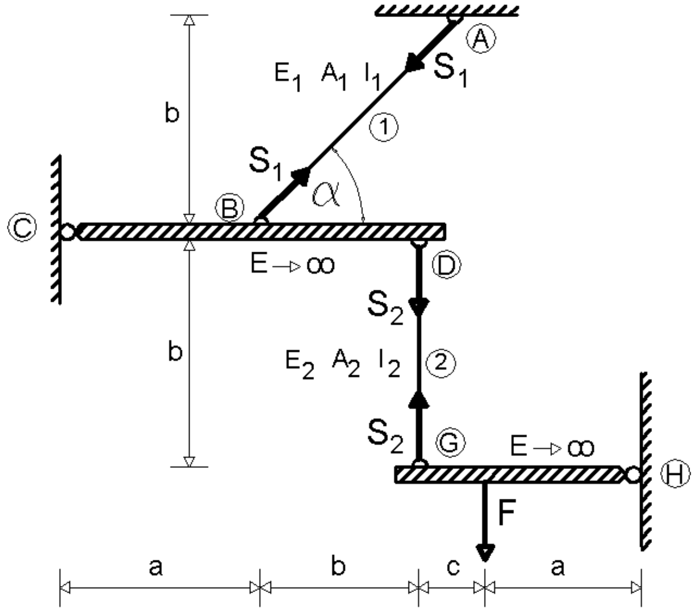
$E = 2 \cdot 10^5 Mpa$

$F = 100 kN$

$a = 3m$

$b = 2m$

$c = 1m$



$$\underline{\sum M_H = 0} \Rightarrow S_2 \cdot 4 - F \cdot 3 = 0 \rightarrow S_2 = \frac{3}{4} F = \frac{3}{4} 100 = 75 kN$$

$$\underline{\sum M_C = 0} \Rightarrow S_2 \cdot 5 - S_1 \cdot \sin 45° \cdot 3 = 0 \rightarrow S_1 = S_2 \frac{5}{3 \cdot \sin 45°} = 75 \frac{5}{3 \cdot \sin 45°} = 176{,}8 kN$$

Dimenzioniranje

$$\sigma_x = \frac{S}{A} \leq \sigma_{dop} \Rightarrow A_{pot} \geq \frac{S}{\sigma_{dop}}$$

$$A_{1pot} \geq \frac{S_1}{\sigma_{dop}} = \frac{176{,}8 \cdot 10^3}{140 \cdot 10^6} = 1{,}26 \cdot 10^{-3} m^2 = 12{,}6 cm^2$$

$$A_1 = \frac{d_1^{\,2} \pi}{4} \Rightarrow d_1 = \sqrt{\frac{4A_1}{\pi}} = 4{,}01 cm \rightarrow usvojeno \rightarrow d_1 = 42 mm \rightarrow (A_1 = 13{,}85 cm^2)$$

$$A_{2pot} \geq \frac{S_2}{\sigma_{dop}} = \frac{75 \cdot 10^3}{140 \cdot 10^6} = 0{,}536 \cdot 10^{-3} m^2 = 5{,}36 cm^2$$

$$A_2 = \frac{d_2^{\,2} \pi}{4} \Rightarrow d_2 = \sqrt{\frac{4A_2}{\pi}} = 2{,}61 cm \rightarrow usvojeno \rightarrow d_1 = 28 mm \rightarrow (A_2 = 6{,}15 cm^2)$$

Kontrola

$$\sigma_{x1} = \frac{S_1}{A_1} = \frac{176{,}8 \cdot 10^3}{13{,}85 \cdot 10^{-4}} = 127{,}6 MPa < \sigma_{dop} = 140 MPa$$

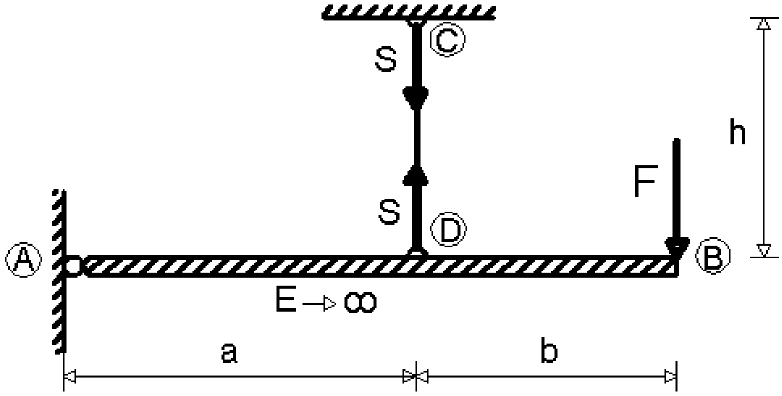
$$\sigma_{x2} = \frac{S_2}{A_2} = \frac{75 \cdot 10^3}{6{,}15 \cdot 10^{-4}} = 121{,}9 MPa < \sigma_{dop} = 140 MPa$$

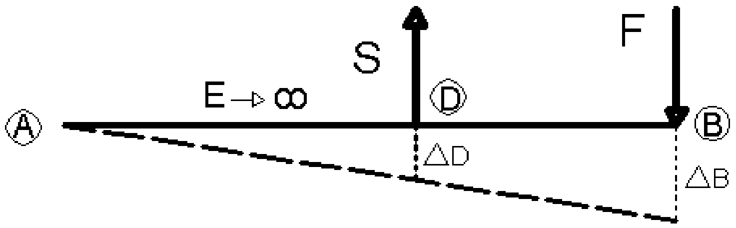
$$\Delta l_1 = \frac{S_1 l_1}{E_1 A_1} = \frac{176{,}8 \cdot 10^3 \cdot 2}{2 \cdot 10^{11} \cdot 13{,}85 \cdot 10^{-4} \cdot 0{,}707} = 18{,}056 \cdot 10^{-4} m = 0{,}18 cm$$

$$\Delta l_2 = \frac{S_2 l_2}{E_2 A_2} = \frac{75 \cdot 10^3 \cdot 2}{2 \cdot 10^{11} \cdot 6{,}15 \cdot 10^{-4}} = 12{,}195 \cdot 10^{-4} m = 0{,}12 cm$$

8. Zadatak

Odrediti pomak točaka B i D na zadanom sustavu.





$$\underline{\Sigma M_A = 0} \Rightarrow F(a+b) - Sa = 0 \rightarrow S = F\,\frac{a+b}{a}$$

$$\Delta h = \frac{S \cdot h}{E \cdot A} = \frac{F \cdot h \cdot (a+b)}{a \cdot E \cdot A}$$

$$\frac{\Delta h}{a} = \frac{\Delta B}{a+b} \Rightarrow \Delta B = \frac{F \cdot h \cdot (a+b)^2}{a^2 \cdot E \cdot A}$$

9. Zadatak

$l_1 = 1m$

$l_2 = 1,8m$

$A_1 = 1cm^2 = 1 \cdot 10^{-4} m^2$

$A_2 = 1,5cm^2 = 1,5 \cdot 10^{-4} m^2$

$E_1 = E_2$

$F = 6000N$

$$\underline{\Sigma M_A = 0} \Rightarrow Fa - S_1 2a - S_2 3a = 0 \ldots (1)$$

$$\frac{\Delta l_1}{2a} = \frac{\Delta l_2}{3a} \Rightarrow 3\Delta l_1 = 2\Delta l_2$$

$$3\frac{S_1 l_1}{E_1 A_1} = 2\frac{S_2 l_2}{E_2 A_2} \Rightarrow S_1 = \frac{2}{3}\frac{S_2 l_2 A_1}{l_1 A_2} \ldots (2)$$

$$(2) \rightarrow (1) \Rightarrow Fa - \frac{4}{3}\frac{S_2 l_2 A_1 a}{l_1 A_2} - S_2 3a = 0 \Rightarrow S_2 = \frac{F}{\frac{4}{3}\frac{l_2 A_1}{l_1 A_2} + 3}$$

$S_1 = 1043,5N$

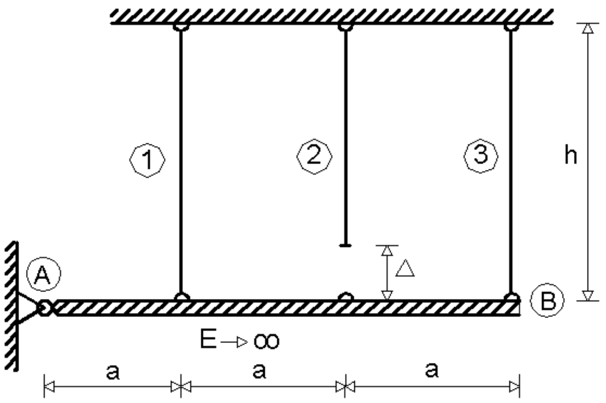
$S_2 = 1304,35N$

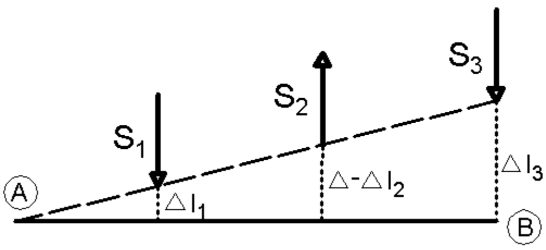
$$\sigma_1 = \frac{S_1}{A_1} = \frac{1043,5}{1 \cdot 10^{-4}} \frac{N}{m^2} = 10,43MPa$$

$$\sigma_2 = \frac{S_2}{A_2} = \frac{1304,35}{1,5 \cdot 10^{-4}} \frac{N}{m^2} = 8,7MPa$$

10. Zadatak

Kruta greda AB obješena je o tri čelična štapa istog poprečnog presjeka površine 10 cm^2. Dužina štapova je h = 1 m, s tim da je srednji štap (2) napravljen kraći od projektirane dužine za Δ = 0,6 mm. Odrediti sile u štapovima i izduženje srednjeg štapa ako je izvršena prinudna montaža sustava.





$$E_1A_1 = E_2A_2 = E_3A_3 = EA$$

$$l_1 = l_2 = l_3 = h$$

$$\Delta l_1 = \frac{S_1 h}{EA}; \Delta l_2 = \frac{S_2 h}{EA}; \Delta l_3 = \frac{S_3 h}{EA}$$

$$\frac{\Delta l_3}{3a} = \frac{\Delta l_1}{a} \Rightarrow \Delta l_3 = 3\Delta l_1 \Rightarrow \frac{S_3 h}{EA} = 3\frac{S_1 h}{EA} \Rightarrow S_3 = 3S_1 \ldots (1)$$

$$\frac{\Delta l_1}{a} = \frac{\Delta - \Delta l_2}{2a} \Rightarrow 2\Delta l_1 = \Delta - \Delta l_2 \Rightarrow \frac{S_2 h}{EA} = \Delta - 2\frac{S_1 h}{EA} \Rightarrow S_2 = \frac{\Delta EA}{h} - 2S_1 \ldots (2)$$

$$\underline{\Sigma M_A = 0} \rightarrow S_1 a - S_2 2a + S_3 3a = 0 \Rightarrow S_1 = 2S_2 - 3S_3 \ldots (3)$$

$$\frac{\Delta EA}{h} = c = \frac{6 \cdot 10^{-4} \cdot 2{,}1 \cdot 10^{11} \cdot 1 \cdot 10^{-3}}{1{,}00} = 126 \cdot 10^3 N = 126 kN$$

$$(1),(2) \rightarrow (3) \Rightarrow S_1 = 2c - 4S_1 - 9S_1 \rightarrow S_1 = \frac{1}{7}c = \frac{126}{7} = 18 kN$$

$$S_1 \rightarrow (1) \Rightarrow S_3 = \frac{3}{7}c = \frac{3}{7}126 = 54 kN$$

$$S_1 \rightarrow (2) \Rightarrow S_2 = c - 2S_1 = c - \frac{2}{7}c = \frac{5}{7}126 = 90 kN$$

$$\underline{S_1 = 18 kN}$$

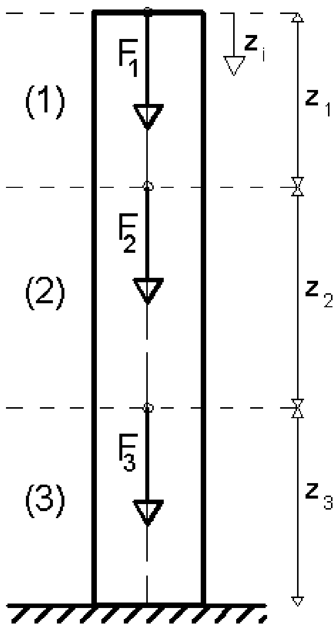
$$\underline{S_2 = 90 kN}$$

$$\underline{S_3 = 54 kN}$$

$$\Delta l_2 = \frac{S_2 h}{EA} = \frac{90 \cdot 10^3 \cdot 1{,}00}{2{,}1 \cdot 10^{11} \cdot 1 \cdot 10^{-3}} = 4{,}3 \cdot 10^{-4} m = 0{,}4 mm$$

11. Zadatak

Cilindrični stup promjera 4 cm, dužine 120 cm opterećen je u presjecima (1), (2) i (3) na udaljenostima z_i = 0; 40 i 80 cm (i=1,2,3) od slobodnog kraja aksijalnim silamaF_i(i=1,2,3) = 15 kN, 10 kN i 5 kN.
Izračumati naprezanje u pojedinim dijelovima stupa i pomak slobodnog kraja.



$$E = 2 \cdot 10^5 \, MPa$$

$$d = 4cm$$

$$L = 120cm; l_1 = l_2 = l_3 = l = \frac{L}{3} = 40cm$$

$$A = \frac{d^2 \pi}{4} = \frac{4^2 \cdot 10^{-4} \pi}{4} = 12{,}56 \cdot 10^{-4} m^2$$

$(1) \ldots 0 \le z_1 < 40cm$

$$N_{(1)} = -F_1 = -15kN$$

$$\sigma_{(1)} = \frac{N_{(1)}}{A} = \frac{-15 \cdot 10^3}{12{,}56 \cdot 10^{-4}} \frac{N}{m^2} = -12 \cdot 10^6 \, Pa = -12MPa$$

$(2) \ldots 40 \le z_2 < 80cm$

$$N_{(2)} = -F_1 - F_2 = -25kN$$

$$\sigma_{(2)} = \frac{N_{(2)}}{A} = \frac{-25 \cdot 10^3}{12{,}56 \cdot 10^{-4}} \frac{N}{m^2} = -20 \cdot 10^6 \, Pa = -20MPa$$

$(3) \ldots 80 \le z_3 < 120cm$

$$N_{(3)} = -F_1 - F_2 - F_3 = -30kN$$

$$\sigma_{(3)} = \frac{N_{(3)}}{A} = \frac{-30 \cdot 10^3}{12{,}56 \cdot 10^{-4}} \frac{N}{m^2} = -24 \cdot 10^6 \, Pa = -24MPa$$

$$\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3 = \frac{l}{AE}\left(N_{(1)} + N_{(2)} + N_{(3)}\right) = \sum_{i=1}^{3} k \cdot N_{(i)}$$

$$k = \frac{l}{AE} = \frac{0{,}4}{12{,}56 \cdot 10^{-4} \cdot 2 \cdot 10^{11}} = 0{,}159 \cdot 10^{-8} \frac{m}{N}$$

$$\Delta l_1 = k \cdot N_{(1)} = 0{,}159 \cdot 10^{-8} \cdot \left(-15 \cdot 10^3\right) = -2{,}415 \cdot 10^{-5} m = -0{,}024mm$$

$$\Delta l_2 = k \cdot N_{(2)} = 0{,}159 \cdot 10^{-8} \cdot \left(-25 \cdot 10^3\right) = -4{,}00 \cdot 10^{-5} m = -0{,}04mm$$

$$\Delta l_3 = k \cdot N_{(3)} = 0{,}159 \cdot 10^{-8} \cdot \left(-30 \cdot 10^3\right) = -4{,}80 \cdot 10^{-5} m = -0{,}048mm$$

$$\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3 = -0{,}112mm$$

12. Zadatak

Dimenzionirati čeličnu zategu kružnog porečnog presjeka (A_1) i drveni kosnik pravokutnog poprečnog presjeka (h=2b) površine $A_2 = 10A_1$ ako je F=155 kN a dopušteni naponi za čelik $12 \cdot 10^7$ N/m^2 i drvo $6 \cdot 10^6$ N/m^2.

$$\sigma_{1dop} = 12 \cdot 10^7 \frac{N}{m^2}$$

$$\sigma_{2dop} = 6 \cdot 10^6 \frac{N}{m^2}$$

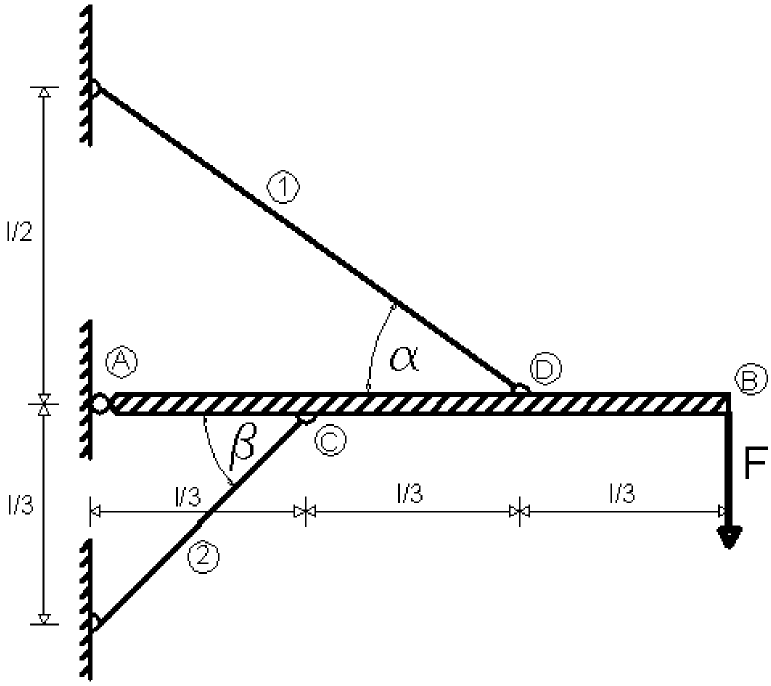
$$E_1 = 2 \cdot 10^{11} \frac{N}{m^2}$$

$$E_2 = 1 \cdot 10^{10} \frac{N}{m^2}$$

$$E_1 = 20E_2$$

$$A_2 = 10A_1$$

$$h = 2b$$



$$\sin\alpha = \frac{l/2}{5l/6} = \frac{3}{5}$$

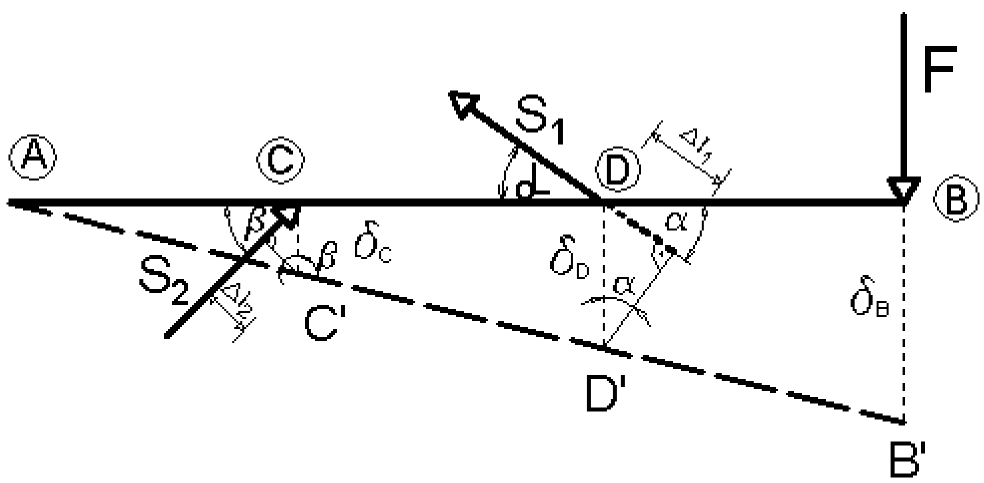
$$\cos\alpha = \frac{2l/3}{5l/6} = \frac{4}{5}$$

$$l_2 = \frac{\sqrt{2}}{3} l$$

$$l_1 = \frac{5}{6} l$$

$$\sin\beta = \cos\beta = \frac{\sqrt{2}}{2}$$

Plan pomaka :



Sile u štapovima :

$$\underline{\sum M_A = 0} \rightarrow Fl - S_1 \sin\alpha \frac{2}{3} l - S_2 \sin\beta \frac{1}{3} l = 0 \ldots (1)$$

$$\frac{\delta_c}{\frac{1}{3}l} = \frac{\delta_D}{\frac{2}{3}l} \rightarrow \delta_D = 2\delta_C \ldots (2)$$

$$\delta_C = \frac{\Delta l_2}{\sin\beta} = \frac{S_2 l_2}{E_2 A_2 \sin\beta} ; \delta_D = \frac{\Delta l_1}{\sin\alpha} = \frac{S_1 l_1}{E_1 A_1 \sin\alpha}$$

$$(2) \Rightarrow \frac{S_1 l_1}{E_1 A_1 \sin\alpha} = 2\frac{S_2 l_2}{E_2 A_2 \sin\beta} \rightarrow S_1 = 2\frac{E_1 A_1 l_2 \sin\alpha}{E_2 A_2 l_1 \sin\beta} S_2$$

$$S_1 = 2\frac{20E_2 \cdot A_1 \cdot 5l \cdot 3 \cdot 3 \cdot 2}{E_2 \cdot 10A_1 \cdot 6 \cdot \sqrt{2}l \cdot 5 \cdot \sqrt{2}} S_2 \Rightarrow S_1 = 6S_2 \rightarrow (1) \Rightarrow$$

$$\Rightarrow S_2 = \frac{F}{\frac{1}{3}\sin\beta + 4\sin\alpha} = \frac{F}{\left(\frac{12}{5} + \frac{1}{6}\sqrt{2}\right)} = 0{,}38F \rightarrow \underline{S_2 = 58{,}9kN}; \underline{S_1 = 353{,}4kN}$$

Dimenzioniranje :

$$\sigma_1 = \frac{S_1}{A_1} \le \sigma_{1dop} \rightarrow A_1 = \frac{d^2\pi}{4} \ge \frac{S_1}{\sigma_{1dop}} \rightarrow d \ge \sqrt{\frac{4S_1}{\pi\sigma_{1dop}}} = \sqrt{\frac{4 \cdot 353{,}4 \cdot 10^3}{\pi \cdot 12 \cdot 10^7}}$$

$$d \ge 0{,}06m = 6cm \rightarrow \underline{d = 7cm}$$

$$A_2 = 10A_1 = 10 \cdot 38{,}48 = 384{,}8cm^2$$

$$A_2 = b \cdot h = 2b^2 = 384{,}8cn^2 \Rightarrow b = \sqrt{\frac{A_2}{2}} = \sqrt{\frac{384{,}8}{2}} = 13{,}87cm$$

$$\underline{b = 13{,}87cm}$$

$$\underline{h = 27{,}74cm}$$

Kontrola napona :

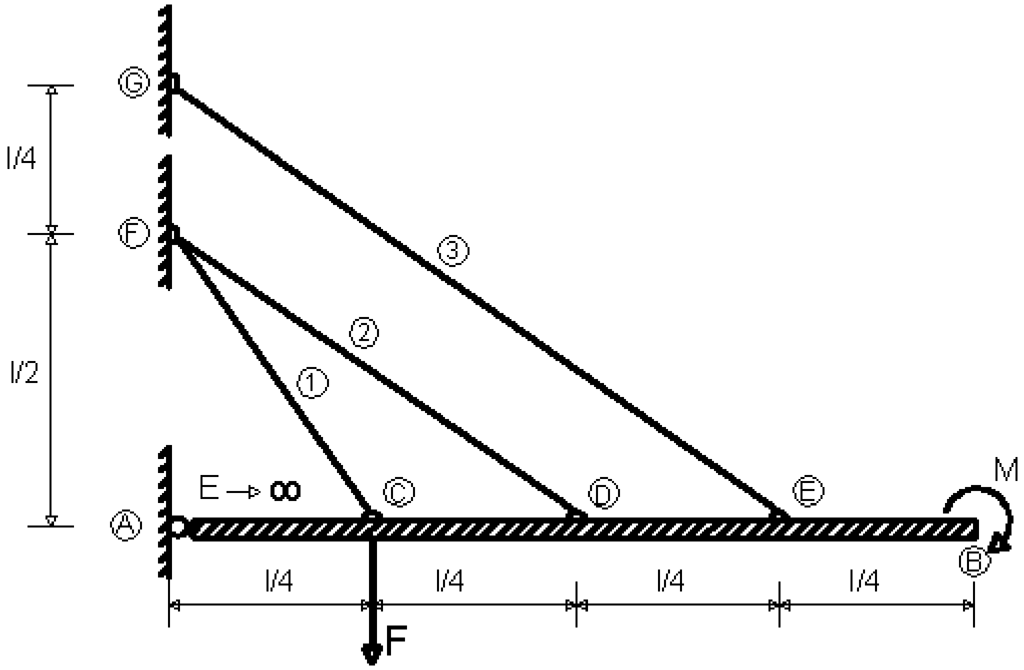
$$\sigma_1 = \frac{S_1}{A_1} = \frac{353{,}4 \cdot 10^3}{\frac{7^2 \cdot 10^{-4}\pi}{4}} = 91{,}84MPa \le \sigma_{1dop} = 120MPa$$

$$\sigma_2 = \frac{S_2}{A_2} = \frac{58{,}9 \cdot 10^3}{2 \cdot 13{,}87^2 \cdot 10^{-4}} = 1{,}53MPa \le \sigma_{2dop} = 6MPa$$

13. Zadatak

Odrediti sile u čeličnim kružnim zategama CF, DF i EG uslijed djelovanja naznačenog opterećenja.

$$M = \frac{Fl}{4}$$



Plan pomaka

$$l_1 = \sqrt{\frac{l^2}{4} + \frac{l^2}{16}} = \sqrt{\frac{5}{16} l^2} = \frac{\sqrt{5}}{4} l$$

$$l_2 = \sqrt{2 \frac{l^2}{4}} = \frac{\sqrt{2}}{2} l$$

$$l_3 = \sqrt{2 \frac{9}{16} l^2} = \frac{3}{4} \sqrt{2} l$$

$$\sin \alpha = \frac{\frac{l}{2}}{\frac{\sqrt{5}}{4} l} = \frac{2}{\sqrt{5}}$$

$$\sin \beta = \frac{\frac{l}{2}}{\frac{\sqrt{2}}{2} l} = \frac{1}{\sqrt{2}}$$

Iz uvjeta ravnoteže sila :

$$\underline{\sum M_A = 0} \rightarrow \frac{Fl}{4} + M - \frac{l}{4}\left[S_1 \sin\alpha + (2S_2 + 3S_3)\sin\beta\right] = 0 \rightarrow 2F - S_1 \sin\alpha - (2S_2 + 3S_3)\sin\beta = 0 \ldots (1)$$

Iz plana pomaka :

$$\frac{\delta_C}{\frac{l}{4}} = \frac{\delta_D}{\frac{l}{2}} = \frac{\delta_E}{3\frac{l}{4}} \ldots (2)$$

$$\delta_C = \frac{\Delta l_1}{\sin\alpha}; \delta_D = \frac{\Delta l_2}{\sin\beta}; \delta_E = \frac{\Delta l_3}{\sin\beta};$$

$$(2) \rightarrow \frac{\delta_C}{\frac{l}{4}} = \frac{\delta_D}{\frac{l}{2}} \rightarrow 2\delta_C = \delta_D \rightarrow \frac{2\Delta l_1}{\sin\alpha} = \frac{\Delta l_2}{\sin\beta} \rightarrow \frac{2S_1 l_1}{\sin\alpha} = \frac{S_2 l_2}{\sin\beta} \rightarrow S_1 = \frac{4}{5}S_2$$

$$\frac{\delta_D}{\frac{l}{2}} = \frac{\delta_E}{3\frac{l}{4}} \rightarrow 3\delta_D = 2\delta_E \rightarrow \frac{3\Delta l_2}{\sin\beta} = \frac{2\Delta l_3}{\sin\beta} \rightarrow 3\Delta l_2 = 2\Delta l_3 \rightarrow S_2 = S_3$$

$$(1) \rightarrow 2F - \frac{4}{5}S_2 \sin\alpha - (2S_2 + 3S_2)\sin\beta = 0 \rightarrow S_2 = \frac{2F}{\frac{4}{5}\sin\alpha + 5\sin\beta} \rightarrow$$

$$\underline{S_2 = S_3 = 0{,}47F}$$

$$\underline{S_1 = \frac{4}{5}S_2 = \frac{4}{5}0{,}47F = 0{,}38F}$$

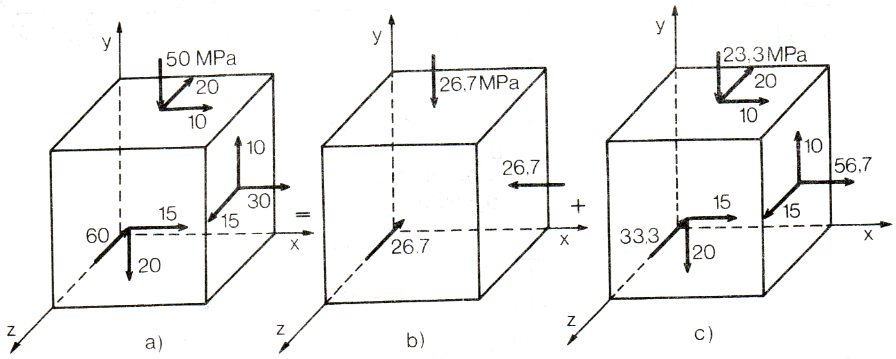
14. Zadatak

Zadani tenzor naprezanja prikazati kao zbroj sfernog tenzora naprezanja i tenzora devijatorskog naprezanja. Odrediti normalno, posmično i puno naprezanje u ravnini s normalom koja ima kosinuse smjera :
cos(x,n)=2/3 ; cos(y,n)=1/3 ; cos(z,n)=2/3.

$$[\sigma_{ij}] = \begin{bmatrix} 30 & 10 & 15 \\ 10 & -50 & -20 \\ 15 & -20 & -60 \end{bmatrix} MPa$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_x - \sigma_s & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_s & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_s \end{bmatrix} + \begin{bmatrix} \sigma_s & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & 0 & \sigma_s \end{bmatrix}$$

srednje naprezanje devijatorski dio sferni dio



$$\sigma_s = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{30 - 50 - 60}{3} = -26{,}7 MPa$$

<u>Komponente devijatorskog tenzora naprezanja :</u>

$$s_{xx} = \sigma_x - \sigma_s = 30 - (-26{,}7) = 56{,}7 MPa$$
$$s_{yy} = \sigma_y - \sigma_s = -50 - (-26{,}7) = -23{,}3 MPa$$
$$s_{zz} = \sigma_z - \sigma_s = -60 - (-26{,}7) = -33{,}3 Pa$$

$$[\sigma_{ij}] = \begin{bmatrix} 56{,}7 & 10 & 15 \\ 10 & -23{,}3 & -20 \\ 15 & -20 & -33{,}3 \end{bmatrix} + \begin{bmatrix} -26{,}7 & 0 & 0 \\ 0 & -26{,}7 & 0 \\ 0 & 0 & -26{,}7 \end{bmatrix}$$

$$\overrightarrow{\rho_n} = [\sigma_{ij}] \cdot \vec{n}$$

$$\begin{bmatrix} \rho_{nx} \\ \rho_{ny} \\ \rho_{nz} \end{bmatrix} = \begin{bmatrix} 30 & 10 & 15 \\ 10 & -50 & -20 \\ 15 & -20 & -60 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 33{,}3 \\ -23{,}3 \\ -36{,}7 \end{bmatrix}$$

$$\rho_n = \sqrt{\rho_{nx}^2 + \rho_{ny}^2 + \rho_{nz}^2} = \sqrt{33{,}3^2 + 23{,}3^2 + 36{,}7^2} = 54{,}8 MPa$$

$$\sigma_n = \overrightarrow{\rho_n} \cdot \vec{n} = 33{,}3 \cdot \frac{2}{3} - 23{,}3 \cdot \frac{1}{3} - 36{,}7 \cdot \frac{2}{3} = -10 MPa$$

$$\tau_n = \sqrt{\rho_n^2 - \sigma_n^2} = \sqrt{54{,}8^2 - 10^2} = 53{,}9 MPa$$

15. Zadatak

Za zadano stanje naprezanja $\sigma_x = 100$ MPa , $\sigma_y = -80$ MPa , $\tau_{xy} = 50$ Mpa odrediti analitički

a) glavna naprezanja
b) maksimalno posmično naprezanje s odgovarajućim normalnim naprezanjem
c) naprezanja σ_n i τ_n u presjeku s normalom koja s osi x zatvara kut $\varphi = -25°$.

<u>Glavna naprezanja i smjerovi naprezanja :</u>

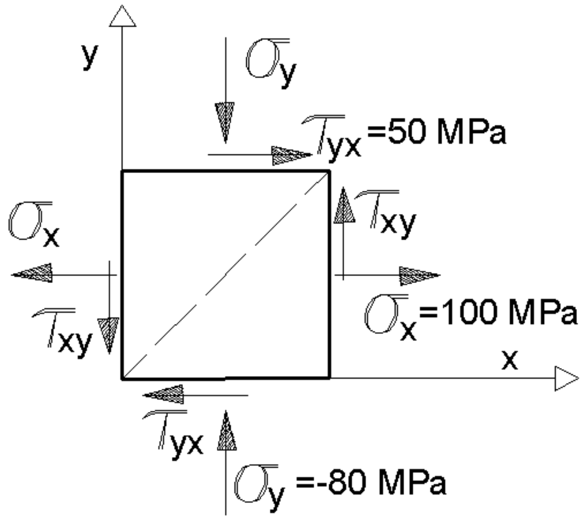
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^{\ 2}} = \frac{100-80}{2} \pm \frac{1}{2}\sqrt{(100+80)^2 + 4\cdot 50^2} = 10 + 103$$

$$\sigma_1 = 113 MPa$$

$$\sigma_2 = -93 MPa$$

$$tg\varphi_{01} = \frac{\tau_{xy}}{\sigma_1 - \sigma_y} = \frac{50}{113+80} = 0{,}2591 \Rightarrow \varphi_{01} = 14{,}5°$$

$$tg\varphi_{02} = \frac{\tau_{xy}}{\sigma_2 - \sigma_y} = \frac{50}{-93+80} = -3{,}8462 \Rightarrow \varphi_{02} = -75{,}5°$$



<u>Kontrola preko invarijante naprezanja :</u>

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \rightarrow 113 - 93 = 10 - 80 \rightarrow 20 = 20$$

$$\left|\varphi_{01}\right| + \left|\varphi_{02}\right| = 14{,}5° + 75{,}5° = 90°$$

<u>Maksimalno posmično naprezanje, smjer normale ravnine i normalno naprezanje :</u>

$$\varphi_1 = \varphi_{01} + 45° = 14{,}5° + 45° = 59{,}5°$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{113+93}{2} = 103 MPa$$

$$\sigma_s = \frac{\sigma_1 + \sigma_2}{2} = \frac{113-93}{2} = 10 MPa$$

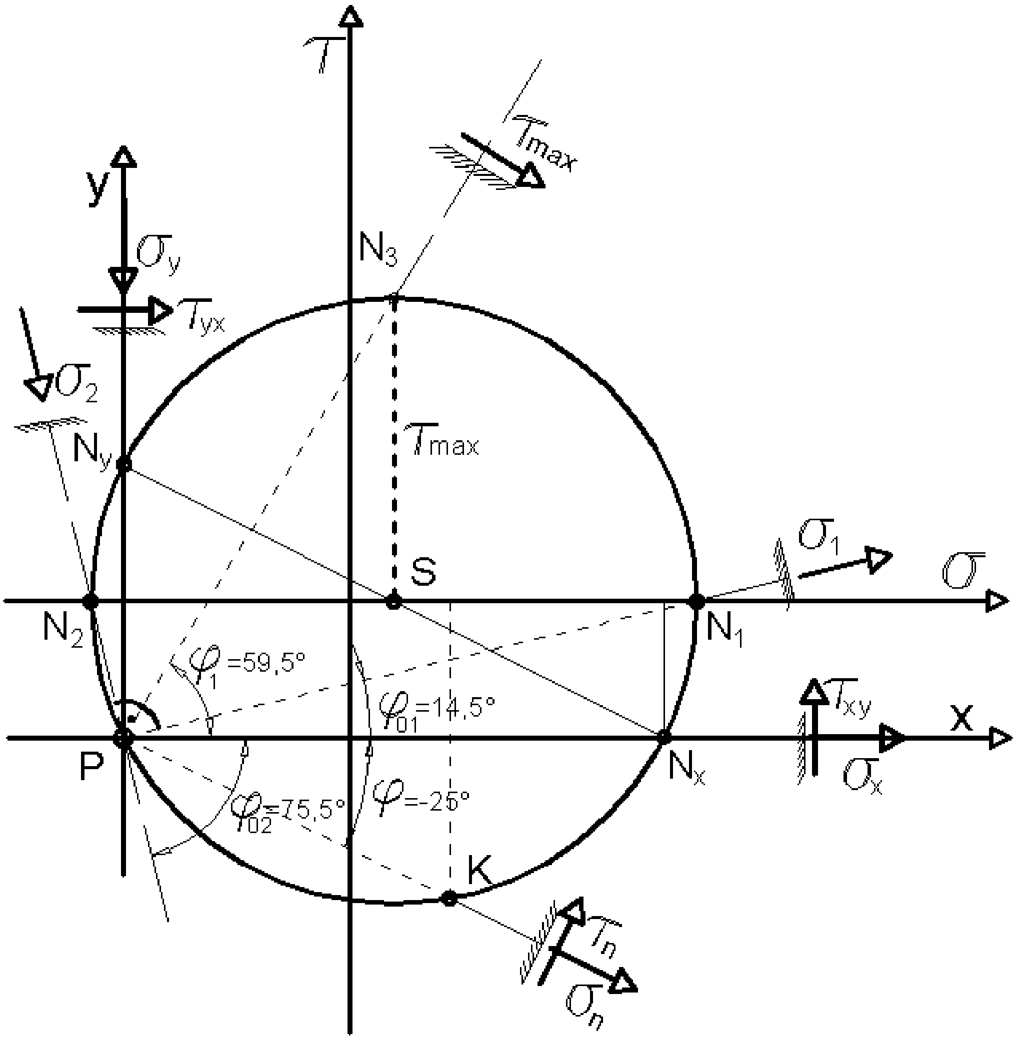
<u>Naprezanja u ravnini pod kutem :</u>

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi + \tau_{xy}\sin 2\varphi$$

$$\sigma_n = \frac{100-80}{2} + \frac{100+80}{2}\cos(-50°) + 50\sin(-50°) = 29{,}6 MPa$$

$$\tau_n = \frac{\sigma_y - \sigma_x}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi = \frac{-80-100}{2}\sin(-50°) + 50\cos(-50°) = 101{,}1 MPa$$

Prikaz Mohrove kružnice naprezanja za zadano stanje naprezanja :



$N_x(\sigma_x,\tau_{xy}) \rightarrow N_x(100,-50)$

$N_y(\sigma_y,\tau_{xy}) \rightarrow N_y(-80,50)$

$\sigma_1 = 113MPa; \varphi_{01} = 14{,}5°; N_1(\sigma_1,0) \rightarrow N_1(113,0)$

$\sigma_2 = -93MPa; \varphi_{02} = -75{,}5°; N_2(\sigma_2,0) \rightarrow N_1(-93,0)$

$\sigma_s = 10MPa; \tau_{\max} = 103MPa; \varphi_1 = 59{,}5°; N_3(\sigma_s,\tau_{\max}) \rightarrow N_1(10,103)$

$\sigma_n = 29{,}6MPa; \tau_n = 101{,}1MPa; K(\sigma_n,\tau_n) \rightarrow K(29{,}6;-101{,}1)$

16. Zadatak

Za zadano stanje naprezanja odrediti glavna normalna naprezanja, glavna posmična naprezanja i oktaedarska naprezanja.

$$\left[\sigma_{ij}\right]=\begin{bmatrix}-80 & 50 & 0\\ 50 & 120 & 0\\ 0 & 0 & 60\end{bmatrix}[MPa]$$

$$\begin{vmatrix}\sigma_x-\sigma_m & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y-\sigma_m & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z-\sigma_m\end{vmatrix}=0\Rightarrow\begin{vmatrix}\sigma_x-\sigma_m & \tau_{xy} & 0\\ \tau_{yx} & \sigma_y-\sigma_m & 0\\ 0 & 0 & \sigma_z-\sigma_m\end{vmatrix}=0$$

$$(\sigma_z-\sigma_m)\left[(\sigma_x-\sigma_m)(\sigma_y-\sigma_m)-\tau_{xy}^{\ 2}\right]=0$$

$$\sigma_z-\sigma_m=0\Rightarrow\sigma_m=\sigma_z=60MPa=\sigma_3{}'$$

$$(\sigma_x-\sigma_m)(\sigma_y-\sigma_m)-\tau_{xy}^{\ 2}=0$$

$$\sigma_m^{\ 2}-(\sigma_x+\sigma_y)\sigma_m+\sigma_x\sigma_y-\tau_{xy}^{\ 2}=0\rightarrow\sigma_m=\frac{\sigma_x+\sigma_y\pm\sqrt{\left[-(\sigma_x+\sigma_y)\right]^2-4\left(-\tau_{xy}^{\ 2}+\sigma_x\sigma_y\right)}}{2}$$

$$\sigma_m=\frac{\sigma_x+\sigma_y}{2}\pm\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^{\ 2}}$$

$$\sigma_{1,2}{}'=\frac{-80+120}{2}\pm\frac{1}{2}\sqrt{\left[(-80-120)\right]^2+4\cdot 50^2}=20\pm 111{,}8$$

$$\sigma_1{}'=131{,}8MPa;\sigma_2{}'=-91{,}8MPa;\sigma_3{}'=60MPa$$

$$\sigma_1=131{,}8MPa$$
$$\sigma_2=60MPa$$
$$\sigma_3=-91{,}8MPa$$

Kontrola naprezanja :

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z$$
$$-80 + 120 + 60 = 131{,}8 + 60 - 91{,}8$$
$$\underline{\underline{100MPa}} = \underline{\underline{100MPa}}$$

Pravac glavnog naprezanja σ_2 podudara se s pravcem osi z. Pravci glavnih naprezanja σ_1 i σ_3 okomiti su na pravac naprezanja σ_2, a njihov se položaj u ravnini okomitoj na pravac σ_2 određuje prema izrazima :

$$tg\varphi_{01} = \frac{\tau_{xy}}{\sigma_1 - \sigma_y} = \frac{50}{131{,}8 - 120} = 4{,}237 \Rightarrow \varphi_{01} = 76{,}7°$$
$$tg\varphi_{03} = \frac{\tau_{xy}}{\sigma_3 - \sigma_y} = \frac{50}{-91{,}8 - 120} = -0{,}236 \Rightarrow \varphi_{03} = -13{,}3°$$
$$|\varphi_{01}| + |\varphi_{03}| = 76{,}7° + 13{,}3° = 90°$$

Maksimalna posmična naprezanja :

$$\tau_1 = \pm\frac{\sigma_2 - \sigma_3}{2} = \pm\frac{60 + 91{,}8}{2} = \pm 75{,}9MPa$$
$$\tau_2 = \pm\frac{\sigma_3 - \sigma_1}{2} = \pm\frac{-91{,}8 - 131{,}8}{2} = \pm 111{,}8MPa$$
$$\tau_3 = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm\frac{131{,}8 - 60}{2} = \pm 35{,}9MPa$$

$$\tau_{\max} = |\tau_2| = 111{,}8MPa$$

Oktaedarska naprezanja :

Budući da normala oktaedarske ravnine zatvara s koordinatnim osima kuteve :

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \to 3\cos^2\alpha = 1 \to \cos\alpha = \pm\frac{1}{\sqrt{3}} \to \alpha = 54{,}7°$$
$$\sigma_{okt} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(131{,}8 + 60 - 91{,}8)$$
$$\sigma_{okt} = 33{,}3MPa$$

$$\tau_{okt} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{1}{3}\sqrt{(131{,}8 - 60)^2 + (60 + 91{,}8)^2 + (-91{,}8 - 131{,})^2}$$
$$\tau_{okt} = 93{,}2MPa$$

$$\rho_{okt} = \sqrt{\sigma_{okt}{}^2 + \tau_{okt}{}^2} = \sqrt{33{,}3^2 + 93{,}2^2} = 99{,}0MPa$$

17. Zadatak

Ploča je izložena djelovanju vlačnog naprezanja u oba pravca i tangencijalnog naprezanja u bočnim ravninama.Odrediti komponente naprezanja u presjeku čija normala zatvara kut od 30° s osi x i smjer i veličinu glavnih naprezanja.

$\sigma_x = 8MPa$

$\sigma_y = 4MPa$

$\tau_{xy} = -2MPa$

$\varphi = 30°$

$$\sigma_n = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi + \tau_{xy}\sin 2\varphi = 8\cos^2 30° + 4\sin^2 30° - 2\sin 60°$$

$\sigma_n = 5{,}27MPa$

$$\tau_n = \frac{\sigma_y - \sigma_x}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi = \frac{4-8}{2}\sin 60° - 2\cos 60°$$

$\tau_n = -2{,}73MPa$

$$\rho_n = \sqrt{\sigma_n^{\ 2} + \tau_n^{\ 2}} = \sqrt{5{,}27^2 + (-2{,}73)^2}$$

$\rho_n = 5{,}93MPa$

<u>Glavna naprezanja :</u>

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^{\ 2}} = \frac{8+4}{2} \pm \frac{1}{2}\sqrt{(8-4)^2 + 4\cdot(-2)^2} = 6 \pm 2{,}83$$

$\sigma_1 = 8{,}83MPa$

$\sigma_2 = 3{,}17MPa$

$$tg\varphi_{01} = \frac{\tau_{xy}}{\sigma_1 - \sigma_y} = \frac{-2}{8{,}83-4} = -0{,}414 \Rightarrow \varphi_{01} = -22{,}5°$$

$$tg\varphi_{02} = \frac{\tau_{xy}}{\sigma_2 - \sigma_y} = \frac{-2}{3{,}17-4} = 2{,}41 \Rightarrow \varphi_{02} = 67{,}5°$$

<u>Kontrola preko invarijante naprezanja :</u>

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \rightarrow 8{,}8 + 3{,}2 = 8 + 4 \rightarrow 12 = 12$$

$$|\varphi_{01}| + |\varphi_{02}| = 22{,}5° + 67{,}5° = 90°$$

<u>Maksimalno posmično naprezanje, smjer normale ravnine i pripadajuće normalno naprezanje :</u>

$$\varphi_1 = \varphi_{01} + 45° = -22{,}6° + 45° = 22{,}4°$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{8{,}8 - 3{,}2}{2} = 2{,}8MPa$$

$$\sigma_s = \frac{\sigma_1 + \sigma_2}{2} = \frac{8{,}8 + 3{,}2}{2} = 6MPa$$

18. Zadatak

Kratki betonski stup, kvadratnog poprečnog presjeka 30x30 cm pritisnut je silom F.Odrediti veličinu ove sile ako je normalno naprezanje u kosom presjeku pod kutem od 30°prema normali 1,5 Mpa.Koliko je tangencijalno naprezanje u tom presjeku?

$$\sigma_n = 1{,}5 MPa$$
$$\varphi = 30°$$
$$a = 30cm = 0{,}3m$$
$$A = a^2 = 0{,}09m^2$$

$$\sigma_n = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi + \tau_{xy}\sin 2\varphi$$
$$\sigma_n = \sigma_x \cos^2\varphi = -\frac{F}{A}\cos^2\varphi \rightarrow F = -\frac{\sigma_n A}{\cos^2\varphi} = -\frac{1{,}5\cdot 10^6 \cdot 0{,}09}{\cos^2 30°}$$
$$F = -180kN$$
$$\sigma_x = \frac{F}{A} = -\frac{180\cdot 10^3}{0{,}09} = -2MPa$$
$$\tau_n = \frac{\sigma_y - \sigma_x}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi$$
$$\tau_n = \frac{-\sigma_x}{2}\sin 2\varphi = -\frac{-2\cdot 10^6}{2}\sin 60° = 0{,}86MPa$$

19. Zadatak

Stup kružnog poprečnog presjeka opterećen je vlačnom silom intenziteta 20 kN. Tangencijalno naprezanje na bilo kojem presjeku (ravnini) ne smije prijeći vrijednost od 7Mpa. Odrediti polumjer stupa.

$$F = 20kN$$
$$\tau_n = 7MPa$$

$$\tau_n \le \frac{\sigma_x}{2}\sin 2\varphi = \frac{F}{2A}\sin 2\varphi = \frac{2F}{d^2\pi}\sin 2\varphi \rightarrow d \ge \sqrt{\frac{2F}{\tau_n \pi}\sin 2\varphi} = \sqrt{\frac{2\cdot 2\cdot 10^4}{7\cdot 10^6\pi}\sin 2\varphi}$$
$$(\sin 2\varphi)_{\max} = 1$$

$$d \ge \sqrt{\frac{2\cdot 2\cdot 10^4}{7\cdot 10^6\pi}} \rightarrow d \ge 4{,}3cm \rightarrow d = 4{,}5cm$$

20. Zadatak

U točki A napregnutog elementa izmjerene su dužinske deformacije u smjerovima x, y i n $\varepsilon_{xx} = -8 \cdot 10^{-4}$, $\varepsilon_{yy} = 4 \cdot 10^{-4}$, $\varepsilon_{nn} = 6 \cdot 10^{-4}$. Odrediti promjene pravog kuta između osi x i y, te veličinu i smjerove glavnih deformacija.

$$\varepsilon_{nn} = \varepsilon_{xx} \cos^2 \varphi + \varepsilon_{yy} \sin^2 \varphi + \varepsilon_{xy} \sin 2\varphi \Rightarrow$$

$$\Rightarrow \varepsilon_{xy} = \frac{1}{\sin 2\varphi}\left(\varepsilon_{nn} - \varepsilon_{xx} \cos^2 \varphi - \varepsilon_{yy} \sin^2 \varphi\right) = \frac{1}{\sin 60°}\left(6 \cdot 10^{-4} + 8 \cdot 10^{-4} \cos 30° - 4 \cdot 10^{-4} \sin 30°\right)$$

$$\varepsilon_{xy} = 12{,}7 \cdot 10^{-4}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = 25{,}4 \cdot 10^{-4}\, rad$$

Glavne deformacije :

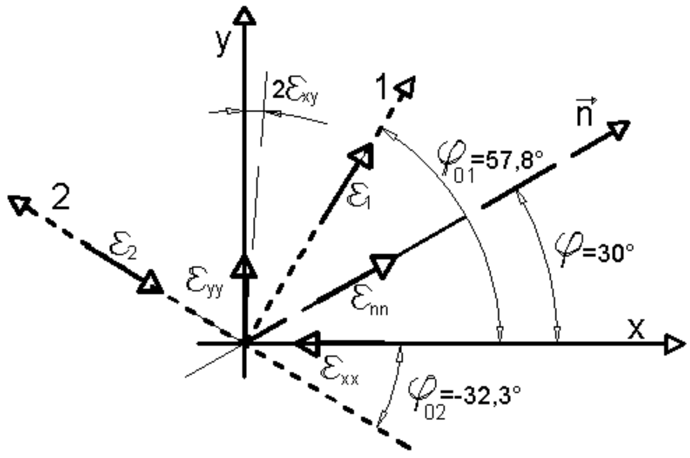
$$\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \frac{1}{2}\sqrt{\left(\varepsilon_{xx} - \varepsilon_{yy}\right)^2 + 4\varepsilon_{xy}{}^2}$$

$$\varepsilon_{1,2} = \left(\frac{-8 + 4}{2} \pm \frac{1}{2}\sqrt{(-8 - 4)^2 + 4(12{,}7)^2}\right) \cdot 10^{-4}$$

$$\varepsilon_{1,2} = -2 \cdot 10^{-4} \pm 14 \cdot 10^{-4}$$

$$\varepsilon_1 = 12 \cdot 10^{-4}$$

$$\varepsilon_2 = -16 \cdot 10^{-4}$$



Kontrola :

$$\varepsilon_{xx} + \varepsilon_{yy} = \varepsilon_1 + \varepsilon_2$$

$$-8 \cdot 10^{-4} + 4 \cdot 10^{-4} = 12 \cdot 10^{-4} - 16 \cdot 10^{-4}$$

$$\underline{\underline{-4 \cdot 10^{-4}}} = \underline{\underline{-4 \cdot 10^{-4}}}$$

Smjerovi glavnih deformacija :

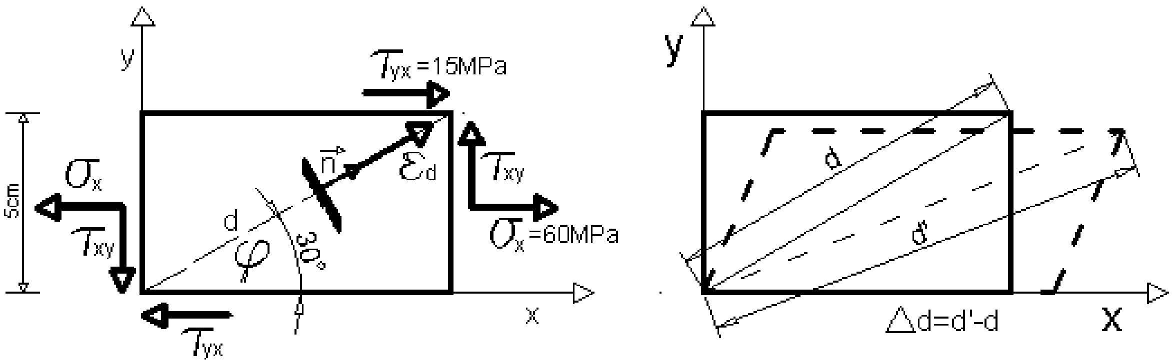
$$tg\varphi_{01} = \frac{\varepsilon_{xy}}{\varepsilon_1 - \varepsilon_{yy}} = \frac{12{,}7 \cdot 10^{-4}}{12 \cdot 10^{-4} - 4 \cdot 10^{-4}} = 1{,}588 \Rightarrow \varphi_{01} = 57{,}8°$$

$$tg\varphi_{02} = \frac{\varepsilon_{xy}}{\varepsilon_2 - \varepsilon_{yy}} = \frac{12{,}7 \cdot 10^{-4}}{-16 \cdot 10^{-4} - 4 \cdot 10^{-4}} = -0{,}635 \Rightarrow \varphi_{02} = -32{,}4°$$

$$|\varphi_{01}| + |\varphi_{02}| = 57{,}8° + 32{,}4° = 90{,}2° \approx 90°$$

21. Zadatak

Odrediti deformaciju dijagonale čeličnog elementa.



$E = 2 \cdot 10^5 \, MPa$

$\nu = 0{,}3$

$\sigma_x = 60 MPa$

$\tau_{xy} = 15 MPa$

$$\varepsilon_{xx} = \frac{\sigma_x}{E} = \frac{60}{2 \cdot 10^5} = 30 \cdot 10^{-5}$$

$$\varepsilon_{yy} = -\nu \frac{\sigma_x}{E} = -0{,}3 \frac{60}{2 \cdot 10^5} = -9 \cdot 10^{-5}$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{2G} = \frac{1+\nu}{E} \tau_{xy} = \frac{1+0{,}3}{2 \cdot 10^5} 15 = 9{,}75 \cdot 10^{-5}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = 19{,}5 \cdot 10^{-5}$$

Relativna deformacija dijagonale :

$$\varepsilon_d = \varepsilon_{xx} \cos^2 \varphi + \varepsilon_{yy} \sin^2 \varphi + \varepsilon_{xy} \sin 2\varphi = 30 \cdot 10^{-5} \cos^2 30° - 9 \cdot 10^{-5} \sin^2 30° + 9{,}75 \cdot 10^{-5} \sin 60°$$

$$\varepsilon_d = 28{,}69 \cdot 10^{-5}$$

$$\sin 30° = \frac{5}{d} \rightarrow d = \frac{5}{\sin 30°} = 10 cm$$

Apsolutna deformacija :

$$\Delta d = \varepsilon_d \cdot d = 28{,}69 \cdot 10^{-5} \cdot 100 = 28{,}69 \cdot 10^{-3} mm$$

22. Zadatak

Pravokutna pločica dimenzija 120x90 mm opterećena je u dva pravca tako da su

$E = 2 \cdot 10^5 MPa$

$\nu = 0{,}3$

$\sigma_x = 10000 \dfrac{N}{cm^2} = 100 MPa$

$\sigma_y = -4000 \dfrac{N}{cm^2} = -40 MPa$

Odrediti kolika su naprezanja u kosoj ravnini koja se poklapa s dijagonalom koja s osi x zatvara oštri kut i za koliko će se promijeniti dužina dijagonale.

$$tg\alpha = \frac{9}{12} = 0{,}75 \rightarrow \alpha = 37° \Rightarrow \varphi = \alpha - 90° = -53°$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi = \frac{100-40}{2} + \frac{100+40}{2}\cos(-106°) = 10{,}7 MPa$$

$$\tau_n = \frac{\sigma_y - \sigma_x}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi = \frac{-40-100}{2}\sin(-106°) = 67{,}2 MPa$$

$$\rho_n = \sqrt{\sigma_n^{\ 2} + \tau_n^{\ 2}} = \sqrt{10{,}7^2 + 67{,}2^2} = 68{,}0 MPa$$

$$\varepsilon_{xx} = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{100}{2 \cdot 10^5} - 0{,}3\frac{(-40)}{2 \cdot 10^5} = 56 \cdot 10^{-5}$$

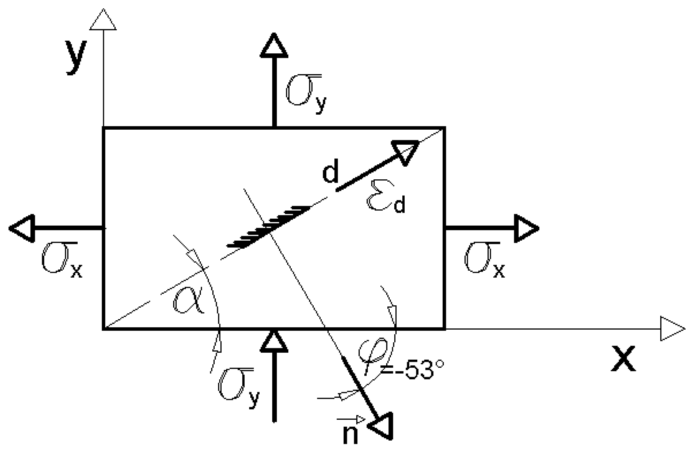
$$\varepsilon_{yy} = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} = \frac{-40}{2 \cdot 10^5} - 0{,}3\frac{100}{2 \cdot 10^5} = -35 \cdot 10^{-5}$$

Budući da tražimo izduženje dijagonale, tražimo deformaciju u smjeru normale ravnine okomite na dijagonalu, te je sada kut φ = α = 37°:

$$\varepsilon_d = \varepsilon_{xx}\cos^2\varphi + \varepsilon_{yy}\sin^2\varphi + \varepsilon_{xy}\sin 2\varphi = 56 \cdot 10^{-5}\cos^2(37°) - 35 \cdot 10^{-5}\sin^2(37°) = 23{,}04 \cdot 10^{-5}$$

$$d = \sqrt{90^2 + 120^2} = 150 mm$$

$$\Delta d = \varepsilon_d \cdot d = 23{,}04 \cdot 10^{-5} \cdot 150 = 3{,}46 \cdot 10^{-2} mm$$



23. Zadatak

Čelična kocka brida a = 10 cm postavljena je bez zazora između dviju krutih stijenki na podlogu i na gornjoj plohi opterećena s q = 60 MPa.

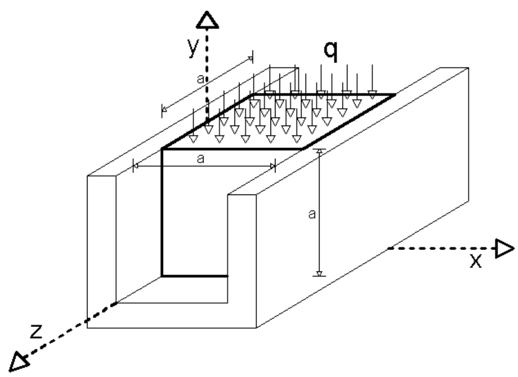
Odrediti : a) naprezanja i deformacije u tri okomita smjera i deformaciju dijagonale kocke
b) promjenu volumena kocke
c) normalno i posmično naprezanje u presjeku pod kutem od 45° prema stjenki.

$E = 2 \cdot 10^5 MPa$

$\nu = 0{,}30$

$\sigma_y = -q = -60 MPa$

$\sigma_z = 0$



$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_z + \sigma_y\right)\right] = 0 \rightarrow \sigma_x = \nu\sigma_y = 0{,}3 \cdot (-60) = -18 MPa$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] = \frac{1}{2 \cdot 10^5}\left[-60 - 0{,}3(-18)\right] = -27{,}3 \cdot 10^{-5}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = \frac{1}{2 \cdot 10^5}\left[-0{,}3(-18-60)\right] = 11{,}7 \cdot 10^{-5}$$

$d = a\sqrt{3}$

$$\varepsilon_d = \varepsilon_{xx}\cos^2\alpha + \varepsilon_{yy}\cos^2\beta + \varepsilon_{zz}\cos^2\gamma = 0 - 27{,}3 \cdot 10^{-5} \cdot \frac{1}{3} + 11{,}7 \cdot 10^{-5} \cdot \frac{1}{3}$$

$\varepsilon_d = -5{,}2 \cdot 10^{-5}$

$\Delta d = \varepsilon_d \cdot d = -5{,}2 \cdot 10^{-5} \cdot 10^2 \cdot \sqrt{3} = -9 \cdot 10^{-3} mm$

Relativna i apsolutna promjena volumena :

$$\varepsilon_V = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0 - 27{,}3 \cdot 10^{-5} + 11{,}7 \cdot 10^{-5} = -15{,}6 \cdot 10^{-5}$$

$\Delta V = \varepsilon_V \cdot V = -15{,}6 \cdot 10^{-5} \cdot 10^3 = -0{,}15 cm$

$$\overrightarrow{\rho_n} = \left[\sigma_{ij}\right] \cdot \vec{n} = \begin{bmatrix} -18 & 0 & 0 \\ 0 & -60 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos 45° \\ \cos 45° \\ \cos 90° \end{bmatrix}$$

$\rho_{nx} = -12{,}72 MPa$

$\rho_{ny} = -42{,}42 MPa$

$\rho_{nz} = 0$

$\sigma_n = \overrightarrow{\rho_n} \cdot \vec{n} = -12{,}72 \cdot \cos 45° - 42{,}42 \cdot \cos 45° = -39 MPa$

$\rho_n = \sqrt{\rho_{nx}^2 + \rho_{ny}^2 + \rho_{nz}^2} = \sqrt{12{,}72^2 + 42{,}42^2 + 0^2} = 44{,}28 MPa$

$\tau_n = \sqrt{\rho_n^2 - \sigma_n^2} = 21 MPa$

24. Zadatak

Odrediti glavna naprezanja na stranicama kvadratnog elementa ako su tenzometri (uređaji za mjerenje dužinske deformacije) A i B pokazali prirast Δn_A = 9,9 mm i Δn_B = 3,1 mm. Tenzometar A postavljen je pod kutem φ = 30° prema pravcu glavnog naprezanja σ_1, a tenzometar B okomito na tenzometar A. Baza tenzometra l_0 = 20 mm, a uvećanje tenzometra k = 1000. E = $0{,}8 \cdot 10^5$ Mpa, ν=0,35

Relativne deformacije :

$$\varepsilon_{nnA} = \frac{\Delta l}{l_0} = \frac{\Delta n_A}{k \cdot l_0} = \frac{9{,}9}{10^3 \cdot 20} = 4{,}95 \cdot 10^{-4}$$

$$\varepsilon_{nnB} = \frac{\Delta n_B}{k \cdot l_0} = \frac{3{,}1}{10^3 \cdot 20} = 1{,}55 \cdot 10^{-4}$$

Za ravninsko stanje deformacija :

$$\sigma_{nA} = \frac{E}{1-\nu^2}\left(\varepsilon_{nnA} + \nu\varepsilon_{nnB}\right) = \frac{0{,}8 \cdot 10^5}{1-0{,}35^2}\left(4{,}95 \cdot 10^{-4} + 0{,}35 \cdot 1{,}55 \cdot 10^{-4}\right) = 50\,MPa$$

$$\sigma_{nB} = \frac{E}{1-\nu^2}\left(\varepsilon_{nnB} + \nu\varepsilon_{nnA}\right) = \frac{0{,}8 \cdot 10^5}{1-0{,}35^2}\left(1{,}55 \cdot 10^{-4} + 0{,}35 \cdot 4{,}95 \cdot 10^{-4}\right) = 30\,MPa$$

$$\sigma_n = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi + \tau_{xy} \sin 2\varphi$$

$$\sigma_{nA} = \sigma_1 \cos^2\varphi + \sigma_2 \sin^2\varphi = \sigma_1 \cos^2 30° + \sigma_2 \sin^2 30° \ \ldots (1)$$

$$\left(\cos(\varphi + 90°)\right)^2 = \left(-\sin\varphi\right)^2 = \sin^2\varphi$$

$$\left(\sin(\varphi + 90°)\right)^2 = \left(\cos\varphi\right)^2 = \cos^2\varphi$$

$$\sigma_{nB} = \sigma_1 \cos^2(\varphi + 90°) + \sigma_2 \sin^2(\varphi + 90°) = \sigma_1 \sin^2 30° + \sigma_2 \cos^2 30° \ \ldots (2)$$

$$0{,}75\sigma_1 + 0{,}25\sigma_2 = 50 \ \ldots (1)$$

$$0{,}25\sigma_1 + 0{,}75\sigma_2 = 30 \ \ldots (2) \Rightarrow$$

$$\sigma_1 = 60\,MPa$$

$$\sigma_2 = 20\,MPa$$

Kontrola :

$$\sigma_{nA} + \sigma_{nB} = 50 + 30 = 60 + 20 = 80\,MPa$$

25. Zadatak

Kocka dužine stranice "a", izložena je djelovanju jednolikog tlaka intenziteta p = - 700 N/cm^2. Odrediti Poissonov koeficijent ako je relativna promjena volumena -9·10^{-5}, a modul elastičnosti materijala 0,7·10^4 kN/cm^2.

$$p = -700 \frac{N}{cm^2} = -7 \cdot 10^6 \frac{N}{m^2}$$
$$E = 0{,}7 \cdot 10^4 \frac{kN}{cm^2} = 0{,}7 \cdot 10^{11} \frac{N}{m^2}$$
$$\varepsilon_V = -9 \cdot 10^{-5}$$

$$\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$
$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma = p$$
$$\varepsilon_V = \frac{1-2\nu}{E} 3p \rightarrow \nu = \frac{1}{2}\left(1 - \frac{\varepsilon_V E}{3p}\right) = \frac{1}{2}\left(1 - \frac{(-9 \cdot 10^{-5}) \cdot 0{,}7 \cdot 10^{11}}{3(-7 \cdot 10^6)}\right)$$
$$\nu = 0{,}35$$

26. Zadatak

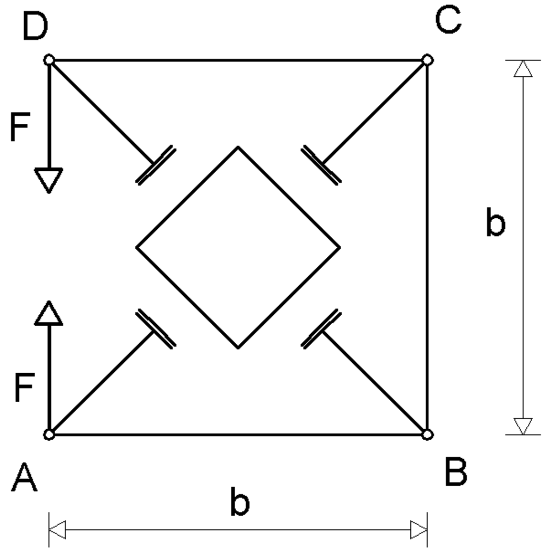
U gredi čeličnog mosta, pri prolazu vlaka, izmjerena su pomoću tenzometra relativna izduženja u horizontalnom pravcu (u smjeru osi x ili paralelno s osi grede) 0,0004 i vertikalnom pravcu (u smjeru osi y ili okomito na os grede) -0,00012. Odrediti normalna naprezanja u pravcu osi grede i okomito na nju.

$$\varepsilon_x = 0{,}0004$$
$$\varepsilon_y = -0{,}00012$$
$$E = 2{,}1 \cdot 10^5 MPa$$
$$\nu = 0{,}3$$
$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \rightarrow \sigma_x = E\varepsilon_x + \nu\sigma_y \ldots (1)$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \rightarrow \sigma_y = E\varepsilon_y + \nu\sigma_x \ldots (2)$$
$$(2) \rightarrow (1) \Rightarrow \sigma_x = \frac{E(\varepsilon_x + \nu\varepsilon_y)}{1-\nu^2} = \frac{2{,}1 \cdot 10^{11}(0{,}0004 - 0{,}3 \cdot 0{,}00012)}{1 - 0{,}3^2}$$
$$\sigma_x = 84 \cdot 10^6 \frac{N}{m^2} = 84 MPa$$

$$\sigma_y = E\varepsilon_y + \nu\sigma_x = -0{,}00012 \cdot 2{,}1 \cdot 10^{11} + 0{,}3 \cdot 84 \cdot 10^6 = 0 MPa$$

27. Zadatak

Za opterećenje na tlak betonske kocke (stranice 7 cm) postavljene su na njene četiri strane papučice od čelika spojene međusobno zglobnim mehanizmom. Dvije sile veličine F djeluju u čvorovima A i D. Odrediti za koliko se promijeni volumen kocke ako je E =4 ·10^3 KN/cm^2, v = 0,3 i F = 50 kN.



čvorA

$$\Sigma y = 0 : F + S_A \sin\alpha = 0 \rightarrow S_A = -\frac{F}{\sin\alpha}$$

$$\Sigma x = 0 : F_{AB} + S_A \cdot \cos\alpha \rightarrow F_{AB} = F$$

čvorB

$$\Sigma x = 0 \rightarrow S_B = -\frac{F}{\cos\alpha} = S_A$$

$$\Sigma y = 0 \rightarrow F_{BC} = F$$

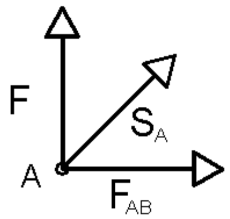
$$\alpha = 45°$$

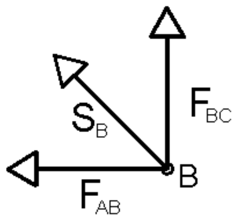
$$\Delta V = \varepsilon_V \cdot V = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)V = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)V = \frac{1-2\nu}{E}2\sigma V$$

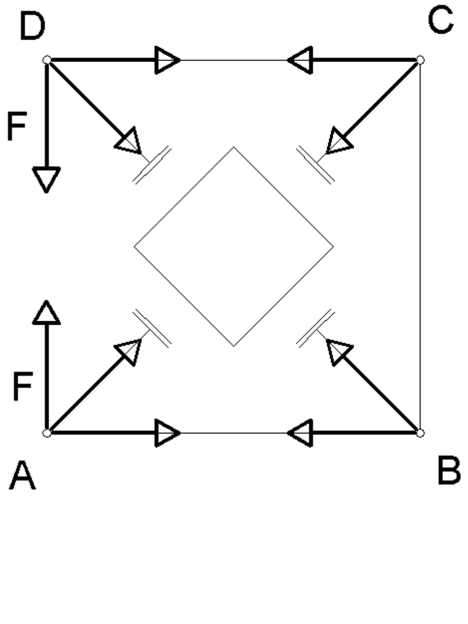
$$\sigma_1 = \sigma_2 = \frac{-S}{a^2} = \frac{-2F}{\sqrt{2}a^2}; \sigma_3 = 0; V = a^3$$

$$\Delta V = \frac{-4(1-2\nu)aF}{\sqrt{2}E} = \frac{-4(1-2\cdot 0,3)\cdot 0,07\cdot 50\cdot 10^3}{\sqrt{2}\cdot 4\cdot 10^{10}}$$

$$\Delta V = -9,9\cdot 10^{-8}\, m^3$$







28. Zadatak

Odrediti potrebne duljine l_1 i l_2 zavara spajanjem kutnih metalnih profila s pločom.

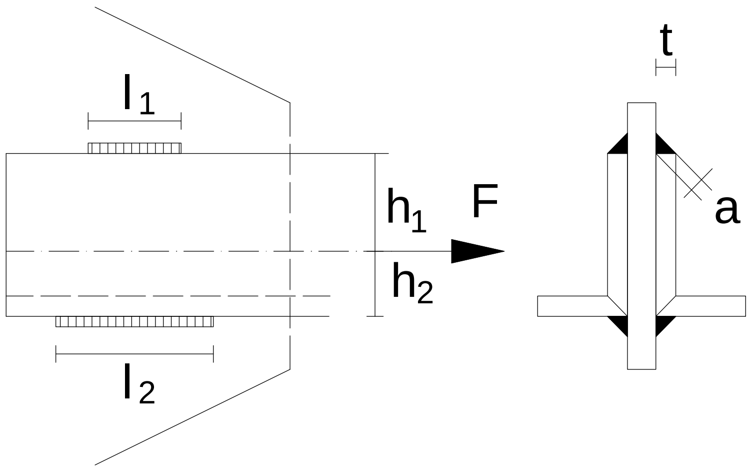
$F = 120kN$

$t = 0{,}5cm$

$h_1 = 3{,}6cm$

$h_2 = 1{,}4cm$

$\tau_{dop} = 90MPa$



$$\tau = \frac{F}{A} = \frac{F}{\sum al} = \frac{F}{2(l_1 + l_2)(t\cos 45°)} \leq \tau_{dop} \Rightarrow l_1 + l_2 \geq \frac{F}{2 \cdot 0{,}7t \cdot \tau_{dop}} = \frac{120 \cdot 10^3}{1{,}4 \cdot 0{,}5 \cdot 10^{-2} \cdot 90 \cdot 10^6} = 19cm$$

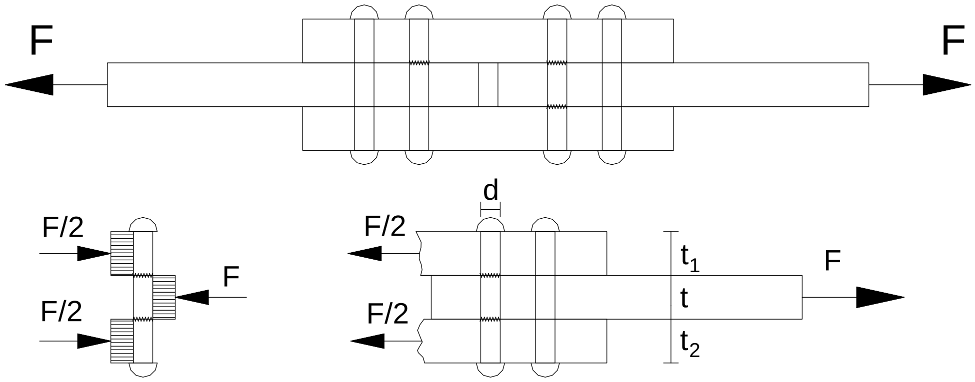
$$\underline{\sum M_A = 0} \rightarrow F_1 \cdot h_1 = F_2 \cdot h_2 \rightarrow (\tau_{dop} 2l_1 a)h_1 = (\tau_{dop} 2l_2 a)h_2 \Rightarrow$$

$$\Rightarrow l_1 \cdot h_1 = l_2 \cdot h_2 \Rightarrow \frac{l_1}{l_2} = \frac{h_2}{h_1} = \frac{1{,}4}{3{,}6} \cong 0{,}4 \rightarrow l_1 = 0{,}4l_2$$

$$l_1 + l_2 = 19cm \rightarrow 0{,}4l_2 + l_2 = 19cm \rightarrow 1{,}4l_2 = 19cm \rightarrow \underline{l_2 = \frac{19}{1{,}4} = 13{,}5cm} \rightarrow \underline{l_1 = 19 - 13{,}5 = 5{,}5cm}$$

29. Zadatak

Odrediti potreban broj zakovica promjera 20 mm, ako je



$F = 300kN$

$t_1 = 8mm$

$t = 12mm$

$\tau_{dop} = 100MPa$

$\sigma_{dop} = 280MPa$

n – broj zakovica

m – reznost zakovica

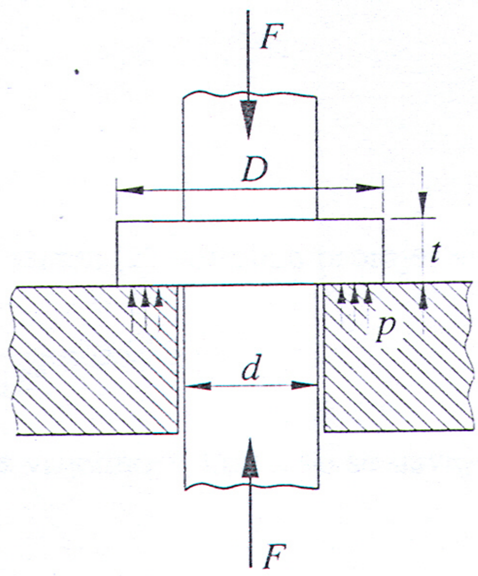
$$\tau = \frac{F}{A_s} = \frac{F}{nmA_1} = \frac{F}{nm\frac{d^2\pi}{4}} \leq \tau_{dop} \rightarrow n \geq \frac{F}{m\frac{d^2\pi}{4}\tau_{dop}} = \frac{300 \cdot 10^3}{2\frac{\left(20 \cdot 10^{-3}\right)^2 \pi}{4} 100 \cdot 10^6}$$

$$n \geq 5 \rightarrow n = 6 \rightarrow 2n = 12 \rightarrow 2reda$$

$$\sigma = \frac{F}{A_p} = \frac{F}{ndt} = \frac{300 \cdot 10^3}{6 \cdot 20 \cdot 10^{-3} \cdot 12 \cdot 10^{-3}} = 208MPa \leq \sigma_{dop} = 280MPa$$

30. Zadatak

Na vijak promjera d djeluje tlačna sila F koja izaziva naprezanje u vijku σ i površinski pritisak p između prstena promjera D i lima. Odrediti promjer D, te posmično naprezanje u prstenu ako je prsten debljine



$d = 10cm$

$\sigma = 100MPa$

$p = 40MPa$

$t = 5cm$

$$p = \frac{F}{A_p} = \frac{4F}{\left(D^2 - d^2\right)\pi} \rightarrow D = \sqrt{d^2 + \frac{4F}{\pi p}}$$

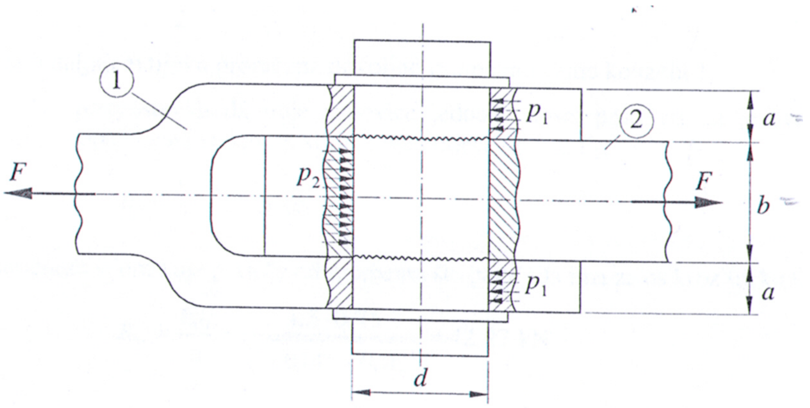
$$\sigma = \frac{F}{A} \rightarrow F = \sigma A = \sigma \frac{d^2 \pi}{4}$$

$$D = \sqrt{d^2 + \frac{4\sigma d^2 \pi}{4\pi p}} = d\sqrt{1 + \frac{\sigma}{p}} = 10\sqrt{1 + \frac{100}{40}} = 18,7cm$$

$$\tau = \frac{F}{A_s} = \frac{\sigma d^2 \pi}{4d\pi t} = \frac{\sigma d}{4t} = \frac{100 \cdot 10^6 \cdot 0,1}{4 \cdot 0,05} = 5 \cdot 10^7 = 50MPa$$

31. Zadatak

Metalna vilica i pločica 1 i 2 spojene su vijkom i opterećene prema slici. Odrediti promjer vijka d, ako je



$F = 150\,kN$

$\tau_{sdop} = 70\,MPa$

$p_{dop} = 150\,MPa$

$a = 15\,mm$

$b = 35\,mm$

$$\tau = \frac{F}{A_s} = \frac{F}{mA} = \frac{F}{m\dfrac{d^2\pi}{4}} \leq \tau_{sdop} \rightarrow d \geq \sqrt{\frac{4F}{m\pi\tau_{sdop}}} = \sqrt{\frac{4 \cdot 150 \cdot 10^3}{2 \cdot \pi \cdot 70 \cdot 10^6}} = 0{,}0369\,m$$

$$p = \frac{F}{A_p} \leq p_{dop}$$

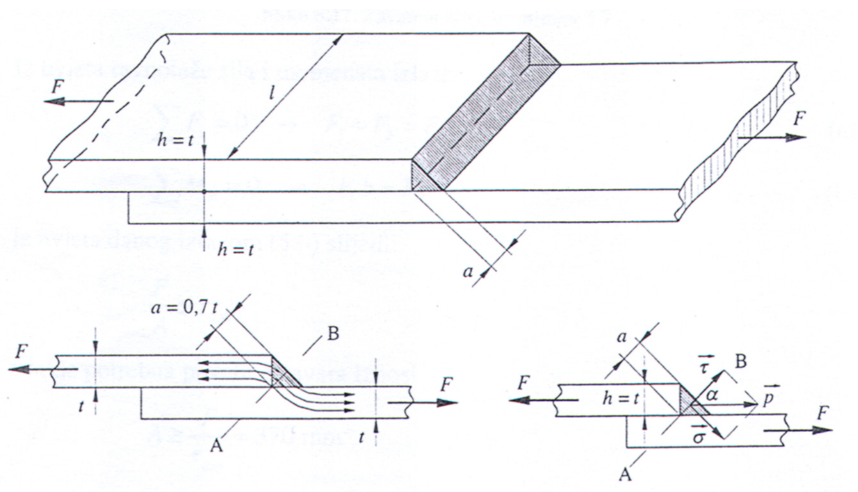
$$p_1 = \frac{F}{A_{p1}} = \frac{F}{2ad}$$

$$p_2 = \frac{F}{A_{p2}} = \frac{F}{bd}$$

$$p_1 > p_2 \Rightarrow d \geq \frac{F}{2ap_{dop}} = \frac{150 \cdot 10^3}{2 \cdot 0{,}015 \cdot 150 \cdot 10^6} = 0{,}033\,m$$

$$0{,}0369\,m > 0{,}033\,m \Rightarrow d = 37\,mm$$

Zavareni spojevi



Lom zavarenih spojeva opterećenih na smicanje nastaje po najslabijem (smičnom) presjeku A-B, tj. po presjeku za koji je smična površima najmanja.

Naprezanja u zavaru iznose :

- normalno naprezanje $\sigma = \frac{F}{al}$
- posmično naprezanje $\tau = \frac{F}{al}$

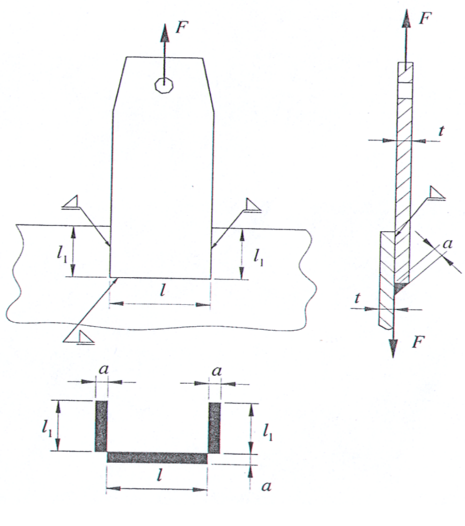
ili općenito $\sigma = \frac{F}{A_\sigma} = \frac{F}{\sum(a \cdot l)_\sigma}, \tau = \frac{F}{A_\tau} = \frac{F}{\sum(a \cdot l)_\tau}$ gdje je A_σ, A_τ … veličina površine zavara koja se odnosi na normalno, odnosno posmično naprezanje.

Za spoj na slici posmično naprezanje je

$$\tau = \frac{F}{A_\tau} = \frac{F}{\sum(a \cdot l)} = \frac{F}{a(2l_1 + l)}$$

Normalno naprezanje jednako je posmičnom.

Kod zavarenih spojeva uobičajeno se provjeravaju normalna naprezanja ako sila djeluje okomito na površinu zavarenog spoja, odnosno na posmična naprezanja dok sila djeluje u samoj površini zavarenog spoja.



33. Zadatak

Odrediti potrebnu duljinu zavara prema slici.

$F = 50kN$

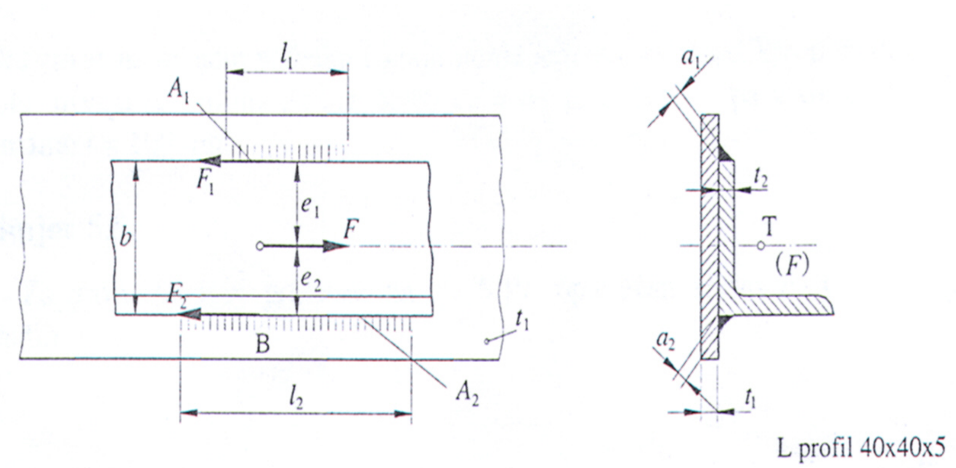
$\tau_{dop} = 135\frac{N}{mm^2}$

$t_1 \geq 1{,}2t_2$

$a_1 = 0{,}7t$

$a_2 = 0{,}84t$

$B40\times40\times5$



Iz priručnika za profile :

$b = 40mm$

$e_2 = 11{,}6mm$

$A = 379mm^2$

$\underline{\sum F_z = 0} \rightarrow F_1 + F_2 = F \ldots (1)$

$\underline{\sum M_B = 0} \rightarrow F_1 b = Fe_2 \ldots (2)$

$$\tau = \frac{F}{A} \leq \tau_{dop} \rightarrow A \geq \frac{F}{\tau_{dop}} = \frac{50\cdot10^3}{135} = 370mm^2$$

$$(2) \rightarrow F_1 = \frac{Fe_2}{b} \rightarrow A_1 = \frac{Ae_2}{b} = \frac{370\cdot11{,}6}{40} = 107mm^2$$

$A_2 = A - A_1 = 263mm^2$

$a_1 = 0{,}7t = 0{,}7\cdot5 = 3{,}5mm$

$a_2 = 0{,}84t = 0{,}84\cdot5 = 4{,}2mm$

$\underline{usvojeno} \rightarrow a_1 = 3mm, a_2 = 4mm$

$$l_1 = \frac{A_1}{a_1} = 30{,}56mm, l_2 = \frac{A_2}{a_2} = 62{,}62mm$$

$\underline{usvojeno} \rightarrow l_1 = 40mm, l_2 = 70mm$

34. Zadatak

Odrediti najveća naprezanja zakovica.

$Q = 20kN$

$d = 1{,}4cm$

$a = 4cm$

$b = 8cm$

$H = 12kN$

Sila na jednu zakovicu od djelovanja sile H

$$F_H = \frac{H}{h}$$

Moment u težištu sustava zakovica

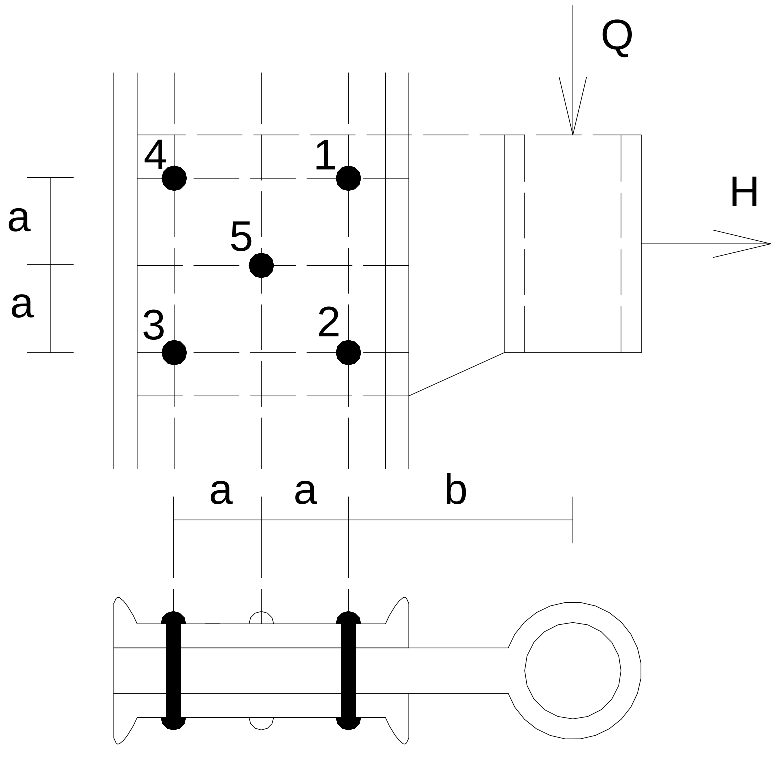
$$M = Q(a+b)$$

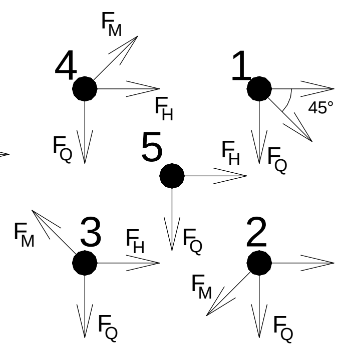
Sila na jednu zakovicu od djelovanja sile Q

$$F_Q = \frac{Q}{h}$$

Sila u i-toj zakovici od momenta M

$$F_{Mi} = k \cdot \rho_i$$





Moment od sile F_{Mi} s obzirom na težište sustava

$$M_i = F_{Mi} \cdot \rho_i = k \cdot \rho_i^2$$

$$M = \sum M_i = \sum_{i=1}^{5} k \cdot \rho_i^2 \Rightarrow k = \frac{M}{\sum_{i=1}^{5} \rho_i^2}$$

$$F_{Mi} = \frac{M}{\sum_{i=1}^{5} \rho_i^2} \cdot \rho_i$$

$$\rho_i = \rho_2 = \rho_3 = \rho_4 = a\sqrt{2}, \rho_5 = 0$$

$$F_{M1} = F_{M2} = F_{M3} = F_{M4} = F_M$$

$$F_M = \frac{Q \cdot (a+b)}{4 \cdot a^2 \cdot 2} \cdot a\sqrt{2} = \frac{\sqrt{2} \cdot Q \cdot (a+b)}{8 \cdot a} = \frac{\sqrt{2} \cdot 20 \cdot (0{,}04 + 0{,}08)}{8 \cdot 0{,}04} = 10{,}6kN$$

U srednjoj zakovici ne djeluje sila od momenta!

$$F_{M5} = 0$$

$$F_H = \frac{12}{5} = 2{,}4kN$$

$$F_Q = \frac{20}{5} = 4kN$$

$$R_1 = \sqrt{\left(F_H + F_M \cdot \frac{\sqrt{2}}{2}\right)^2 + \left(F_Q + F_M \cdot \frac{\sqrt{2}}{2}\right)^2} = 15{,}15kN$$

$$R_2 = \sqrt{\left(F_H - F_M \cdot \frac{\sqrt{2}}{2}\right)^2 + \left(F_Q + F_M \cdot \frac{\sqrt{2}}{2}\right)^2} = 12{,}57kN$$

$$R_3 = \sqrt{\left(F_H - F_M \cdot \frac{\sqrt{2}}{2}\right)^2 + \left(F_Q - F_M \cdot \frac{\sqrt{2}}{2}\right)^2} = 6{,}17kN$$

$$R_4 = \sqrt{\left(F_H + F_M \cdot \frac{\sqrt{2}}{2}\right)^2 + \left(F_Q - F_M \cdot \frac{\sqrt{2}}{2}\right)^2} = 10{,}49kN$$

$$R_5 = \sqrt{F_H^2 + F_Q^2} = 4{,}66kN$$

Najveće opterećenje ima zakovica 1 : R_1 = 15,15 kN

$$\tau = \frac{F}{mA_s} = \frac{R_1}{2\frac{d^2\pi}{4}} = \frac{15150}{2\frac{0{,}014^2\pi}{4}} = 49{,}20MPa$$

35. Zadatak

Unutarnji promjer vanjskog čeličnog prstena je za D manji od vanjskog promjera unutarnjeg aluminijskog valjka, pa je čelični prsten u zagrijanom stanju nataknut na aluminijski valjak. Odrediti pritisak prstena na valjak i naprezanje u prstenu nakon hlađenja prstena.
Kolika sila djeluje u prstenu?

$D = 1{,}1mm$

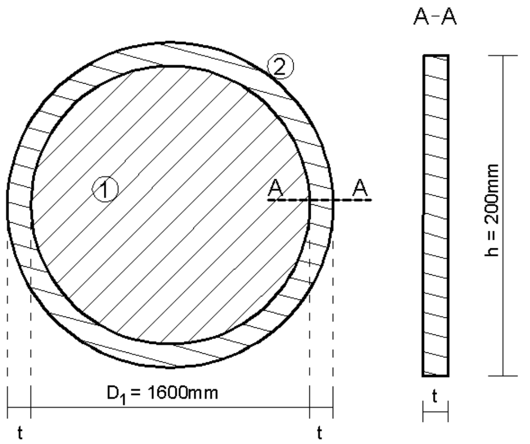
$E_1 = E_{Al} = 0{,}7 \cdot 10^5 \dfrac{N}{mm^2}$

$E_2 = E_{Č} = 2 \cdot 10^5 \dfrac{N}{mm^2}$

$\nu_{Al} = 0{,}34 = \nu_1$

$\nu_{Č} = 0{,}3 = \nu_2$

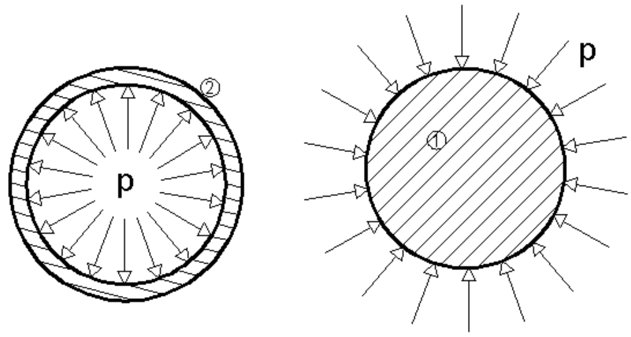
$t = 5mm$



Uvjet deformacija :

$\Delta D = |\Delta D_1| + |\Delta D_2| \ldots (1)$

$\Delta D = D_1 - D_2 = 1{,}1mm$



Uvjet ravnoteže :

$2 \cdot S_2 - p \cdot D_2 = 0 \ldots (2)$

$S_2 = \dfrac{p \cdot D_2 \cdot t}{2}$

$R = p \cdot D_2$

Izraz za deformacije :

$$\varepsilon_1 = \frac{\sigma_1^{(1)}}{E_1} - \nu_1 \frac{\sigma_2^{(1)}}{E_1} = \frac{p}{E_1} - \nu_1 \frac{p}{E_1} = \frac{p}{E_1}(1 - \nu_1) = \frac{\Delta D_1}{D_1}$$

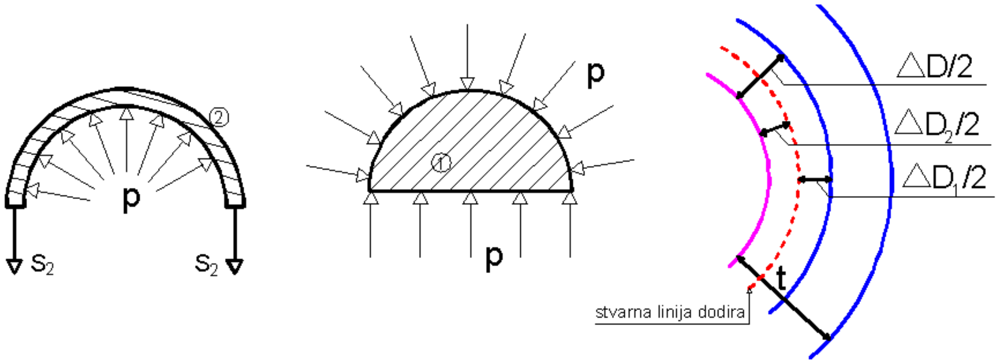
Za valjak :

$$\varepsilon_2 = \frac{\Delta D_2}{D_2}, \frac{\sigma^{(2)}}{E} = \frac{S_2}{F \cdot E_1} \Rightarrow \frac{\Delta D_2}{D_2} = \frac{S_2}{E_1 \cdot F} \xrightarrow{(2)} \Delta D_2 = \frac{D_2^{\,2} p}{2tE_1}$$

$F = D_2 \cdot h$

Za prsten :

$$\varepsilon_2 = \frac{D_2^{\,2} p}{2tE_2 D_2} = \frac{D_2 p}{2tE_2} = \frac{\Delta D_2}{D_2}$$



$$\Delta D_1 = \varepsilon_1 D_1 = \frac{pD_1}{E_1}(1-\nu_1)$$

$$\Delta D_2 = \varepsilon_2 D_2 = \frac{pD_2^{\,2}}{2tE_2}$$

$$\xrightarrow{(1)} \frac{pD_1}{E_1}(1-\nu_1) + \frac{pD_2^{\,2}}{2tE_2} = 1{,}1 \rightarrow p = \frac{1{,}1}{\frac{D_1}{E_1}(1-\nu_1) + \frac{D_2^{\,2}}{2tE_2}}$$

$$D_1 \cong D_2 \cong 1600mm$$

$$p = \frac{1{,}1}{\frac{1600}{0{,}7\cdot 10^5}(1-0{,}34) + \frac{1600^2}{2\cdot 5\cdot 2\cdot 10^5}} = 0{,}85\frac{N}{mm^2} = \sigma_1^{(1)} = \sigma_2^{(1)}$$

Naprezanje u prstenu (homogeno stanje naprezanja) :

$$\sigma_1^{(2)} = \frac{pD_2}{2t} = \frac{0{,}85\cdot 1600}{2\cdot 5} = 136{,}0\frac{N}{mm^2}$$

Sila u prstenu :

$$S = t\cdot h\cdot \sigma_1^{(2)} = 5\cdot 200\cdot 136{,}0 = 136kN$$

Kontrola :

$$\Delta D_2 = D_2\frac{\sigma_1^{(2)}}{E_2} = 1600\frac{136{,}0}{2\cdot 10^5} = 1{,}087mm$$

$$\Delta D_1 = \frac{pD_1}{E_1}(1-\nu_1) = \frac{0{,}85\cdot 1600}{0{,}7\cdot 10^5}(1-0{,}34) = 0{,}013mm$$

$$\Delta D = \Delta D_1 + \Delta D_2 = 1{,}087 + 0{,}013 = 1{,}1mm$$

36. Zadatak

U parnom kotlu promjera D djeluje pritisak p. Kotlovski lim debljine t spojen je u uzdužnom smjeru dvorednim zakovicama promjera d na međusobnom razmaku e. Odrediti :

a) naprezanja u sastavu

b) koliki maksimalni tlak pare može kotao podnijeti ako su zadana dopuštena naprezanja.

$D = 1600 mm$

$p = 1{,}0 \frac{N}{mm^2}$

$t = 10 mm$

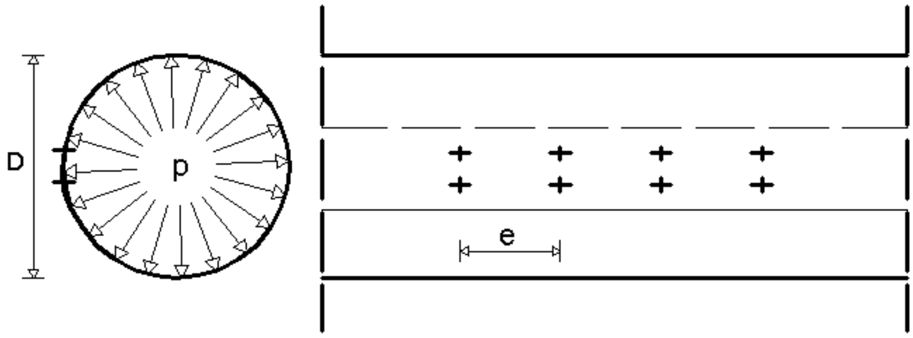
$d = 20 mm$

$e = 100 mm$

$\tau_{dop} = 70 \frac{N}{mm^2}$

$\sigma_{dop} = 100 \frac{N}{mm^2}$

$\sigma_{O\,dop} = 160 \frac{N}{mm^2}$



a) Naprezanje stjenke kotla (u smjeru tangente na stjenku)

$$\sigma = \frac{pD}{2\delta} = \frac{1{,}0 \cdot 1600}{2 \cdot 10} = 80 \frac{N}{mm^2}$$

Sila na razmaku e (sila koja djeluje na jedan red zakovica)

$$S = \sigma \cdot \delta \cdot e = 80 \cdot 10 \cdot 100 = 80 kN$$

Naprezanje zakovice

posmik $$\tau = \frac{S}{m\frac{d^2\pi}{4}} = \frac{80000}{2\frac{20^2\pi}{4}} = 127{,}32 \frac{N}{mm^2}$$

bočni pritisak $$\sigma_O = \frac{S}{2d\delta} = \frac{80000}{2 \cdot 20 \cdot 10} = 200 \frac{N}{mm^2}$$

Vlak na oslabljenom presjeku

$$\sigma = \frac{S}{(e-d)\delta} = \frac{80000}{(100-20)10} = 100 \frac{N}{mm^2}$$

b) Nosivost zakovice

$$N_S = \tau_{dop} \frac{d^2\pi}{4} = 70 \frac{20^2 \pi}{4} = 21{,}991 kN \cong 22 kN$$

$$N_O = \sigma_{Odop} \delta d = 160 \cdot 10 \cdot 20 = 32 kN$$

$$N_V = \sigma_{dop} (e - d) \delta = 100(100 - 20)10 = 80 kN$$

$$N_{\max} = N_V = 80 kN$$

$$\sigma = \frac{pD}{2\delta} \Rightarrow p = \frac{2\delta\sigma}{D} = \frac{2 \cdot 10 \cdot 100}{1600} 160 = 200 \frac{N}{mm^2}$$

$$p_{\max} = 200 \frac{N}{mm^2}$$

37. Zadatak

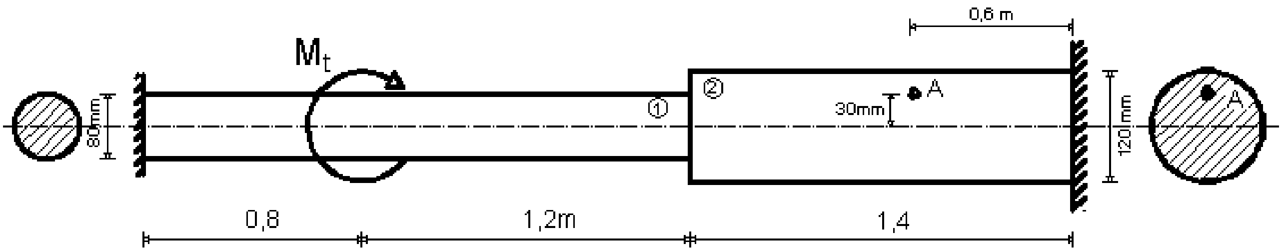
Za satavljeni štap na slici potrebno je odrediti najveća posmična naprezanja i potencijalnu energiju deformacija ako su u točki A zadana glavna naprezanja.

$$\sigma_1 = 8\frac{N}{mm^2}$$

$$\sigma_2 = -8\frac{N}{mm^2}$$

$$E = 2\cdot 10^5\ \frac{N}{mm^2}$$

$$\nu = 0{,}30$$



$$I_{p1} = \frac{D_1^{\ 4}\pi}{32} = \frac{80^4\pi}{32} = 4{,}02\cdot 10^6\, mm^4 = 4{,}02\cdot 10^{-6}\, m^4$$

$$I_{p2} = \frac{D_2^{\ 4}\pi}{32} = \frac{120^4\pi}{32} = 20{,}3\cdot 10^6\, mm^4 = 20{,}3\cdot 10^{-6}\, m^4$$

$$\tau_A = \sigma_1 = \frac{M_C}{I_{p2}}\rho_A \rightarrow M_C = \frac{\tau_A\cdot I_{p2}}{\rho_A} = \frac{8\cdot 20{,}3\cdot 10^6}{30} = 5{,}4\cdot 10^6\, Nmm = 5{,}4 kNm$$

Uvjet ravnoteže

$$M_B + M_C = M_T$$

Uvjet deformacije

$$\varphi_C = 0 \rightarrow \frac{M_T\cdot 0{,}8}{G\cdot I_{p1}} - \frac{M_C\cdot 1{,}4}{G\cdot I_{p2}} - \frac{M_C\cdot 2{,}0}{G\cdot I_{p1}} = 0$$

$$M_T = \frac{I_{p1}}{0{,}8}M_C\left[\frac{1{,}4}{I_{p2}} + \frac{2{,}0}{I_{p1}}\right] = \frac{M_C}{0{,}8}\left[1{,}4\frac{I_{p1}}{I_{p2}} + 2{,}0\right] = \frac{5{,}413}{0{,}8}\left[1{,}4\frac{4{,}02}{20{,}3} + 2{,}0\right]$$

$$M_T = 15{,}4 kNm$$

$$M_B = M_T - M_C = 10 kNm$$

$$\tau_{\max} = \frac{M_B}{I_{p1}}\frac{D_1}{2} = \frac{10 \cdot 10^6}{4{,}02 \cdot 10^6}\frac{80}{2} = 99{,}5\frac{N}{mm^2}$$

$$G = \frac{E}{2(1+\nu)} = \frac{2 \cdot 10^5}{2(1+0{,}3)} = 0{,}77 \cdot 10^5 \frac{N}{mm^2} = 0{,}77 \cdot 10^8 \frac{kN}{m^2}$$

$$U = \frac{{M_B}^2 \cdot 0{,}8}{2 \cdot G \cdot I_{p1}} + \frac{{M_C}^2 \cdot 1{,}2}{2 \cdot G \cdot I_{p1}} + \frac{{M_C}^2 \cdot 1{,}4}{2 \cdot G \cdot I_{p2}}$$

$$U = \frac{10^2 \cdot 0{,}8}{2 \cdot 0{,}77 \cdot 10^8 \cdot 4{,}02 \cdot 10^{-6}} + \frac{5{,}4^2 \cdot 1{,}2}{2 \cdot 0{,}77 \cdot 10^8 \cdot 4{,}02 \cdot 10^{-6}} + \frac{5{,}4^2 \cdot 1{,}4}{2 \cdot 0{,}77 \cdot 10^8 \cdot 20{,}3 \cdot 10^{-6}}$$

$$U = 0{,}129 + 0{,}056 + 0{,}013 = 0{,}198 kNm = 198000 Nmm$$

38. Zadatak

Osovina kružnog poprečnog presjeka opterećena je momentom torzije M_t. Temperatura osovine se promijeni za Δt. Odrediti smjer i veličinu glavnih naprezanja u točki C.

$$E = 2{,}0 \cdot 10^5 \ \frac{N}{mm^2}$$

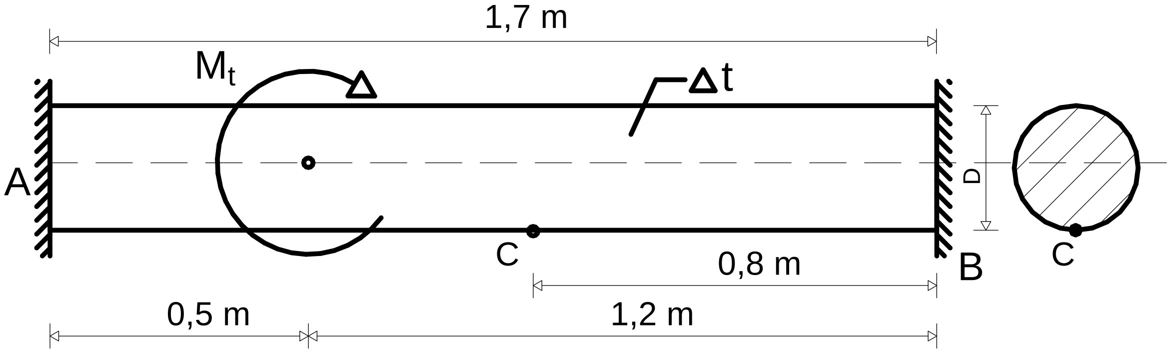
$$\nu = 0{,}30$$

$$\alpha_t = 1{,}25 \cdot 10^{-5} K^{-1}$$

$$\Delta t = +30K$$

$$D = 120mm$$

$$M_t = 16kNm$$



$$G = \frac{E}{2(1+\nu)} = \frac{2 \cdot 10^5}{2(1+0{,}3)} = 0{,}77 \cdot 10^5 \ \frac{N}{mm^2} = 0{,}77 \cdot 10^8 \ \frac{kN}{m^2}$$

$$I_p = \frac{D^4 \pi}{32} = \frac{120^4 \pi}{32} = 20{,}4 \cdot 10^6 mm^4 = 20{,}36 \cdot 10^{-6} m^4$$

$$A = \frac{D^2 \pi}{4} = \frac{120^2 \pi}{4} = 1{,}1 \cdot 10^4 mm^2 = 1{,}1 \cdot 10^{-2} m^2$$

Uvjet ravnoteže :

$$M_A + M_B = M_t \ldots (1)$$

Uvjet deformacije :

$$\varphi_A = \frac{M_A \cdot l}{G \cdot I_p} - \frac{M_t \cdot b}{G \cdot I_p} = 0 \ldots (2)$$

$$M_A = M_t \frac{b}{l} = 16 \frac{1{,}2}{1{,}7} = 11{,}29 kNm$$

$$M_B = M_t - M_A = 4{,}71 kNm$$

$$\Delta l = \alpha_t \cdot \Delta t \cdot l = \frac{F \cdot l}{E \cdot A} \Rightarrow F = \alpha_t \cdot \Delta t \cdot E \cdot A$$

$$F = 1{,}25 \cdot 10^{-5} \cdot 30 \cdot 2 \cdot 10^5 \cdot 1{,}1 \cdot 10^4 = 848{,}23 kN$$

$$\tau_C = \frac{M_B}{I_p}\frac{D}{2} = \frac{4{,}71 \cdot 10^6}{20{,}36 \cdot 10^6}\frac{120}{2} = 13{,}88 \frac{N}{mm^2}$$

$$\sigma_C = -\frac{F}{A} = -\frac{848{,}23 \cdot 10^3}{1{,}1 \cdot 10^4} = -74{,}52 \frac{N}{mm^2}(tlak)$$

$$\sigma_{1,2}{}^C = \frac{\sigma_C}{2} \pm \frac{1}{2}\sqrt{\sigma_C{}^2 + 4\tau_C{}^2} = \frac{-74{,}52}{2} \pm \frac{1}{2}\sqrt{(-74{,}52)^2 + 4 \cdot 13{,}88^2} = -37{,}26 \pm 39{,}76$$

$$\sigma_1{}^C = +2{,}50 \frac{N}{mm^2}$$

$$\sigma_2{}^C = -77{,}02 \frac{N}{mm^2}$$

$$tg 2\varphi = \frac{2 \cdot 13{,}88}{-74{,}52} = -0{,}3725 \rightarrow 2\varphi = -20{,}43° \rightarrow \varphi = -10{,}215°$$

39. Zadatak

Za spoj dva vratila odrediti potreban broj vijaka krute spojke. Zanemariti utjecaj spojke na uvijanje.

$d_V = 12mm$

$\tau_{dop} = 50MPa$

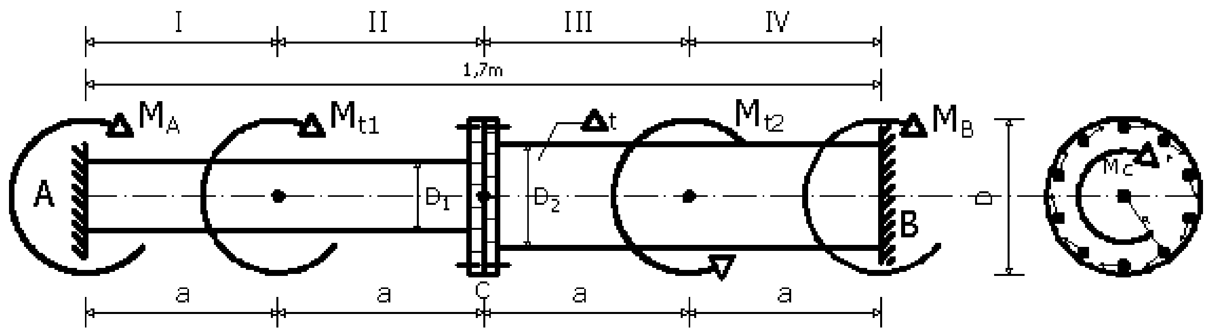
$M_{t1} = 10kNm$

$M_{t2} = 15kNm$

$D_1 = 100mm$

$D_2 = 140mm$

$2R = 0{,}2m$



$$I_{pI} = I_{pII} = \frac{D_1^4\pi}{32} = \frac{0{,}1^4\pi}{32} = 0{,}9817\cdot 10^{-5}m^4$$

$$I_{pIII} = I_{pIV} = \frac{D_2^4\pi}{32} = \frac{0{,}14^4\pi}{32} = 3{,}771\cdot 10^{-5}m^4$$

<u>Uvjet ravnoteže :</u>

$$\sum M_z = 0 \rightarrow M_A + M_B + M_{t1} - M_{t2} = 0 \ldots (1)$$

<u>Uvjet deformacije :</u>

$$\varphi_{AB} = \sum_{i=1}^{i=4}\varphi_i = 0 \rightarrow \varphi_I + \varphi_{II} + \varphi_{III} + \varphi_{IV} = 0 \ldots (2)$$

$(2) \rightarrow$

$$-\left(\frac{M_A}{GI_{pI}}+\frac{M_A+M_{t1}}{GI_{pII}}+\frac{M_A+M_{t1}}{GI_{pIII}}+\frac{M_A+M_{t1}-M_{t2}}{GI_{pIV}}\right)\cdot a=0$$

$I_{pI}=I_{pII}, I_{pIII}=I_{pIV}$

$$M_A=\frac{M_{t2}I_{pI}-M_{t1}\left(2I_{pI}+I_{pIII}\right)}{2\left(I_{pI}+I_{pIII}\right)}$$

$$M_A=\frac{15\cdot 0{,}9817-10\left(2\cdot 0{,}9817+3{,}771\right)}{2\left(0{,}9817+3{,}771\right)}=-4{,}483kNm$$

Torzijski moment u presjeku C :

$$M_C=M_A+M_{t1}=4{,}483-10=-5{,}517kNm$$

Zbog djelovanja momenta M_C vijci spojke su opterećeni na smicanje :

$M_C=n\cdot F\cdot R$

n – broj vijaka

F – posmična sila koja djeluje na jedan vijak

R – krak djelovanja sile F

Potreban broj vijaka u spoju :

$$\tau_s=\frac{F}{A_v}=\frac{4F}{d_v{}^2\pi}\le\tau_{sdop}\rightarrow F\le\frac{d_v{}^2\pi}{4}\tau_{sdop}$$

$$n\ge\frac{4M_C}{d_v{}^2\pi\tau_{sdop}R}=\frac{4\cdot 5517}{0{,}012^2\cdot\pi\cdot 50\cdot 10^6\cdot 0{,}1}=9{,}76\rightarrow \underline{n=10vijaka}$$

40. Zadatak

Sastavljena osovina kružnog poprečnog presjeka opterećena je momentom torzije M_t . Odrediti najveća naprezanja u pojedinim dijelovima osovine i potencijalnu energiju deformacije.

$M_t = 15 kNm$

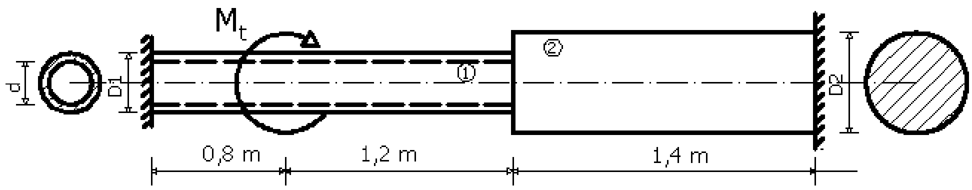
$E = 2 \cdot 10^5 \frac{N}{mm}$

$\nu = 0,3$

$D_1 = 80mm$

$D_2 = 120mm$

$d = 64mm$



$$I_{p1} = \frac{\left(D_1^{\;4} - d^4\right)\pi}{32} = \frac{\left(80^4 - 64^4\right)\pi}{32} = 2,4 \cdot 10^6\, mm^4$$

$$I_{p2} = \frac{D_2^{\;4}\pi}{32} = \frac{120^4 \pi}{32} = 20,3 \cdot 10^6\, mm^4$$

Uvjet ravnoteže

$$\sum M_z = 0 \rightarrow M_A + M_B = M_t \ \ldots (1)$$

Uvjet deformacije

$$\varphi_A = 0 \rightarrow \frac{M_t \cdot 0,8}{GI_{p1}} - \frac{M_B \cdot 1,4}{GI_{p2}} - \frac{M_B \cdot 2,0}{GI_{p1}} = 0 \ \ldots (2)$$

$$M_B = M_t \frac{0,8}{I_{p1}} \frac{1}{\frac{1,4}{I_{p2}} + \frac{2,0}{I_{p1}}} = 5,54\, kNm$$

$$M_A = M_t - M_B = 9,458\, kNm$$

$$\tau_{\max 2} = \frac{M_B}{I_{p2}} \frac{D_2}{2} = \frac{5,54 \cdot 10^6}{20,3 \cdot 10^6} \frac{120}{2} = 16,37 \frac{N}{mm^2}$$

$$\tau_{\max 1} = \frac{M_A}{I_{p1}} \frac{D_1}{2} = \frac{9,458 \cdot 10^6}{2,4 \cdot 10^6} \frac{80}{2} = 157,63 \frac{N}{mm^2}$$

$$G = \frac{E}{2(1+\nu)} = \frac{2 \cdot 10^5}{2(1+0,3)} = 0,77 \cdot 10^5 \frac{N}{mm^2} = 0,77 \cdot 10^8 \frac{kN}{m^2}$$

$$E_p = U = \frac{M_A^{\;2} \cdot 0,8}{2GI_{p1}} + \frac{M_B^{\;2} \cdot 1,2}{2GI_{p1}} + \frac{M_B^{\;2} \cdot 1,4}{2GI_{p2}} = 0,1936 + 0,0996 + 0,0137 = 0,307 kNm$$

41. Zadatak

Odrediti potrebne dimenzije pojedinih odsječaka te kut uvijanja slobodnog kraja.

$a = 400mm$

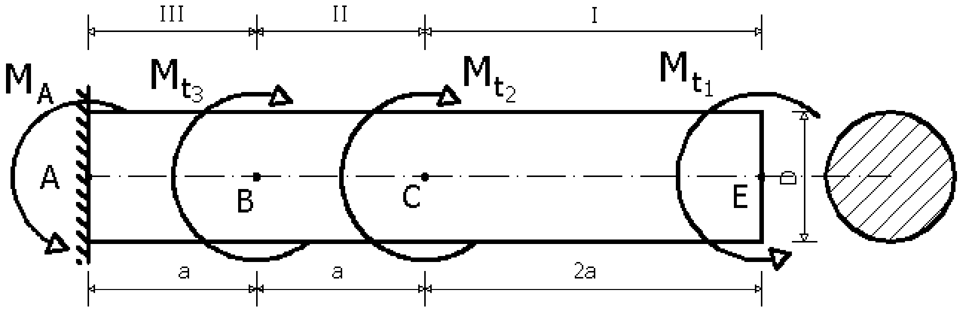
$G = 80GPa$

$M_{t1} = 4kNm$

$M_{t2} = 7kNm$

$M_{t3} = 3kNm$

$\tau_{dop} = 60MPa$



Uvjet ravnoteže :

$$\sum M_z = 0 \rightarrow M_{t1} - M_{t2} - M_{t3} + M_A = 0 \ldots (1)$$

$$(1) \rightarrow M_A = M_{t3} + M_{t2} - M_{t1} = 3 + 7 - 4 = 6kNm$$

$$\underline{M_A = 6kNm}$$

Vrijednost momenata torzije na pojedinim odsječcima :

$$M_I = M_{t1} = 4kNm$$

$$M_{II} = M_{t1} - M_{t2} = 4 - 7 = -3kNm$$

$$M_{III} = M_{t1} - M_{t2} - M_{t3} = 4 - 7 - 3 = -6kNm$$

Kriterij čvrstoće :

$$\tau_i = \frac{M_i}{W_{pi}} = \frac{16M_i}{d_i^{\,3}\pi} \le \tau_{dop}, i = I, II, III, IV \rightarrow D_i \ge \sqrt[3]{\frac{16M_i}{\pi\tau_{dop}}}$$

Dimenzioniranje :

$$D_I \geq \sqrt[3]{\frac{16M_I}{\pi\tau_{dop}}} = \sqrt[3]{\frac{16\cdot 4\cdot 10^3}{\pi\cdot 60\cdot 10^6}} = 0{,}0698m \rightarrow D_I = 70mm$$

$$D_{II} \geq \sqrt[3]{\frac{16M_{II}}{\pi\tau_{dop}}} = \sqrt[3]{\frac{16\cdot 3\cdot 10^3}{\pi\cdot 60\cdot 10^6}} = 0{,}0634m \rightarrow D_{II} = 64mm$$

$$D_{III} \geq \sqrt[3]{\frac{16M_{III}}{\pi\tau_{dop}}} = \sqrt[3]{\frac{16\cdot 6\cdot 10^3}{\pi\cdot 60\cdot 10^6}} = 0{,}0798m \rightarrow D_{III} = 80mm$$

Kutevi uvijanja :

$$\varphi_I = \varphi_{CD} = \frac{M_I\cdot 2a}{G\cdot I_{p\,I}} = \frac{64M_I\cdot a}{G\cdot D_I{}^4\pi} = \frac{64\cdot 4\cdot 10^3\cdot 0{,}4}{80\cdot 10^9\cdot 0{,}07^4\cdot\pi} = 16{,}99\cdot 10^{-3}rad$$

$$\varphi_{II} = \varphi_{BC} = \frac{M_{II}\cdot a}{G\cdot I_{p\,II}} = \frac{32M_{II}\cdot a}{G\cdot D_{II}{}^4\pi} = \frac{32\cdot(-3)\cdot 10^3\cdot 0{,}4}{80\cdot 10^9\cdot 0{,}064^4\cdot\pi} = -9{,}07\cdot 10^{-3}rad$$

$$\varphi_{III} = \varphi_{AB} = \frac{M_{III}\cdot a}{G\cdot I_{p\,III}} = \frac{32M_{III}\cdot a}{G\cdot D_{III}{}^4\pi} = \frac{32\cdot(-6)\cdot 10^3\cdot 0{,}4}{80\cdot 10^9\cdot 0{,}08^4\cdot\pi} = -7{,}46\cdot 10^{-3}rad$$

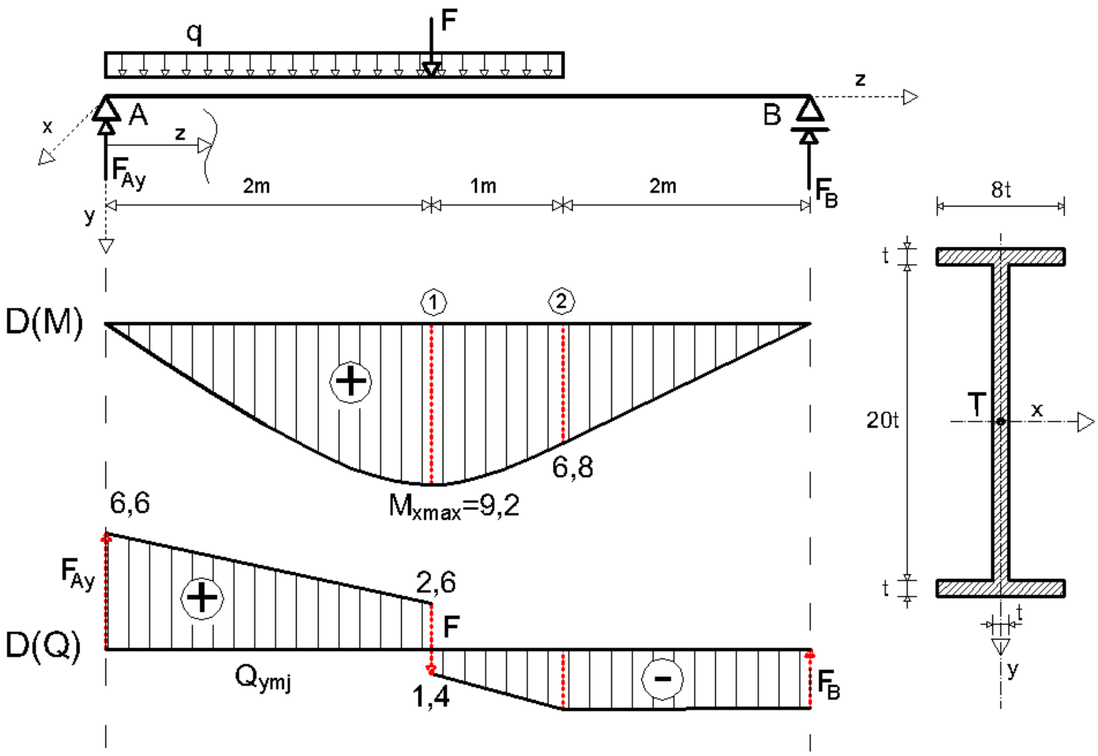
Kut uvijanja slobodnog kraja nosača :

$$\varphi_{uk} = \varphi_{AD} = \varphi_I + \varphi_{II} + \varphi_{III} = (16{,}99 - 9{,}07 - 7{,}46)\cdot 10^{-3} = 0{,}46\cdot 10^{-3}rad$$

42. Zadatak

Za nosač oblika I-profila odrediti dimenzije poprečnog presjeka prema maksimalnim normalnim i tangencijalnim naprezanjima.

$$F = 4kN$$
$$q = 2\frac{kN}{m}$$
$$h = 20t$$
$$b = 8t$$
$$\sigma_{dop} = 120MPa$$
$$\tau_{dop} = 60MPa$$



Reakcije oslonaca :

$$\underline{\sum F_y = 0} \rightarrow -F_{Ay} - F_B + F + q \cdot 3 = 0$$
$$\underline{\sum M_A = 0} \rightarrow F_B \cdot 5 - F \cdot 2 - q \cdot 3 \cdot 1{,}5 = 0$$
$$F_B = \frac{1}{5}(2F + 4{,}5q) = 3{,}4kN$$
$$F_{Ay} = F_A = F + 3q = 6{,}6kN$$

Momenti savijanja :

$$M_{x1} = M_{x1}^{l} = F_{Ay} \cdot 2 - q \cdot 2 \cdot 1 = 9{,}2 kNm$$
$$M_{x2} = F_B \cdot 2 = 6{,}8 kNm$$

Ekstremni momenti savijanja na udaljenosti z od oslonca A :

$$z = 0 \ldots 2m$$
$$M_x(z) = F_{Ay} z - qz\frac{z}{2} = F_{Ay} z - \frac{1}{2} qz^2$$
$$\frac{dM_x}{dz} = F_{Ay} - qz = 0 \rightarrow z = \frac{F_{Ay}}{q} = 3{,}3m$$

$$z = 2 \ldots 3m$$
$$M_x(z) = F_{Ay} z - \frac{1}{2} qz^2 - F(z-2)$$
$$\frac{dM_x}{dz} = F_{Ay} - qz - F = 0 \rightarrow z = \frac{F_{Ay} - F}{q} = 1{,}3m$$

Iz dijagrama D(M) i D(Q) je vidljivo da se maksimalni moment savijanja javlja u presjeku 1, dok se maksimalna poprečna sila javlja u presjeku A

$$M_{x\max} = M_{x1} = 9{,}2 kNm$$
$$Q_{y\max} = Q_{yA} = 6{,}6 kN$$

Aksijalni moment površine drugog reda i moment otpora

$$I_x = \frac{8t \cdot (22t)^3}{12} - \frac{7t \cdot (20t)^3}{12} = 2432 t^4$$
$$W_x = \frac{I_x}{y_{\max}} = \frac{2432 t^4}{\left(\frac{h}{2} + t\right)} = 221{,}09 t^3$$

Iz kriterija čvrstoće za maksimalna normalna i posmična naprezanja :

$$\sigma_{z\max} = \frac{M_{x\max}}{W_x} = \frac{M_{x\max}}{221{,}09t^3} \le \sigma_{dop}$$

$$t \ge \sqrt[3]{\frac{M_{x\max}}{221{,}09 \cdot \sigma_{dop}}} = 7{,}03 \cdot 10^{-3}\,m$$

$$\tau_{zy\max} = \frac{Q_{y\max}}{I_x}\left(\frac{S_x}{b}\right)_{\max} \le \tau_{dop}$$

$$\left(\frac{S_x}{b}\right)_{\max} = \frac{bt\frac{h+t}{2} + \frac{h}{2}t\frac{h}{4}}{t} = 134t^2$$

$$\tau_{zy\max} = \frac{Q_{y\max}}{2432t^4}134t^2 \le \tau_{dop}$$

$$t \ge \sqrt{\frac{134Q_{y\max}}{2432\tau_{dop}}} = 2{,}46 \cdot 10^{-3}\,m$$

Zaključuje se da je za dimenzioniranje mjerodavan kriterij maksimalnih normalnih naprezanja i usvaja se :

$$t = 7{,}1mm, h = 20t = 142mm, b = 8t = 56{,}8mm, I_x = 6{,}18 \cdot 10^{-6}\,mm^4$$

Provjera naprezanja u presjecima 1 i 2 :

$$M_{x1mj} = 9{,}2kNm$$

$$Q_{y1mj} = 2{,}6kN$$

$$\sigma_{z1mj} = \frac{M_{x1mj}}{I_x}y_{mj} = \frac{9200}{6{,}18 \cdot 10^{-6}} \cdot 0{,}0781 = 116{,}26MPa \le \sigma_{dop} = 120MPa$$

$$\tau_{zymj} = \frac{Q_{y1mj}}{I_x}\left(\frac{S_x}{b}\right)_{\max} = \frac{6600}{6{,}18 \cdot 10^{-6}} \cdot 134 \cdot 0{,}0071^3 = 0{,}051MPa \le \tau_{dop} = 60MPa$$

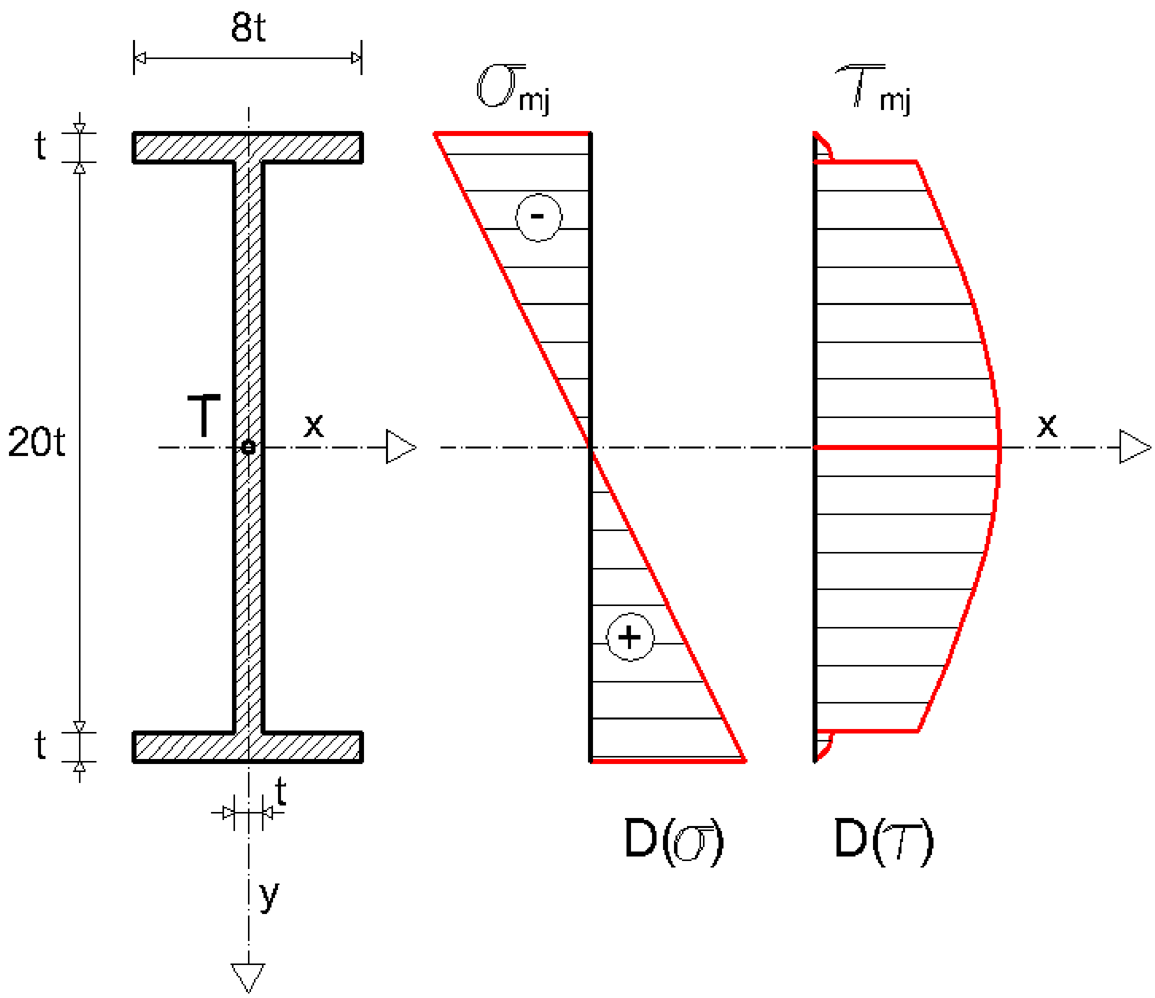
$$M_{x2mj} = 6{,}8kNm$$

$$Q_{y1mj} = 3{,}4kN$$

$$\sigma_{z2mj} = \frac{M_{x2mj}}{I_x}y_{mj} = \frac{6800}{6{,}18 \cdot 10^{-6}} \cdot 0{,}0781 = 85{,}94MPa \le \sigma_{dop} = 120MPa$$

$$\tau_{zymj} = \frac{Q_{y2mj}}{I_x}\left(\frac{S_x}{b}\right)_{\max} = \frac{3400}{6{,}18 \cdot 10^{-6}} \cdot 84 \cdot 0{,}0071^3 = 0{,}026MPa \le \tau_{dop} = 60MPa$$

Dijagrami naprezanja :

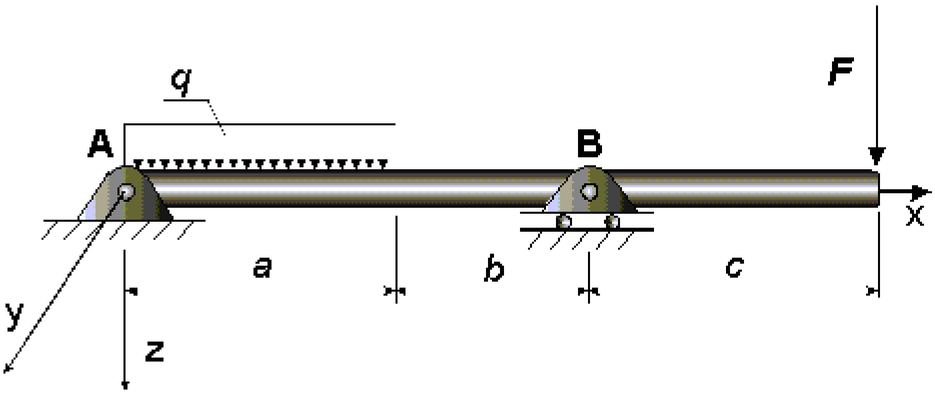


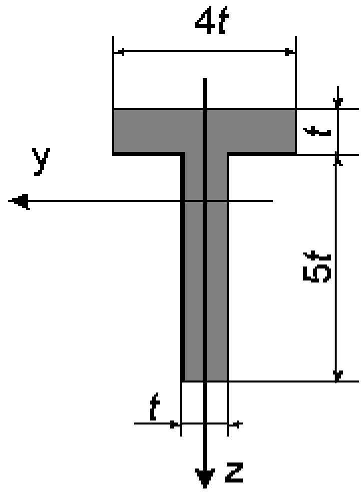
43. Zadatak

Za nosač zadan i opterećen prema slici potrebno je:

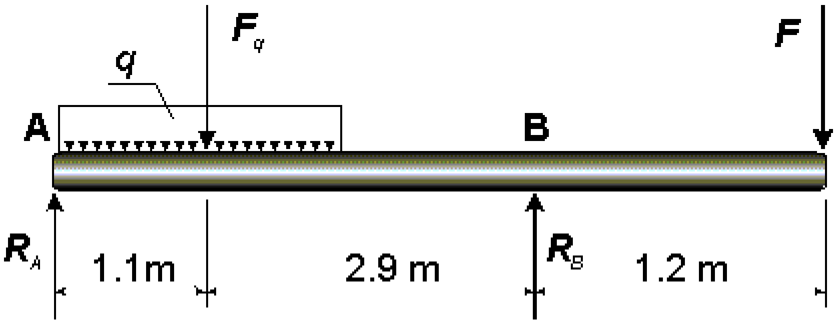
a) odrediti reakcije u osloncima
b) skicirati dijagrame poprečnih sila i momenta savijanja
c) dimenzionirati nosač (t zaokružiti na prvi veći cijeli broj u milimetrima)
d) u presjeku x skicirati raspodjelu normalnog i tangencijalnog naprezanja po visini presjeka

a = 2.2 m
b = 1.8 m
c = 1.2 m
x = a / 2
F = 8 kN
q = 7 kN/m
σ_{DOP} = 180 Mpa





a) <u>reakcije u osloncima</u>



$$\sum M_A \equiv -1.1\,F_q + 4R_B - 5.2\,F = 0$$

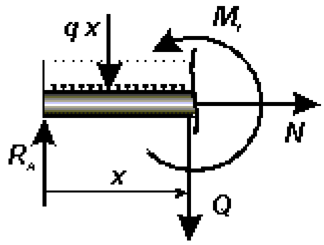
$$R_B = \frac{1.1\,F_q + 5.2\,F}{4} = 14.635\ \text{kN}$$

$$\sum M_B \equiv -4\,R_A + 2.9\,F_q - 1.2\,F = 0$$

$$R_A = \frac{2.9\,F_q - 1.2\,F}{4} = 8.765\ \text{kN}$$

b) <u>dijagram poprečnih sila i momenta savijanja</u>

I. područje: 0 m <x < 2.2 m



$$\sum F_x = 0 \quad \Rightarrow \quad N = 0\ \text{kN}$$

$$\sum F_y \equiv Q + q\,x - R_A = 0$$

$$Q = -q\,x + R_A$$

$$Q = -7\,x + 8.765$$

$$Q(0) = 8.765\ \text{kN}$$

$$Q(2.2) = -6.635\ \text{kN}$$

$$Q = 0\ \text{kN} \quad \Rightarrow \quad x = \frac{R_A}{q} = 1.252\ \text{m}$$

$$\sum M_x \equiv M_y - R_A\,x + q\,x\frac{x}{2} = 0$$

$$M_y = -\frac{q}{2}x^2 + R_A\,x$$

$$M_y = -3.5\,x^2 + 8.765\,x$$

$$M_y(0) = 0\ \text{kNm}$$

$$M_y(1.252) = 5.488\ \text{kNm}$$

$$M_y(2.2) = 2.343\ \text{kNm}$$

II. područje: 2.2 m < x < 4 m

$$\sum F_x = 0 \quad \Rightarrow \quad N = 0 \text{ kN}$$

$$\sum F_y \equiv Q + q\,a - R_A = 0$$

$$Q = -q\,a + R_A$$

$$Q = -6.635 \text{ kN}$$

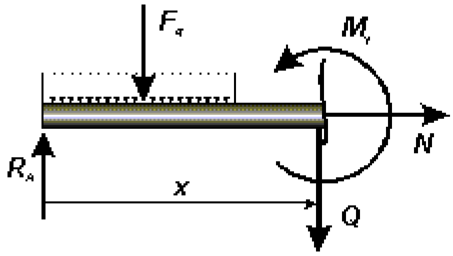
$$\sum M_x \equiv M_y - R_A\,x + F_q\,(x - 1.1) = 0$$

$$M_y = R_A\,x - F_q\,x + 1.1 F_q$$

$$M_y = -6.635\,x + 16.94$$

$$M_y(2.2) = 2.343 \text{ kNm}$$

$$M_y(4) = -9.600 \text{ kNm}$$



III. područje: 4 m < x < 5.2 m

$$\sum F_x = 0 \quad \Rightarrow \quad N = 0 \text{ kN}$$

$$\sum F_y \equiv -Q + F = 0$$

$$Q = F$$

$$Q = 8 \text{ kN}$$

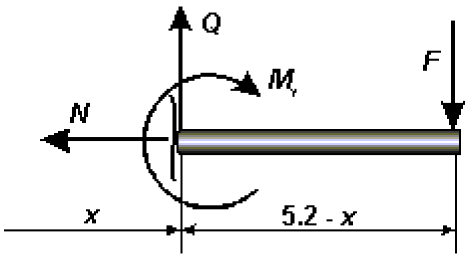
$$\sum M_x \equiv -M_y - F\,(5.2 - x) = 0$$

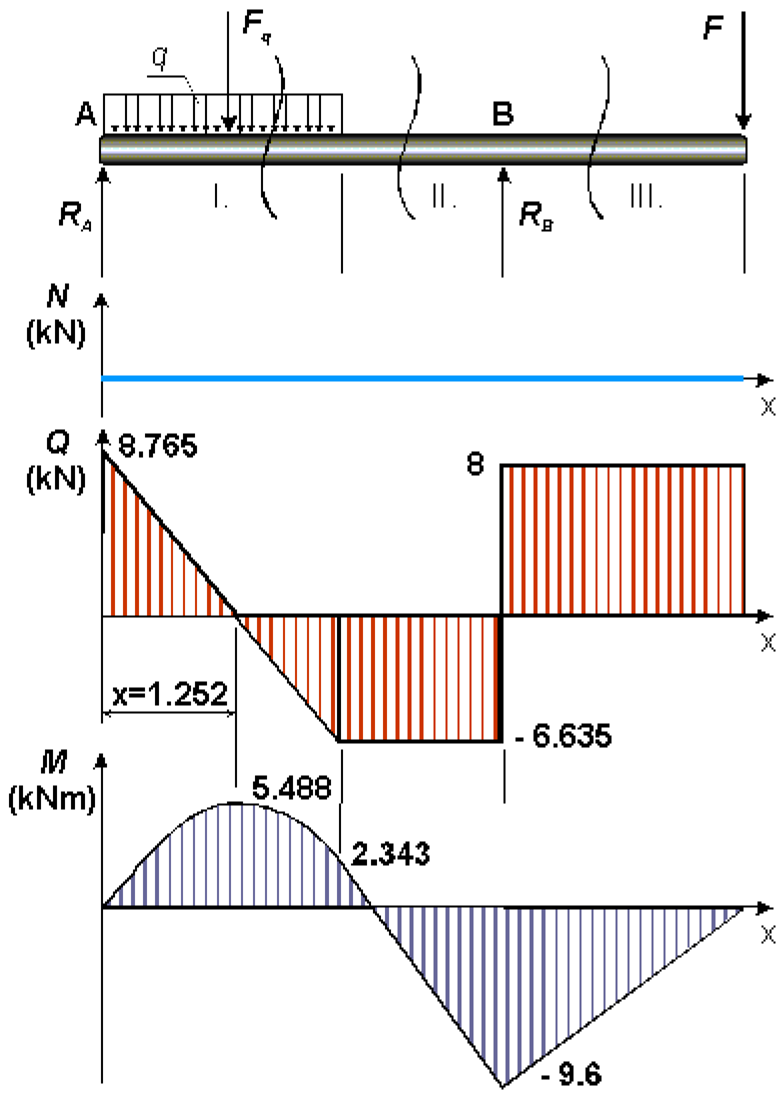
$$M_y = F\,x - 5.2\,F$$

$$M_y = 8x - 41.6$$

$$M_y(4) = -9.600 \text{ kNm}$$

$$M_y(5.2) = 0 \text{ kNm}$$



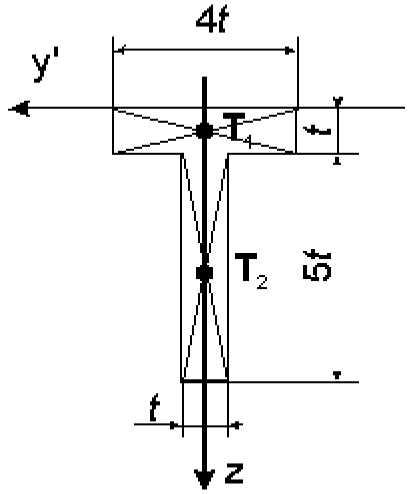


Mmax = 5.488 kNm
Mmin = -9.6 kNm

Najveći moment po apsolutnoj vrijednosti je u točki x = 4 m i iznosi My = - 9.6 kNm .
Stoga treba dimenzionirati nosač za iznos momenta |My| = 9.6 kNm !

c) dimenzioniranje nosača

- statički moment površine i težište:



$$S'_y = A z_t \qquad S'_y = 4t \cdot 6t \cdot 3t - 3t \cdot 5t \cdot 3.5t = 19.5t^3$$

$$z_t = \frac{S'_y}{A}$$

$$S'_y = A_1\, z_{T_1} + A_2\, z_{T_2} =$$

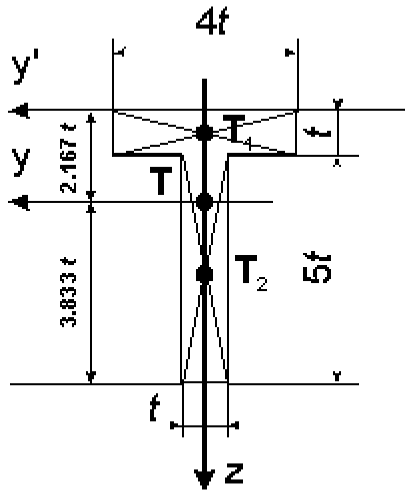
$$= 4t\, t\, \frac{t}{2} + 5t\, t\, (1+2.5)t =$$

$$= 19.5t^3$$

$$z_T = \frac{S'_y}{A} = \frac{19.5t^3}{4t^2 + 5t^2} = 2.167t$$

$$\underline{z_T = 2.167t}$$

- moment tromosti presjeka



$$I_y = I_{y_1} + A_1 d_1^2 \quad + \quad I_{y_2} + A_2 d_2^2 =$$

$$= \frac{4t \cdot t^3}{12} + 4t^2 \cdot (2.167t - 0.5t)^2 + \frac{t \cdot 125t^3}{12} + 5t^2 \cdot (3.833t - 2.5t)^2 =$$

$$= 30.75\, t^4$$

$$\underline{I_y = 30.75\, t^4}$$

$$\sigma = \frac{M_y}{I_y} z \leq \sigma_{dop}$$

$$I_y = 30.75 t^4$$

$$z_{max} = 3.833 t$$

$$M_y = 9.6 \text{ kNm}$$

$$\sigma_{dop} = 180 \text{ MPa}$$

$$\frac{M_y}{I_y} z \leq \sigma_{dop}$$

$$\frac{9.6 \cdot 10^3}{30.75 t^4} \cdot 3.833 t \leq 180 \cdot 10^6$$

$$t^3 \geq \frac{9.6 \cdot 10^3 \cdot 3.833}{30.75 \cdot 180 \cdot 10^6}$$

$$t \geq 18.803 \text{ mm}$$

$$\underline{t = 19 \text{ mm}}$$

d) raspodjela normalnog i tangencijalnog naprezanja u presjeku za x = 1.1 m

- normalno naprezanje

$$Q(1.1) = 1.065 \text{ kN}$$

$$M_y(1.1) = 5.407 \text{ kNm}$$

$$t = 19 \text{ mm} = 1.9 \text{ cm}$$

$$I_y = 30.75 t^4 = 400.737 \text{ cm}^4$$

$$z_E = 2.167 t = 4.117 \text{ cm}$$

$$\underline{z_F = 3.833 t = 7.283 \text{ cm}}$$

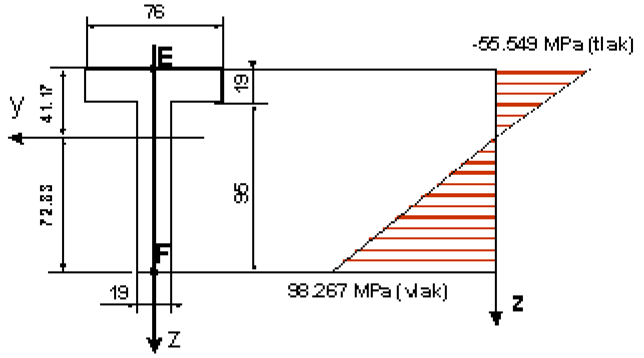
$$\sigma = \frac{M_y}{I_y} z$$

$$\sigma_E = \frac{5.407 \cdot 10^3 \cdot 10^2 \text{ N cm}}{400.737 \text{ cm}^4} (-4.117) \text{ cm} = -5554.920 \frac{\text{N}}{\text{cm}^2}$$

$$\underline{\sigma_E = -55.549 \text{ MPa}}$$

$$\sigma_F = \frac{5.407 \cdot 10^3 \cdot 10^2 \text{ N cm}}{400.737 \text{ cm}^4} 7.283 \text{ cm} = 9826.690 \frac{\text{N}}{\text{cm}^2}$$

$$\underline{\sigma_F = 98.267 \text{ MPa}}$$



- tangencijalno naprezanje

$$\tau = \frac{Q\ S_y^*}{b\ I_y}$$

$I_y = 400.737\ \text{cm}^4$

$Q(1.1) = 1.065\ \text{kN}$

"P":

$$S_{yP}^* = 7.6 \cdot 1.9 \cdot (4.117 - 0.8) = 47.897\ \text{cm}^3$$

$$\tau_{P-iznad} = \frac{1.065 \cdot 10^3\ \text{N} \cdot 47.897\ \text{cm}^3}{7.6\ \text{cm} \cdot 400.737\ \text{cm}^4} = 16.749\ \frac{\text{N}}{\text{cm}^2}$$

$\underline{\tau_{P-iznad} = 0.168\ \text{MPa}}$

$$\tau_{P-ispod} = \frac{1.065 \cdot 10^3\ \text{N} \cdot 47.897\ \text{cm}^3}{1.9\ \text{cm} \cdot 400.737\ \text{cm}^4} = 66.995\ \frac{\text{N}}{\text{cm}^2}$$

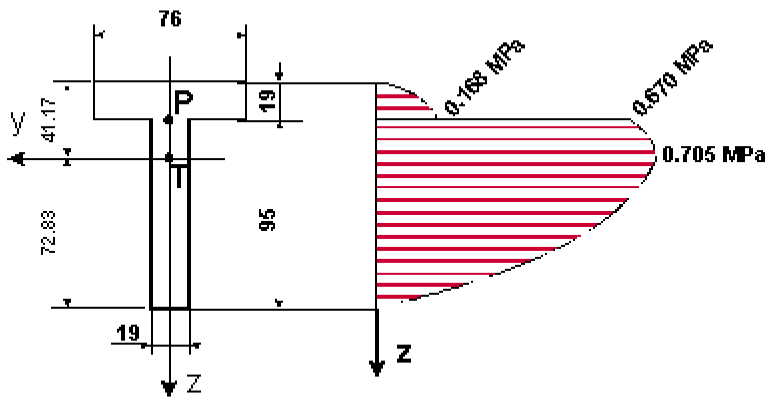
$\underline{\tau_{P-ispod} = 0.670\ \text{MPa}}$

"T":

$$S_{yT}^* = 1.9 \cdot 7.283 \cdot \frac{7.283}{2} = 50.390\ \text{cm}^3$$

$$\tau_T = \frac{1.065 \cdot 10^3\ \text{N} \cdot 50.390\ \text{cm}^3}{1.9\ \text{cm} \cdot 400.737\ \text{cm}^4} = 70.482\ \frac{\text{N}}{\text{cm}^2}$$

$\underline{\tau_T = 0.705\ \text{MPa}}$



44. Zadatak

Za nosač sa slike odrediti dimenzije poprečnog presjeka ako je on:

a) kružnog oblika
b) pravokutnog oblika
c) oblika I-profila

prema uvjetu maksimalnih normalnih naprezanja.
Za I-profil napraviti kontrolu dobivenih dimenzija prema kriteriju dopuštenih tangencijalnih naprezanja.

$F = 4kN$

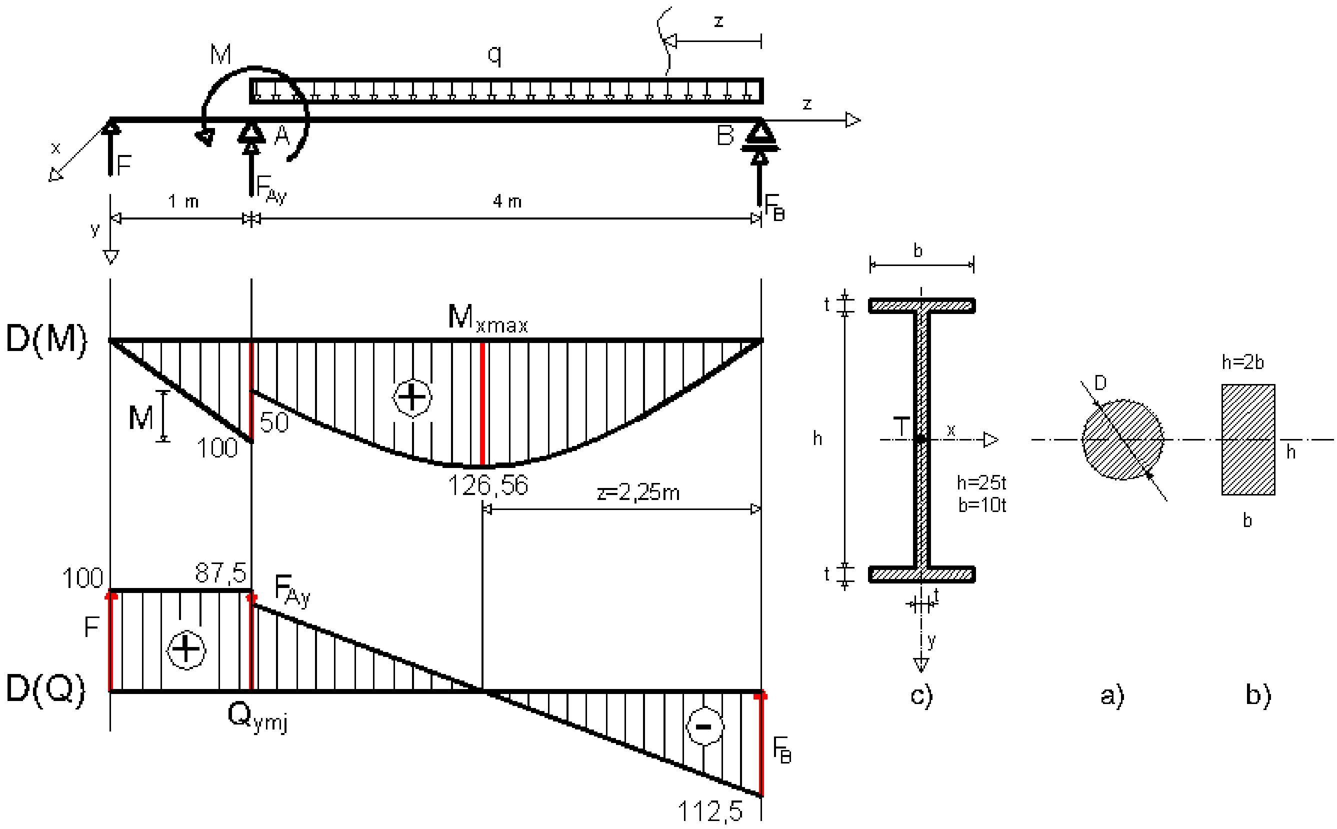
$q = 2\frac{kN}{m}$

$h = 20t$

$b = 8t$

$\sigma_{dop} = 120MPa$

$\tau_{dop} = 60MPa$



<u>Reakcije oslonaca :</u>

$\underline{\sum F_y = 0} \rightarrow -F_{Ay} - F_B + F + q \cdot 4 = 0$

$\underline{\sum M_A = 0} \rightarrow F_B \cdot 4 - F \cdot 1 - q \cdot 4 \cdot 2 + M = 0$

$F_{Ay} = F_A = -12{,}5kN$

$F_B = 112{,}5kN$

<u>Momenti savijanja :</u>

$M_{x1} = M_{x1}^{\,l} = F_{Ay} \cdot 2 + q \cdot 2 \cdot 1 = 9{,}2kNm$

$M_{x2} = F_B \cdot 2 = 6{,}8kNm$

Maksimalni moment savijanja na udaljenosti z od oslonca B :

$$M_x(z) = F_B z - \frac{1}{2} q z^2$$

$$\frac{dM_x}{dz} = F_B - qz = 0 \rightarrow z = \frac{F_B}{q} = 2{,}25m$$

$$M_{\max} = 112{,}5 \cdot 2{,}25 - 50 \cdot 2{,}25 \cdot \frac{2{,}25}{2} = 126{,}56kNm$$

Moment savijanja u točki A :

$$M_{xA} = F \cdot 1 = 100kNm$$

Za dimenzioniranje prema najvećim normalnim naprezanjima koristi se maksimalni moment savijanja :

$$M_{\max} = M_{x1} = 126{,}56kNm$$

Za kontrolu dimenzija prema dopuštenim tangencijalnim naprezanjima koristi se vrijednost poprečne sile :

$$Q_{y\max} = \left|Q_{yB}\right| = F_B = 112{,}5kN$$

a) kružni poprečni presjek

$$W_{x\min} \geq \frac{M_{x\max}}{\sigma_{dop}} = \frac{D^3 \pi}{32} \rightarrow D \geq \sqrt[3]{\frac{32 M_{x\max}}{\pi \sigma_{dop}}} = \sqrt[3]{\frac{32 \cdot 126{,}56 \cdot 10^6}{\pi 100}}$$

$$D \geq 234{,}47mm$$

$$USVOJENO \rightarrow D = 235mm$$

b) pravokutni poprečni presjek

$$W_x = \frac{h^2 b}{6} = \frac{h^3}{12} \geq \frac{M_{x\max}}{\sigma_{dop}} \rightarrow h \geq \sqrt[3]{\frac{12 M_{x\max}}{\sigma_{dop}}} = \sqrt[3]{\frac{12 \cdot 126{,}56 \cdot 10^6}{100}}$$

$$h \geq 247{,}6mm$$

$$USVOJENO \rightarrow h = 248mm, b = 124mm$$

c) poprečni presjek oblika I-profila

$$I_x = \frac{10t(27t)^3}{12} - \frac{9t(25t)^3}{12} = 4683{,}8t^4$$

$$W_x = \frac{I_x}{y_{\max}} = \frac{4683{,}8t^4}{13{,}5t} = 346{,}9t^3 \geq \frac{M_{x\max}}{\sigma_{dop}} \to t \geq \sqrt[3]{\frac{M_{x\max}}{346{,}9\sigma_{dop}}} = \sqrt[3]{\frac{126{,}56 \cdot 10^6}{346{,}9 \cdot 100}}$$

$$t \geq 15{,}39mm$$

$$USVOJENO \to t = 16mm, h = 25t = 400mm, b = 160mm$$

Provjera dimenzija prema dopuštenom tangencijalnom naprezanju :

$$\tau_{zy\max} = \frac{Q_{y\max}}{I_x}\left(\frac{S_x}{b}\right)_{\max} \leq \tau_{dop}$$

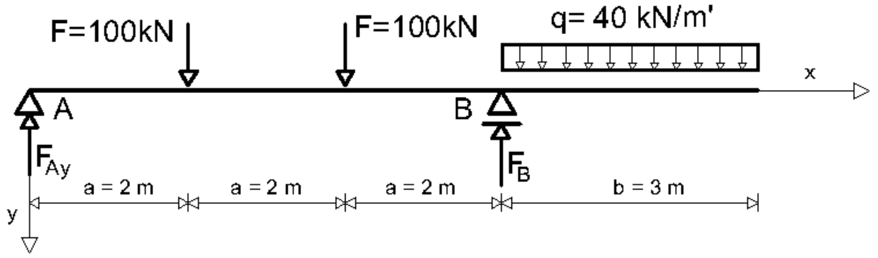
$$\left(\frac{S_x}{b}\right)_{\max} = \frac{bt\frac{h+t}{2} + \frac{h}{2}t\frac{h}{4}}{t} = 53280mm^2$$

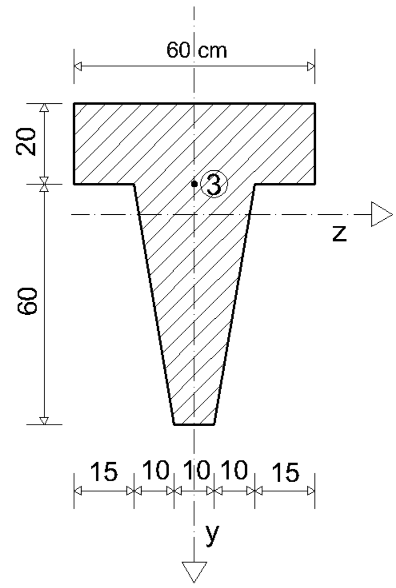
$$\tau_{zy\max} = \frac{112{,}5 \cdot 10^3}{4683{,}3 \cdot 16^4} \cdot 53280 \frac{N}{mm^4} mm^2$$

$$\tau_{zy\max} = 19{,}53 \frac{N}{mm^2} < \tau_{dop} = 60 \frac{N}{mm^2}$$

45. Zadatak

Za zadani nosač i opterećenje odrediti maksimalna normalna i posmična naprezanja u visini osi “z“te smjer i veličinu glavnih naprezanja u točki 3 poprečnog presjeka nad ležajem B.





Položaj težišta poprečnog presjeka nosača :

$$y_T = \frac{\sum A_i y_i}{\sum A_i} = \frac{20\cdot 60\cdot 70 + 2\left(\frac{10\cdot 60}{2}\right)\cdot 40 + 10\cdot 60\cdot 30}{20\cdot 60 + 2\left(\frac{10\cdot 60}{2}\right) + 10\cdot 60} = \frac{126000}{2400} = 52{,}5cm$$

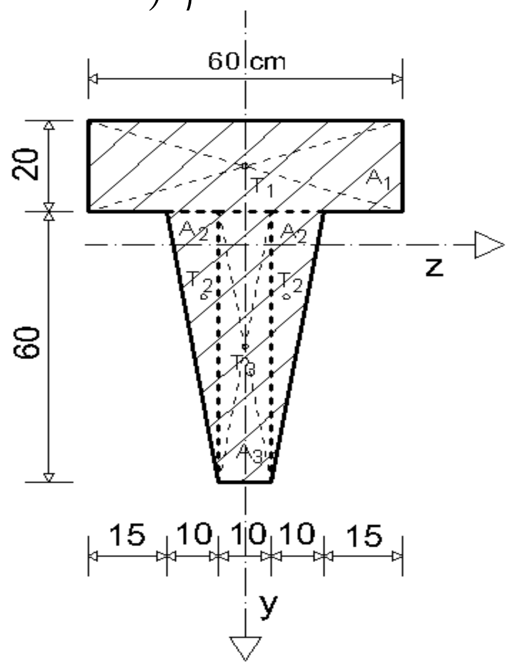
$$T(x_T, y_T) = T(30;52{,}5)$$

Moment površine drugog reda :

$$I_z = I_{z1} + A_1 y_1^{\ 2} + 2\left(I_{z2} + A_2 y_2^{\ 2}\right) + I_{z3} + A_3 y_3^{\ 2}$$

$$I_z = \frac{60\cdot 20^3}{12} + 60\cdot 20\cdot (70-52{,}5)^2 + 2\left(\frac{10\cdot 60^3}{36} + \frac{10\cdot 60}{2}\cdot\left(\frac{2}{3}60 - 52{,}5\right)^2\right) + \frac{10\cdot 60^3}{12} + 10\cdot 60\cdot(30-52{,}5)^2$$

$$\underline{I_z = 110{,}5\cdot 10^4\, cm^4 = 110{,}5\cdot 10^8\, mm^4}$$



Reakcije i unutarnje sile :

$$\underline{\sum M_B = 0} \rightarrow F_A\cdot 3a - F\cdot 2a - F\cdot a + q\cdot b\cdot\frac{b}{2} = 0$$

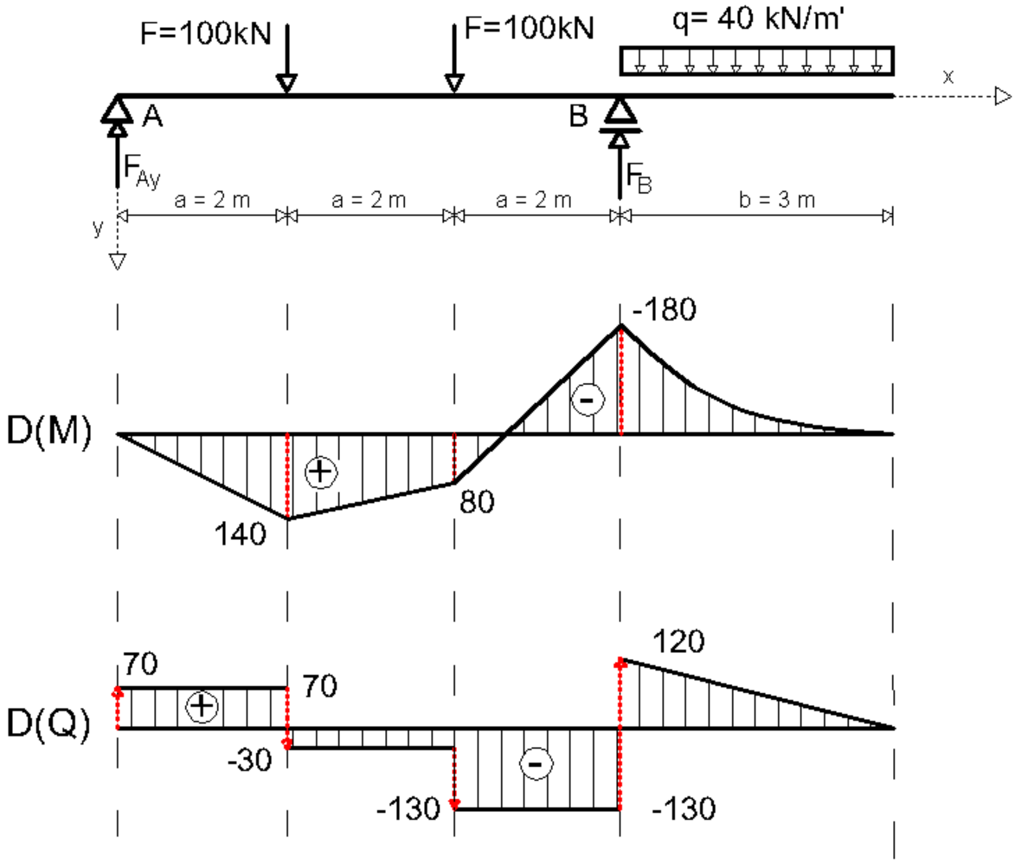
$$F_A = \frac{3Fa - q\frac{b^2}{2}}{3a} = \frac{3\cdot 100\cdot 2 - 40\frac{3^2}{2}}{3\cdot 2} \rightarrow \underline{F_A = 70kN}$$

$$\underline{\sum F_y = 0} \rightarrow F_A + F_B - 2F - q\cdot b = 0$$

$$\underline{F_B = 2F + qb - F_A = 2\cdot 100 + 40\cdot 3 - 70 \rightarrow \underline{F_B = 250kN}}$$

$$M_1 = F_A \cdot a = 70 \cdot 2 = 140 kNm$$

$$M_B = -q \cdot b \cdot \frac{b}{2} = -40 \cdot 3 \cdot \frac{3}{2} = -180 kNm = M_{\max}$$



$$M_{\max} = -180 kNm$$
$$T_{\max} = -130 kN$$

<u>Maksimalno normalno naprezanje :</u>

$$\sigma_1^{\,g} = \frac{M_{\max}}{I_z} y_1 = +\frac{180 \cdot 10^6}{110{,}5 \cdot 10^8}(80 - 52{,}5) \cdot 10 = +4{,}48 \frac{N}{mm^2}$$

$$\sigma_1^{\,d} = \frac{M_{\max}}{I_z} y_2 = -\frac{180 \cdot 10^6}{110{,}5 \cdot 10^8}(52{,}5) \cdot 10 = -8{,}55 \frac{N}{mm^2}$$

Posmično naprezanje u osi "z" :

$$S_z = 10 \cdot 52{,}5 \cdot \frac{52{,}5}{2} + 2\left(\frac{8{,}75 \cdot 52{,}5}{2}\right)\left(\frac{52{,}5}{3}\right) = 21820 cm^3 = 21{,}8210^6 mm^3$$

$$\frac{x}{52{,}5} = \frac{10}{60} \rightarrow x = \frac{52{,}5 \cdot 10}{60} = 8{,}75 cm$$

$$b = 2x + 10 = 2 \cdot 8{,}75 + 10 = 27{,}5 cm = 275 mm$$

$$\tau_z = \frac{T_{\max} \cdot S_z}{I_z \cdot b} = \frac{130 \cdot 10^3 \cdot 21{,}82 \cdot 10^6}{110{,}5 \cdot 10^8 \cdot 275} = 0{,}933 \frac{N}{mm^2}$$

Glavna naprezanja i njihov smjer u točki 3 :

$$S_z^{\,3} = 20 \cdot 60 \cdot (70 - 52{,}5) = 21000 cm^3 = 21 \cdot 10^6 mm^3$$

$$\tau_3 = \frac{T_{\max} \cdot S_z}{I_z \cdot b} = \frac{130 \cdot 10^3 \cdot 21 \cdot 10^6}{110{,}5 \cdot 10^8 \cdot 300} = 0{,}82 \frac{N}{mm^2}$$

$$\sigma_3 = \frac{M_{\max}}{I_z} y_3 = + \frac{180 \cdot 10^6}{110{,}5 \cdot 10^8} (60 - 52{,}5) \cdot 10 = +1{,}22 \frac{N}{mm^2}$$

$$\sigma_{1,2}^{(3)} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = \frac{1{,}22}{2} \pm \frac{1}{2}\sqrt{1{,}22^2 + 4 \cdot 0{,}82^2} = 0{,}61 \pm 1{,}022$$

$$\sigma_1^{(3)} = +1{,}632 \frac{N}{mm^2}$$

$$\sigma_2^{(3)} = -0{,}412 \frac{N}{mm^2}$$

$$tg 2\varphi = \frac{2 \cdot 0{,}82}{1{,}22} = 1{,}34 \rightarrow 2\varphi = 53{,}35° \rightarrow \varphi = 26{,}67°$$

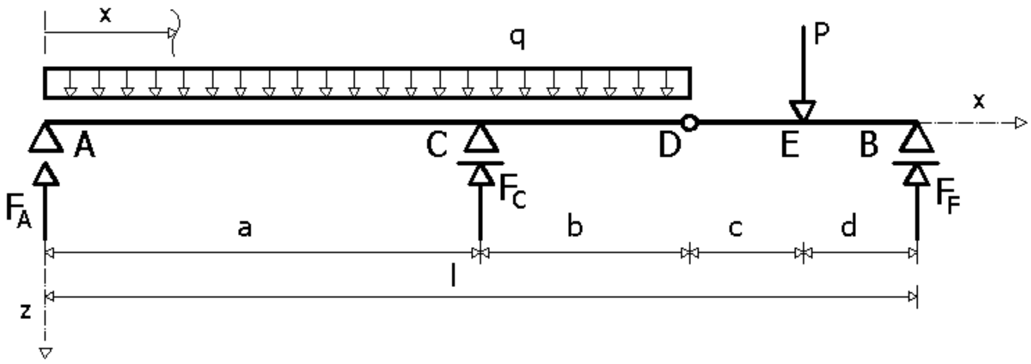
46. Zadatak

Koristeći analitički postupak određivanja elastične linije nosača odrediti kut zaokreta u točkama D i F te progib u točki E.

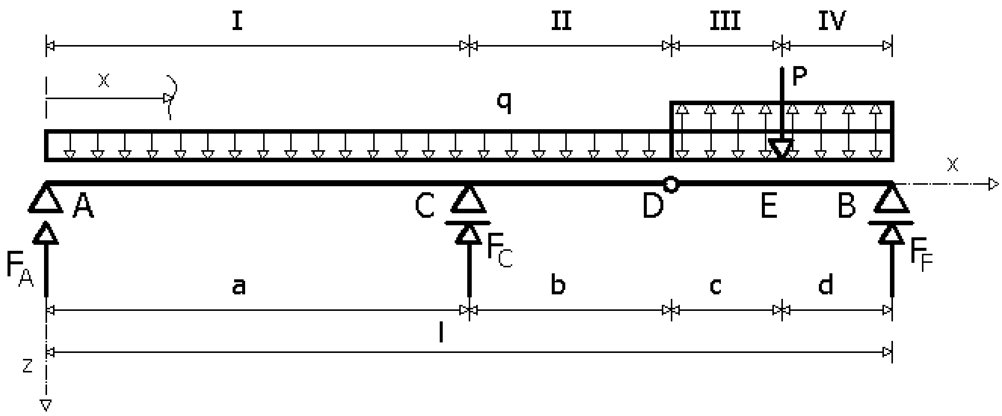
$a = 5m$ $\qquad q = 10\dfrac{kN}{m}$

$b = 2m$ $\qquad P = 50kN$

$c = d = 1m$ $\qquad EI_y = 8 \cdot 10^5\, Nm^2$



Ekvivalentni sustav



Reakcije u osloncima

$$\underline{\sum M_D^{\,d} = 0} \rightarrow F_F \cdot 2 - P \cdot 1 = 0 \rightarrow F_F = \frac{P}{2} = \frac{50}{2} = 25kN$$

$$\underline{\sum M_A = 0} \rightarrow q \cdot 7 \cdot 3{,}5 + P \cdot 8 - F_C \cdot 5 - F_F \cdot 9 = 0 \rightarrow F_C = \frac{420}{5} = 84kN$$

$$\underline{\sum F_z = 0} \rightarrow F_A - q \cdot 7 - P + F_F + F_C = 0 \rightarrow F_A = 11kN$$

Momenti savijanja na pojedinim segmentima :

$$(I)\, 0 \le x \le a \rightarrow M(x) = F_A \cdot x - q \cdot \frac{x^2}{2}$$

$$(II)\, a \le x \le (a+b) \rightarrow M(x) = F_A \cdot x - q \cdot \frac{x^2}{2} + F_C \cdot (x-5)$$

$$(III)\, (a+b) \le x \le (a+b+c) \rightarrow M(x) = F_A \cdot x - q \cdot \frac{x^2}{2} + F_C \cdot (x-5) + q \cdot \frac{(x-7)^2}{2}$$

$$(IV)\, (a+b+c) \le x \le (a+b+c+d) \rightarrow M(x) = F_A \cdot x - q \cdot \frac{x^2}{2} + F_C \cdot (x-5) + q \cdot \frac{(x-7)^2}{2} - P \cdot (x-8)$$

Iz diferencijalne jednadžbe elastične linije nosača slijedi za moment savijanja :

$$EI \frac{d^2 w}{dx^2} = -M(x)$$

$$(I)\, EI \frac{d^2 w}{dx^2} = -F_A \cdot x + q \cdot \frac{x^2}{2}$$

$$(II)\, EI \frac{d^2 w}{dx^2} = -F_A \cdot x + q \cdot \frac{x^2}{2} - F_C \cdot (x-5)$$

$$(III)\, EI \frac{d^2 w}{dx^2} = -F_A \cdot x + q \cdot \frac{x^2}{2} - F_C \cdot (x-5) - q \cdot \frac{(x-7)^2}{2}$$

$$(IV)\, EI \frac{d^2 w}{dx^2} = -F_A \cdot x + q \cdot \frac{x^2}{2} - F_C \cdot (x-5) - q \cdot \frac{(x-7)^2}{2} + P \cdot (x-8)$$

Za kut zaokreta :

$$(I)\, EI \frac{dw}{dx} = -F_A \cdot \frac{x^2}{2} + q \cdot \frac{x^3}{6} + C_1$$

$$(II)\, EI \frac{dw}{dx} = -F_A \cdot \frac{x^2}{2} + q \cdot \frac{x^3}{6} - F_C \cdot \frac{(x-5)^2}{2} + C_2$$

$$(III)\, EI \frac{dw}{dx} = -F_A \cdot \frac{x^2}{2} + q \cdot \frac{x^3}{6} - F_C \cdot \frac{(x-5)^2}{2} - q \cdot \frac{(x-7)^3}{6} + C_3$$

$$(IV)\, EI \frac{dw}{dx} = -F_A \cdot \frac{x^2}{2} + q \cdot \frac{x^3}{6} - F_C \cdot \frac{(x-5)^2}{2} - q \cdot \frac{(x-7)^3}{6} + P \cdot \frac{(x-8)^2}{2} + C_4$$

Za progib nosača :

$$(I)\, EIw = -F_A \cdot \frac{x^3}{6} + q \cdot \frac{x^4}{24} + C_1 \cdot x + D_1$$

$$(II)\, EIw = -F_A \cdot \frac{x^3}{6} + q \cdot \frac{x^4}{24} - F_C \cdot \frac{(x-5)^3}{6} + C_2 \cdot x + D_2$$

$$(III)\, EIw = -F_A \cdot \frac{x^3}{6} + q \cdot \frac{x^4}{24} - F_C \cdot \frac{(x-5)^3}{6} - q \cdot \frac{(x-7)^4}{24} + C_3 \cdot x + D_3$$

$$(IV)\, EIw = -F_A \cdot \frac{x^3}{6} + q \cdot \frac{x^4}{24} - F_C \cdot \frac{(x-5)^3}{6} - q \cdot \frac{(x-7)^4}{24} + P \cdot \frac{(x-8)^3}{6} + C_4 \cdot x + D_4$$

Uvjeti kompatibilnosti deformacija :

$$\varphi^{(I)}(a)=\varphi^{(II)}(a)\rightarrow \underline{C_1=C_2}$$

$$\varphi^{(III)}(a+b+c)=\varphi^{(IV)}(a+b+c)\rightarrow \underline{C_3=C_4}$$

$$w^{(I)}(a)=w^{(II)}(a)\rightarrow \underline{D_1=D_2}$$

$$w^{(II)}(a+b)=w^{(III)}(a+b)\rightarrow \underline{C_2\cdot 7+D_2=C_3\cdot 7+D_3} \ldots (1)$$

$$w^{(III)}(a+b+c)=w^{(IV)}(a+b+c)\rightarrow C_3\cdot 8+D_3=C_4\cdot 8+D_4\rightarrow \underline{D_3=D_4}$$

Rubni uvjeti :

$$w(0)=0\rightarrow D_1=0\rightarrow \underline{D_2=D_1=0}$$

$$w(a)=w^{(I)}(x=5)=0\rightarrow -11\frac{5^3}{6}+10\frac{5^4}{24}+C_1\cdot 5=0\rightarrow \underline{C_1=-6{,}25=C_2}$$

$$w(l)=w^{(IV)}(x=9)=0\rightarrow -11\cdot\frac{9^3}{6}+10\cdot\frac{9^4}{24}-84\cdot\frac{(9-5)^3}{6}-10\cdot\frac{(9-7)^4}{24}+50\cdot\frac{(9-8)^3}{6}+C_4\cdot 9+D_4=0$$

$$\underline{502{,}917+9\cdot C_4+D_4=0} \ldots (2)$$

$$(1)\Rightarrow -6{,}25\cdot 7+0=C_4\cdot 7+D_4\rightarrow \underline{D_4=-43{,}75-7\cdot C_4} \ldots (3)$$

$$(3)\rightarrow(2)\Rightarrow \underline{C_4=-229{,}58=C_3}$$

$$\underline{D_4=1563{,}31=D_3}$$

Progib u točki E :

$$w_E=w^{(III)}(x=8)=w^{(IV)}(x=8)=\frac{1}{EI_y}\left[-F_A\cdot\frac{x^3}{6}+q\cdot\frac{x^4}{24}-F_C\cdot\frac{(x-5)^3}{6}-q\cdot\frac{(x-7)^4}{24}+C_3\cdot x+D_3\right]$$

$$w_E=\frac{10^3}{8\cdot 10^6}\left[-11\cdot\frac{8^3}{6}+10\cdot\frac{8^4}{24}-84\cdot\frac{(8-5)^3}{6}-10\cdot\frac{(8-7)^4}{24}-229{,}58\cdot 8+1563{,}31\right]$$

$$\underline{w_E=14{,}53\cdot 10^{-3}m=14{,}53mm}$$

Kut zaokreta u točki F :

$$\varphi_F=\varphi^{(IV)}(x=9)=\frac{1}{EI_y}\left[-F_A\cdot\frac{x^2}{2}+q\cdot\frac{x^3}{6}-F_C\cdot\frac{(x-5)^2}{2}-q\cdot\frac{(x-7)^3}{6}+P\cdot\frac{(x-8)^2}{2}+C_4\right]$$

$$\varphi_F=\frac{10^3}{8\cdot 10^6}\left[-11\cdot\frac{9^2}{2}+10\cdot\frac{9^3}{6}-84\cdot\frac{(9-5)^2}{2}-10\cdot\frac{(9-7)^3}{6}+50\cdot\frac{(9-8)^2}{2}-229{,}58\right]$$

$$\underline{\varphi_F=-15{,}05\cdot 10^{-3}rad=-0{,}01505rad}$$

Kut zaokreta u točki D :

$$\varphi_D = \varphi_D{}^l + \varphi_D{}^d = \varphi^{(II)}(x=7) + \varphi^{(III)}(x=7)$$

$$\varphi_D{}^l = \frac{1}{EI_y}\left[-F_A \cdot \frac{x^2}{2} + q \cdot \frac{x^3}{6} - F_C \cdot \frac{(x-5)^2}{2} + C_2\right]$$

$$\varphi_D{}^d = \frac{1}{EI_y}\left[-F_A \cdot \frac{x^2}{2} + q \cdot \frac{x^3}{6} - F_C \cdot \frac{(x-5)^2}{2} - q \cdot \frac{(x-7)^3}{6} + C_3\right]$$

$$\varphi_D{}^l = \frac{10^3}{8 \cdot 10^6}\left[-11 \cdot \frac{7^2}{2} + 10 \cdot \frac{7^3}{6} - 84 \cdot \frac{(7-5)^2}{2} - 6{,}25\right]$$

$$\varphi_D{}^l = 15{,}99 \cdot 10^{-3} rad = 0{,}01599 rad$$

$$\varphi_D{}^d = \frac{10^3}{8 \cdot 10^6}\left[-11 \cdot \frac{7^2}{2} + 10 \cdot \frac{7^3}{6} - 84 \cdot \frac{(7-5)^2}{2} - 10 \cdot \frac{(7-7)^3}{6} - 229{,}58\right]$$

$$\varphi_D{}^d = -11{,}93 \cdot 10^{-3} rad = -0{,}01193$$

$$\underline{\varphi_D = 0{,}01599 - 0{,}01193 = 0{,}00406 rad}$$

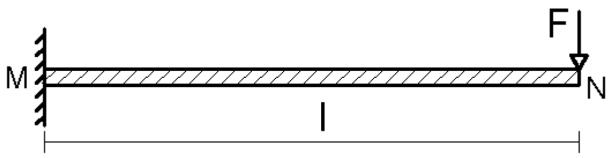
47. Zadatak

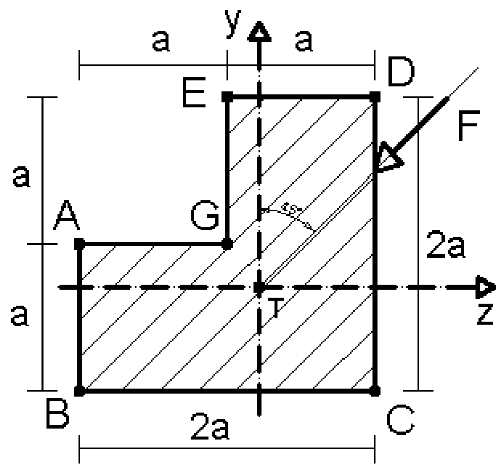
Odrediti točke presjeka u kojima se javljaju ekstremna naprezanja i njihove vrijednosti.

$F = 10kN$

$l = 5m$

$a = 20cm = 0{,}2m$





$$A = 2a \cdot a + a^2 = 3a^2$$

$$y_T = \frac{a^2 \cdot \frac{a}{2} + 2a^2 \cdot a}{3a^2} = \frac{5}{6}a$$

$$z_T = \frac{a^2 \cdot \frac{a}{2} + 2a^2 \cdot \frac{3}{2}a}{3a^2} = \frac{7}{6}a$$

$$I_y^{'} = \left[\frac{a \cdot a^3}{12} + a^2 \cdot \left(\frac{a}{2} - \frac{7}{6}a\right)^2\right] + \left[\frac{2a \cdot a^3}{12} + 2a^2 \cdot \left(\frac{3}{2}a - \frac{7}{6}a\right)^2\right] = \frac{11}{12}a^4$$

$$I_z^{'} = \left[\frac{a \cdot a^3}{12} + a^2 \cdot \left(\frac{a}{2} - \frac{5}{6}a\right)^2\right] + \left[\frac{a \cdot (2a)^3}{12} + 2a^2 \cdot \left(a - \frac{5}{6}a\right)^2\right] = \frac{11}{12}a^4$$

$$I_{yz}^{'} = \left[0 + a^2 \cdot \left(\frac{a}{2} - \frac{7}{6}a\right) \cdot \left(\frac{a}{2} - \frac{5}{6}a\right)\right] + \left[0 + 2a^2 \cdot \left(\frac{3}{2}a - \frac{7}{6}a\right) \cdot \left(a - \frac{5}{6}a\right)\right] = \frac{1}{3}a^4$$

<u>Glavne osi momenata površine drugog reda :</u>

$$tg2\alpha = -\frac{2I_{yz}^{'}}{I_y^{'} - I_z^{'}} = -\frac{2 \cdot \frac{1}{3}a^4}{\frac{11}{12}a^4 - \frac{11}{12}a^4} = -\frac{2 \cdot \frac{1}{3}a^4}{0} = \infty \rightarrow 2\alpha = 90° \rightarrow \alpha = 45°$$

$$I_{1,2} = \frac{1}{2}\left(I_y^{'} + I_z^{'}\right) \pm \frac{1}{2}\sqrt{\left(I_y^{'} - I_z^{'}\right)^2 + 4\left(I_{yz}^{'}\right)^2} = \frac{1}{2} \cdot 2 \cdot \frac{11}{12}a^4 \pm \frac{1}{2} \cdot 2\frac{1}{3}a^4 = \left(\frac{11}{12} \pm \frac{1}{3}\right)a^4$$

$$I_1 = \left(\frac{11}{12} + \frac{1}{3}\right)a^4 = \frac{5}{4}a^4 = I_y$$

$$I_2 = \left(\frac{11}{12} - \frac{1}{3}\right)a^4 = \frac{7}{12}a^4 = I_z$$

$$I_{yz} = 0$$

Položaj neutralne osi :

$$tg\varphi = -\frac{I_y}{I_z} tg\alpha = -\frac{\frac{5}{4}a^4}{\frac{7}{12}a^4} tg45° = -\frac{15}{7} = -2{,}14286 \rightarrow \varphi = -65°$$

Ekstremna naprezanja su u točkama B i D :

$$\sigma_B = M\left(\frac{\cos 45°}{I_y} z_B + \frac{\sin 45°}{I_z} y_B\right) = Fl\frac{\sqrt{2}}{2}\left(\frac{-\sqrt{2}a}{\frac{5}{4}a^4} + \frac{\frac{\sqrt{2}}{6}a}{\frac{7}{12}a^4}\right) = \frac{Fl}{a^3}\left(-\frac{4}{5}+\frac{2}{7}\right) = -\frac{18}{35}\frac{Fl}{a^3} = -\frac{18}{35}\frac{10\cdot 10^3\cdot 5}{0{,}2^3}$$

$$\sigma_B = -3{,}21 MPa$$

$$\sigma_D = M\left(\frac{\cos 45°}{I_y} z_D + \frac{\sin 45°}{I_z} y_D\right) = Fl\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}a}{\frac{5}{4}a^4} + \frac{\frac{\sqrt{2}}{6}a}{\frac{7}{12}a^4}\right) = \frac{Fl}{a^3}\left(\frac{4}{5}+\frac{2}{7}\right) = \frac{38}{35}\frac{Fl}{a^3} = \frac{38}{35}\frac{10\cdot 10^3\cdot 5}{0{,}2^3}$$

$$\sigma_D = 6{,}78 MPa$$

$$z_B = \frac{-7}{6}a\cos 45° + \frac{-5}{6}a\sin 45° = \frac{-7}{6}\frac{\sqrt{2}}{2}a + \frac{-5}{6}\frac{\sqrt{2}}{2}a = -\sqrt{2}a$$

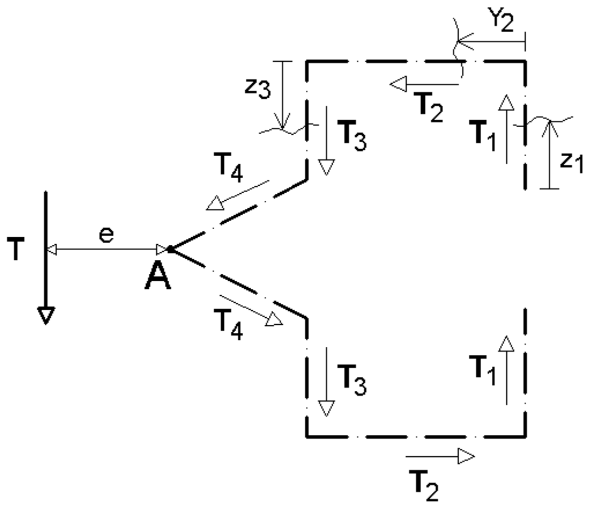
$$y_B = \frac{-5}{6}a\cos 45° - \frac{-7}{6}a\sin 45° = \frac{-5}{6}\frac{\sqrt{2}}{2}a + \frac{7}{6}\frac{\sqrt{2}}{2}a = \frac{\sqrt{2}}{6}a$$

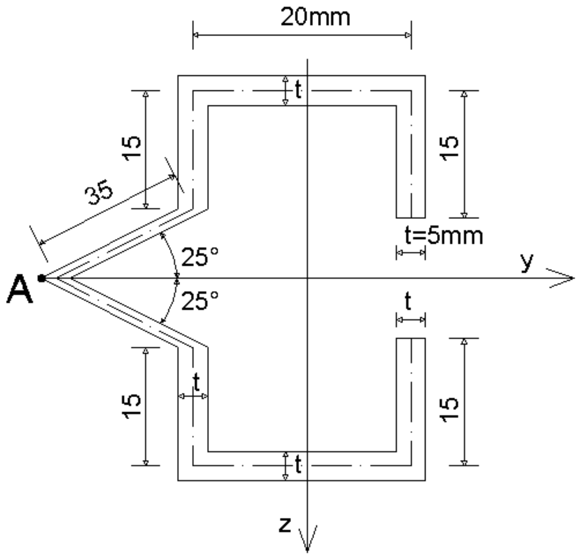
$$z_D = \frac{5}{6}a\cos 45° + \frac{7}{6}a\sin 45° = \frac{5}{6}\frac{\sqrt{2}}{2}a + \frac{7}{6}\frac{\sqrt{2}}{2}a = \sqrt{2}a$$

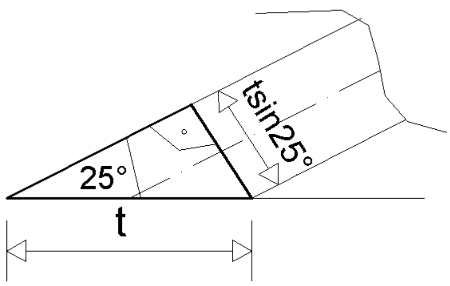
$$y_D = \frac{7}{6}a\cos 45° - \frac{5}{6}a\sin 45° = \frac{7}{6}\frac{\sqrt{2}}{2}a - \frac{5}{6}\frac{\sqrt{2}}{2}a = \frac{\sqrt{2}}{6}a$$

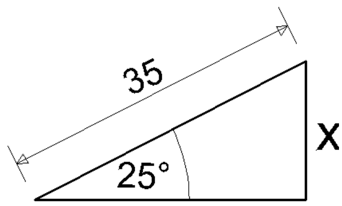
48. Zadatak

Za zadani tankostijeni presjek prikazan na slici potrebno je odrediti središte posmika u odnosu na točku A.









$\underline{x = 35 \cdot \sin 25° = 14{,}8mm}$

$$I_y = 2\left[2\left(\frac{5\cdot 15^3}{12} + 5\cdot 15\cdot(29{,}8-7{,}5)^2\right) + \frac{20\cdot 5^3}{12} + 20\cdot 5\cdot(29{,}8)^2 + \frac{5\cdot 14{,}8^3}{12} + 5\cdot 14{,}8\cdot\left(\frac{14{,}8}{2}\right)^2\right]$$

$$\underline{I_y = 3{,}4364\cdot 10^5\, mm^4}$$

$$\tau_{xz1} = \frac{T\cdot S_{y1}}{I_y\cdot b} = \frac{T}{I_y\cdot t}\left(z_1\cdot t\cdot\left(14{,}8+\frac{z_1}{2}\right)\right) = \frac{T}{I_y}\left(14{,}8z_1 + \frac{z_1{}^2}{2}\right)$$

$$T_1 = \int_0^{15}\frac{T}{I_y}\left(14{,}8z_1 + \frac{z_1{}^2}{2}\right)\cdot t\cdot dz_1 = \frac{T\cdot t}{I_y}\int_0^{15}\left(14{,}8z_1 + \frac{z_1{}^2}{2}\right)\cdot dz_1 = \frac{T\cdot t}{I_y}\left(14{,}8\frac{z_1{}^2}{2} + \frac{z_1{}^3}{6}\right)\Bigg|_0^{15}$$

$$\underline{T_1 = 0{,}0324\cdot T}$$

$$\tau_{xy2} = \frac{T\cdot S_{y2}}{I_y\cdot b} = \frac{T}{I_y\cdot t}\left(S_{y1} + S_{y2}\right) = \frac{T}{I_y\cdot t}\left(1672{,}5 + y_2\cdot t\cdot 29{,}8\right)$$

$$S_{y1} = 5\cdot 15\cdot(14{,}8 + 15/2) = 1672{,}5 mm^3$$

$$T_2 = \frac{T}{I_y\cdot t}\int_0^{20}\left(1672{,}5 + 149y_2\right)\cdot t\cdot dy_2 = \frac{T}{I_y}\left(1672{,}5y_2 + 149\frac{y_2{}^2}{2}\right)\Bigg|_0^{20}$$

$$\underline{T_2 = 0{,}184\cdot T}$$

$$\tau_{xz3} = \frac{T\cdot S_{y3}}{I_y\cdot b} = \frac{T}{I_y\cdot t}\left(S_{y1} + S_{y2} + S_{y3}\right) = \frac{T}{I_y\cdot t}\left(1672{,}5 + 2980 + z_3\cdot t\cdot\left(29{,}8 - \frac{z_3}{2}\right)\right) =$$

$$= \frac{T}{I_y\cdot t}\left(4652{,}5 + 149z_3 - 5\frac{z_3{}^2}{2}\right)$$

$$S_{y2} = 5\cdot 20\cdot 29{,}8 = 2980 mm^3$$

$$T_3 = \int_0^{15}\frac{T}{I_y\cdot t}\left(4652{,}5 + 149z_3 - 5\frac{z_3{}^2}{2}\right)\cdot t\cdot dz_3 = \frac{T}{I_y}\left(4652{,}5z_3 + 149\frac{z_3{}^2}{2} - 5\frac{z_3{}^3}{6}\right)\Bigg|_0^{15}$$

$$\underline{T_3 = 0{,}244\cdot T}$$

$$T\cdot e = 2\cdot T_1\cdot\left(20 + 35\cdot\cos 25°\right) + 2\cdot T_2\cdot 29{,}8 - 2\cdot T_3\cdot\left(35\cdot\cos 25°\right)$$

$$T\cdot e = 103{,}44\cdot 0{,}0324T + 59{,}6\cdot 0{,}184T - 31{,}72\cdot 2\cdot 0{,}244T$$

$$\underline{e = -1{,}16 mm}$$

49. Zadatak

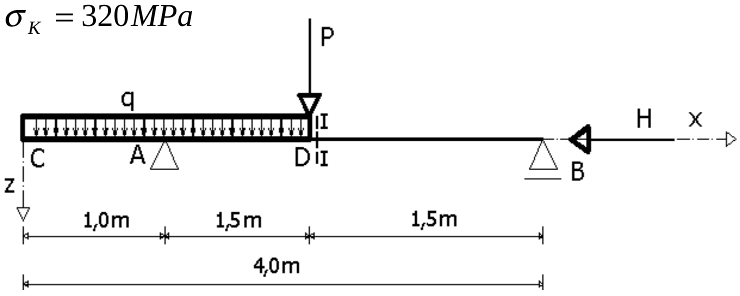
Za zadani nosač i opterećenje odrediti mjerodavan koeficijent sigurnosti prema teoriji najvećih normalnih naprezanja u točkama 1 i 2 presjeka I-I ako je zadano kritično naprezanje materijala σ_K.

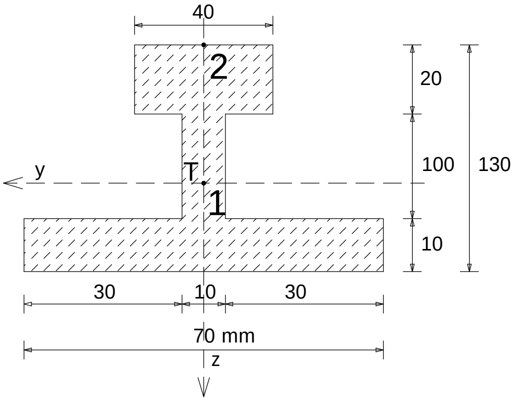
$P = 30kN$

$H = 25kN$

$q = 15kN/m$

$\sigma_K = 320MPa$





$\Sigma F_x = 0:$

$R_{Ax} - H = 0 \Rightarrow \underline{R_{Ax} = H = 25kN}$

$\Sigma M_A = 0:$

$$-q \cdot 1 \cdot \frac{1}{2} + q \cdot \frac{1{,}5^2}{2} + P \cdot 1{,}5 - R_B \cdot 3 = 0 \Rightarrow \underline{R_B = 18{,}125kN}$$

$\Sigma F_y = 0:$

$R_{Ay} + R_B - q \cdot 2{,}5 - P = 0 \Rightarrow \underline{R_{Ay} = 49{,}375kN}$

<u>Presjek I – I :</u>

$N = -25kN$

$T = -18{,}125kN$

$M = 27{,}1875kNm$

$$y_T = \frac{10 \cdot 70 \cdot 5 + 10 \cdot 100 \cdot 60 + 20 \cdot 40 \cdot 120}{10 \cdot 70 + 10 \cdot 100 + 20 \cdot 40} = \frac{159500}{2500} \Rightarrow \underline{y_T = 63{,}8mm}$$

$\underline{A = 2500mm^2}$

$$I_y = \frac{70 \cdot 10^3}{12} + 70 \cdot 10 \cdot (63{,}8 - 5)^2 + \frac{10 \cdot 100^3}{12} + 10 \cdot 100 \cdot (63{,}8 - 60)^2 + \frac{40 \cdot 20^3}{12} + 40 \cdot 20 \cdot (66{,}2 - 10)^2$$

$\underline{I_y = 5{,}83 \cdot 10^6 mm^4}$

$$W_{y\max} = \frac{I_y}{z_{\max}} = \frac{5{,}83 \cdot 10^6}{66{,}2} \Rightarrow \underline{W_{y\max} = 88{,}1 \cdot 10^3 mm^3}$$

$$S_{y\max} = 40 \cdot 20 \cdot (66{,}2 - 10) + (66{,}2 - 20) \cdot 10 \cdot \frac{66{,}2 - 20}{2} \Rightarrow \underline{S_{y\max} = 55{,}6 \cdot 10^3 mm^3}$$

Točka 1 :

$$\sigma_x^{\,1} = \frac{N}{A} = \frac{-25\cdot 10^3}{2500} = -10\frac{N}{mm^2} \Rightarrow \underline{\sigma_x^{\,1} = -10MPa}$$

$$\tau_{xz}^{\,1} = \frac{T\cdot S_{y\max}}{I_y\cdot b} = \frac{18{,}125\cdot 10^3\cdot 55{,}6\cdot 10^3}{5{,}83\cdot 10^6\cdot 10} = 17{,}29\frac{N}{mm^2} \Rightarrow \underline{\tau_{xz}^{\,1} = 17{,}29MPa}$$

Točka 2 :

$$\sigma_x^{\,2} = \frac{M}{W_y} + \frac{N}{A} = \frac{-27{,}1875\cdot 10^6}{88{,}1\cdot 10^3} + \frac{-25\cdot 10^3}{2500} = -308{,}6 - 10 = -318{,}6\frac{N}{mm^2} \Rightarrow \underline{\sigma_x^{\,2} = -318{,}6MPa}$$

$$\tau_{xz}^{\,2} = \frac{T\cdot S_y^{\,2}}{I_y\cdot b} = \frac{18{,}125\cdot 10^3\cdot 0}{5{,}83\cdot 10^6\cdot 40} = 0 \Rightarrow \underline{\tau_{xz}^{\,2} = 0}$$

$$S_y^{\,2} = 0$$

Provjera po teoriji najvećih normalnih naprezanja (1. teorija čvrstoće) :

Točka 1 :

$$\sigma_{ek}^{\,1} = 0{,}5\cdot\sigma_x^{\,1} + 0{,}5\cdot\sqrt{\left(\sigma_x^{\,1}\right)^2 + 4\cdot\left(\tau_{xz}^{\,1}\right)^2}$$

$$\sigma_{ek}^{\,1} = 0{,}5\cdot(-10) + 0{,}5\cdot\sqrt{(-10)^2 + 4\cdot(17{,}29)^2} \Rightarrow \underline{\sigma_{ek}^{\,1} = 13{,}00MPa}$$

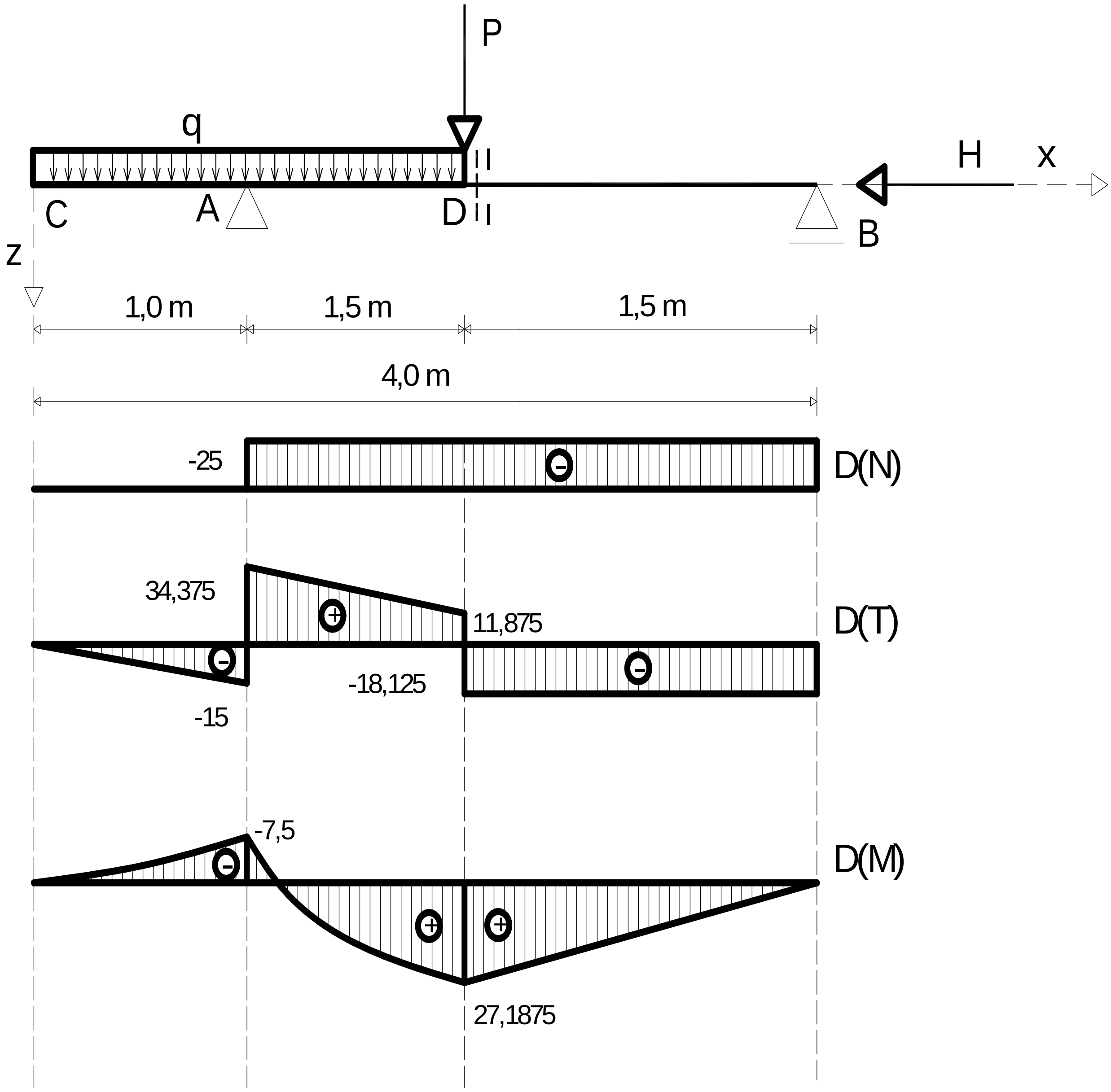
$$k^1 = \frac{\sigma_K}{\sigma_{ek}^{\,1}} = \frac{320}{13} \Rightarrow \underline{k^1 = 24{,}6}$$

Točka 2 :

$$\sigma_{ek}^{\,2} = 0{,}5\cdot\sigma_x^{\,2} + 0{,}5\cdot\sqrt{\left(\sigma_x^{\,2}\right)^2 + 4\cdot\left(\tau_{xz}^{\,2}\right)^2}$$

$$\sigma_{ek}^{\,1} = 0{,}5\cdot(-318{,}6) + 0{,}5\cdot\sqrt{(-318{,}6)^2 + 4\cdot(0)^2} \Rightarrow \underline{\sigma_{ek}^{\,1} = \sigma_x^{\,2} = -318{,}6MPa}$$

$$k^2 = \frac{\sigma_K}{\sigma_{ek}^{\,2}} = \frac{320}{318{,}6} \Rightarrow \underline{k^2 = 1{,}0}$$


P
q
H
x
C
A
D
B
z
1,0 m
1,5 m
1,5 m
4,0 m
-25
D(N)
34,375
11,875
-18,125
-15
D(T)
-7,5
D(M)
27,1875