# 1. MATEMATIČKE OSNOVE

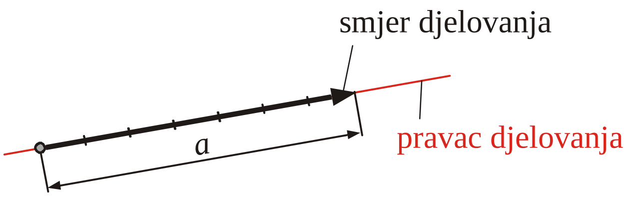
## 1.1 Veličine u fizici

U fizici razlikujemo skalarne, vektorske i tenzorske veličine

- skalarne: $m$, $V$, $T$, $\rho$,...
- vektorske: $\vec{r}$ , $\vec{v}$ , $\vec{a}$ , $\vec{F}$ , $\vec{M}$ , …
- tenzorske: $\mathbf{T}$ (tenzor naprezanja), $\mathbf{D}$ (tenzor deformacije), $\mathbf{I}$ (tenzor momenata inercije),

## 1.2 Vektori i operacije s vektorima

Vektori su određeni modulom te smjerom i pravcem djelovanja.

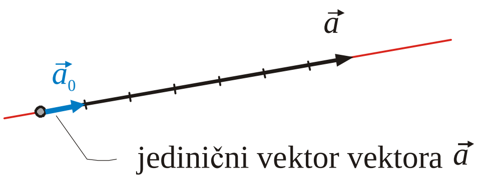



$a$ - modul ili veličina vektora, $a = |\vec{a}|$

➢ suprotan vektor

-$\vec{a}$ suprotan vektor vektoru $\vec{a}$

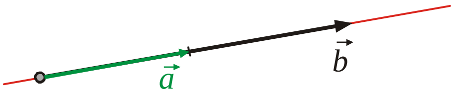
➢ jedinični vektor (ort)




$$\vec{a}_0 = \frac{\vec{a}}{a} \text{ ili } \vec{a} = a\,\vec{a}_0\,, \quad |\vec{a}_0| = a_0 = 1$$

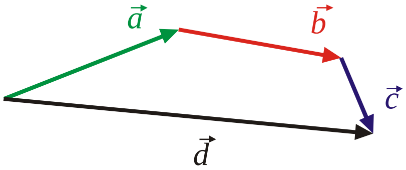
- Množenje vektora skalarom, $\vec{b} = \lambda\vec{a}$

Umnožak skalara i vektora je vektor.
Npr. $\vec{b} = 2\vec{a}$



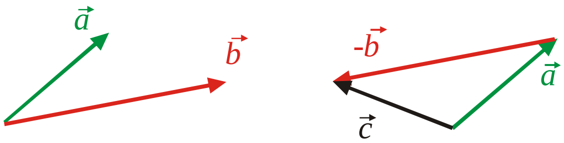

- Zbrajanje vektora, $\vec{d} = \vec{a} + \vec{b} + \vec{c}$

Zbroj dvaju ili više vektora je vektor.

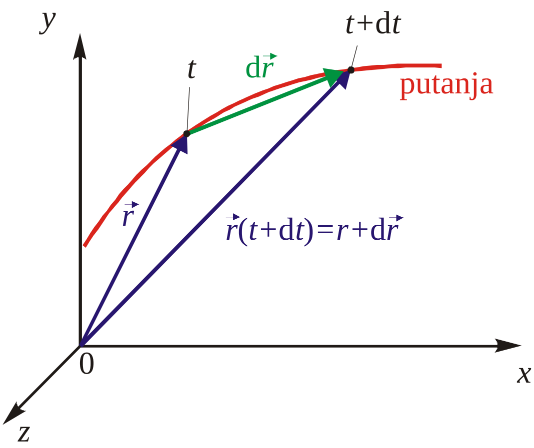



- Oduzimanje vektora = zbrajanje suprotnog vektora

$$\vec{c} = \vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$$



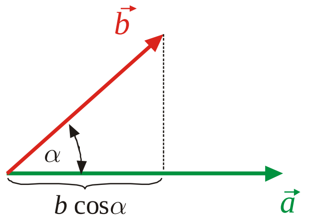

Npr.




- nul vektor , (za $\vec{a} = \vec{b}$ ) $\vec{a} - \vec{b} = \vec{0}$ , $\left|\vec{0}\right| = 0$

- Skalarni umnožak vektora: $\lambda = \vec{a} \cdot \vec{b}$

Skalarni umnožak dvaju vektora je skalar čiji je modul jednak umnošku modula obaju vektora i kosinusa kuta između njih.




$$\vec{a} \cdot \vec{b} = a \cdot b \cos\alpha$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

- Ako su oba vektora jedinična ( $a = 1$ , $b = 1$): $\vec{a}_0 \cdot \vec{b}_0 = \cos\alpha$
  Skalarni umnožak dvaju jediničnih vektora jednak je kosinusu kuta među njima.

- Za $a = 1$ $\quad \vec{a}_0 \cdot \vec{b} = b\cos\alpha = b_a$ $\quad$ projekcija vektora $\vec{b}$ na smjer vektora $\vec{a}_0$
- Za $b = 1$ $\quad \vec{a} \cdot \vec{b}_0 = a\cos\alpha = a_b$ $\quad$ projekcija vektora $\vec{a}$ na smjer vektora $\vec{b}_0$

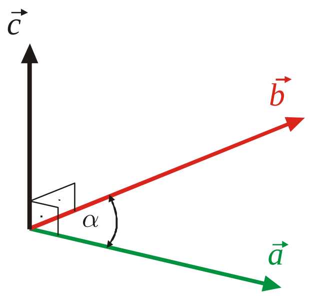
$$\vec{a} \cdot \vec{b} = a \cdot \underbrace{b\cos\alpha}_{b_a} = \underbrace{a\cos\alpha}_{a_b} \cdot b$$

$\vec{a} \cdot \vec{b} =$ umnožak modula vektora $\vec{a}$ i projekcije vektora $\vec{b}$ na vektor $\vec{a}$ ili
umnožak modula $\vec{b}$ i projekcije vektora $\vec{a}$ na vektor $\vec{b}$

Npr. $\quad \vec{a} \cdot \vec{a} = a \cdot a \cdot \cos(0) = a^2$

- Vektorski umnožak vektora, $\vec{c} = \vec{a} \times \vec{b}$

Vektorski (vanjski) umnožak dvaju vektora je vektor, koji je okomit na oba vektora koja čine produkt, a po veličini je jednak umnošku modula tih vektora te sinusa kuta između vektora.

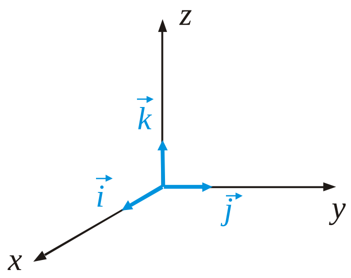



1) $c = ab\sin\alpha$
2) pravac $c$ je okomit na $\vec{a}$ i $\vec{b}$
3) smjer $\vec{c}$ po pravilu desne ruke

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

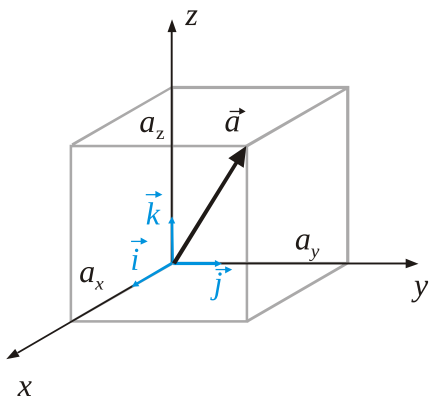
## 1.3 Prikaz vektora komponentama

Prethodne definicije su pogodne za geometrijsku interpretaciju operacija s vektorima. Za potrebe analitičkog računanja s vektorima nužno je uvesti koordinatni sustav i prikazati vektore komponentama.




Ortovi $\vec{i}, \vec{j}, \vec{k}$ čine bazu vektorskog prostora.

- Svaki se vektor u prostoru može prikazati linearnom kombinacijom baznih ortova. Koeficijenti te linearne kombinacije su komponente vektora.




$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \quad \vec{a} = \left(a_x, a_y, a_z\right)$$

$$\left.\begin{aligned} a_x &= \vec{a}\cdot\vec{i} = a\cos\left(x,\vec{a}\right) \\ a_y &= \vec{a}\cdot\vec{j} = a\cos\left(y,\vec{a}\right) \\ a_z &= \vec{a}\cdot\vec{k} = a\cos\left(z,\vec{a}\right) \end{aligned}\right\}$$

Komponente vektora su projekcije vektora $\vec{a}$ na smjerove ortova

➢ Analitički izrazi za operacije s vektorima

- množenje vektora skalarom, $\vec{b} = \lambda\cdot\vec{a} = \begin{cases} b_x = \lambda a_x \\ b_y = \lambda a_y \\ b_z = \lambda a_z \end{cases}$

- zbrajanje vektora, $\vec{c} = \vec{a} + \vec{b} = \begin{cases} c_x = a_x + b_x \\ c_y = a_y + b_y \\ c_z = a_z + b_z \end{cases}$

- skalarni umnožak vektora

$$\lambda = \vec{a} \cdot \vec{b} = \left( a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \right) \cdot \left( b_x \vec{i} + b_y \vec{j} + b_z \vec{k} \right)$$

$$\lambda = \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Npr. $a^2 = \vec{a} \cdot \vec{a} = a_x^2 + a_y^2 + a_z^2$

$$\vec{a}_0 = \frac{\vec{a}}{a} = \frac{a_x}{a} \vec{i} + \frac{a_y}{a} \vec{j} + \frac{a_z}{a} \vec{k}$$

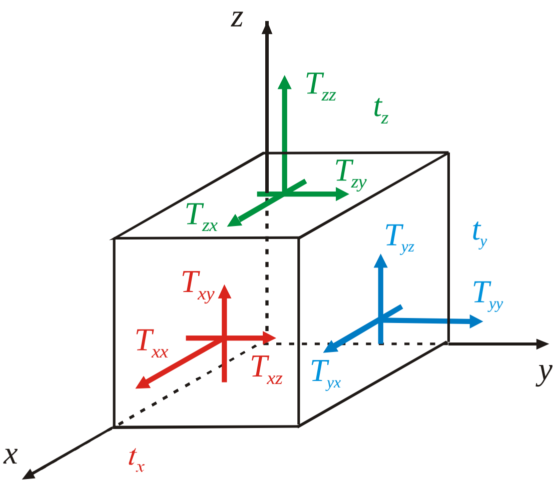
$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \left( a_y b_z - a_z b_y \right) \vec{i} + \left( a_z b_x - a_x b_z \right) \vec{j} + \left( a_x b_y - a_y b_x \right) \vec{k}$$

- mješoviti umnožak , $\lambda = \vec{d} \cdot (\vec{a} \times \vec{b})$

Skalarni umnožak vektora $\vec{d}$ s vektorom $(\vec{a} \times \vec{b})$ je skalar. Geometrijski gledano skalarno vektorski umnožak je volumen paralelopipeda kojemu su bridovi $\vec{d}, \vec{a}$ i $\vec{b}$

$$\lambda = \vec{d} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} d_x & d_y & d_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

## 1.3 Tenzori i operacije s tenzorima




baza tenzorskog prostora

$$\begin{bmatrix} \vec{i}\vec{i} & \vec{i}\vec{j} & \vec{i}\vec{k} \\ \vec{j}\vec{i} & \vec{j}\vec{j} & \vec{j}\vec{k} \\ \vec{k}\vec{i} & \vec{k}\vec{j} & \vec{k}\vec{k} \end{bmatrix}$$

- Označavanje komponenti tenzora $\mathbf{T}_{\underset{\text{površina}}{\cdot}\overset{\text{smjer}}{\cdot}}$

$$\begin{aligned}\mathbf{T} = {} & T_{xx}\vec{i}\vec{i} + T_{xy}\vec{i}\vec{j} + T_{xz}\vec{i}\vec{k} \\ & + T_{yx}\vec{j}\vec{i} + T_{yy}\vec{j}\vec{j} + T_{yz}\vec{j}\vec{k} \\ & + T_{zx}\vec{k}\vec{i} + T_{zy}\vec{k}\vec{j} + T_{zz}\vec{k}\vec{k}\end{aligned}$$

ili

$$\mathbf{T} = \begin{array}{c} \left[\begin{matrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{matrix}\right] \end{array} \begin{matrix} \vec{t}_x \\ \vec{t}_y \\ \vec{t}_z \end{matrix}$$

glavna dijagonala tenzora

- Množenje tenzora skalarom

$$\mathbf{B} = \lambda \mathbf{A} \quad \text{ili} \quad \begin{aligned} B_{xx} &= \lambda A_{xx} \\ B_{xy} &= \lambda A_{xy} \\ &\vdots \\ B_{zz} &= \lambda B_{zz} \end{aligned}$$

- Zbrajanje tenzora

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{ili} \quad \begin{aligned} C_{xx} &= A_{xx} + B_{xx} \\ C_{xy} &= A_{xy} + B_{xy} \\ &\vdots \\ C_{zz} &= A_{zz} + B_{zz} \end{aligned}$$

- Tenzorski produkt vektora – dijada

$$\mathbf{A} = \vec{a}\vec{b} = \left(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\right)\left(b_x\vec{i} + b_y\vec{j} + b_z\vec{k}\right)$$

$$\mathbf{A} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

- Skalarni produkt vektora i tenzora

$$\vec{t} = \vec{n} \cdot \mathbf{T} = \left(n_x, n_y, n_z\right) \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix},$$

prema pravilu za množenje matrica

$$\vec{t} = \left( n_x T_{xx} + n_y T_{yx} + n_z T_{zx}, \quad n_x T_{xy} + n_y T_{yy} + n_z T_{zy}, \quad n_x T_{xz} + n_y T_{yz} + n_z T_{zz} \right)$$

Oprez! $\vec{n} \cdot \mathbf{T} \neq \mathbf{T} \cdot \vec{n}$

- Dvostruki skalarni umnožak tenzora $\lambda = \mathbf{A} : \mathbf{B}$

$$\lambda = \mathbf{A} : \mathbf{B} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix} : \begin{bmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{yx} & B_{yy} & B_{yz} \\ B_{zx} & B_{zy} & B_{zz} \end{bmatrix}$$

$$\lambda = A_{xx}B_{xx} + A_{xy}B_{xy} + A_{xz}B_{xz} + A_{yx}B_{yx} + A_{yy}B_{yy} + A_{yz}B_{yz} + A_{zx}B_{zx} + A_{zy}B_{zy} + A_{zz}B_{zz}$$

- trag tenzora = zbroj članova na glavnoj dijagonali ( I invarijanta tenzora)

- transponirani tenzor $\mathbf{T}^{\mathrm{T}}$ - zamjena redaka u tablici stupcima

$$\mathbf{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}, \qquad \mathbf{T}^{\mathrm{T}} = \begin{bmatrix} T_{xx} & T_{yx} & T_{zx} \\ T_{xy} & T_{yy} & T_{zy} \\ T_{xz} & T_{yz} & T_{zz} \end{bmatrix}$$

- simetrični tenzor $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$

ili

$$S_{xy} = S_{yx}, \quad S_{xz} = S_{zx}, \quad S_{yz} = S_{zy}$$

npr.: $\mathbf{S} = \begin{bmatrix} 3 & 2 & -4 \\ 2 & -1 & 6 \\ -4 & 6 & 5 \end{bmatrix}$

- antisimetrični tenzori $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$

ili $A_{xx} = A_{yy} = A_{zz} = 0$

$$A_{xy} = -A_{yx}, \quad A_{xz} = -A_{zx}, \quad A_{yz} = -A_{zy}$$

npr.: $\mathbf{A} = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & -5 \\ 1 & 5 & 0 \end{bmatrix}$

- Pravilo: Dvostruki skalarni umnožak simetričnog i antisimetričnog tenzora jednak je nuli.
  $\mathbf{S} : \mathbf{A} = 0$

Poopćenje: skalari i vektori su tenzori nultog i prvog reda.

Broj komponenti tenzora n-tog reda u trodimenzijskom prostoru je $3^n$.

skalar n = 0 $3^0 = 1$ komponenta
vektor n = 1 $3^1 = 3$ komponente

tenzor n = 2 $3^2 = 9$ komponenti
tenzor n = 3 $3^3 = 27$ komponenti

- Skalarni (unutarnji) produkt → red tenzora rezultata jednak je zbroju redova faktora umanjen za 2

-Vektorski produkt → red tenzora rezultata jednak je zbroju redova faktora umanjen za 1
- Tenzorski produkt → red tenzora rezultata jednak je zbroju redova faktora

Primjeri:

$\vec{a} \cdot \vec{b} = \lambda$
1+1-2 = 0

$\vec{a} \times \vec{b} = \vec{c}$
1+1-1 = 1

$\mathbf{A} : \mathbf{B} = \lambda$
2+2-2x2 = 0

$\mathbf{A} \times \mathbf{B} = \mathbf{C}$ (tenzor trećeg reda)
2+2-1 = 3

$\vec{a}\mathbf{B} = \mathbf{C}$ (tenzor trećeg reda)
1+2-0 = 3

$\vec{a} \cdot \mathbf{B} = \vec{c}$ (vektor)

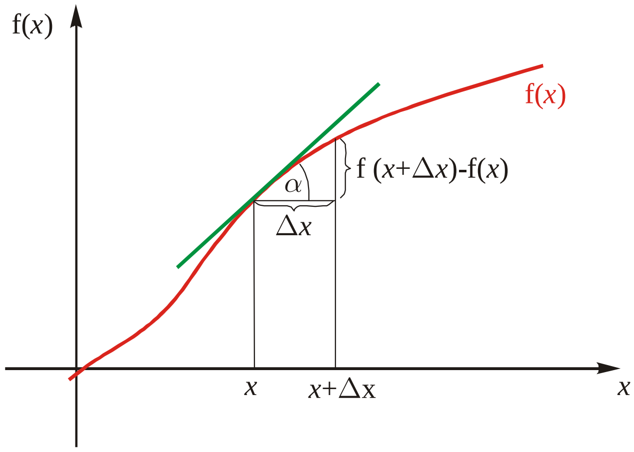
1+2-2 = 1

## 1.4 Diferencijalni račun

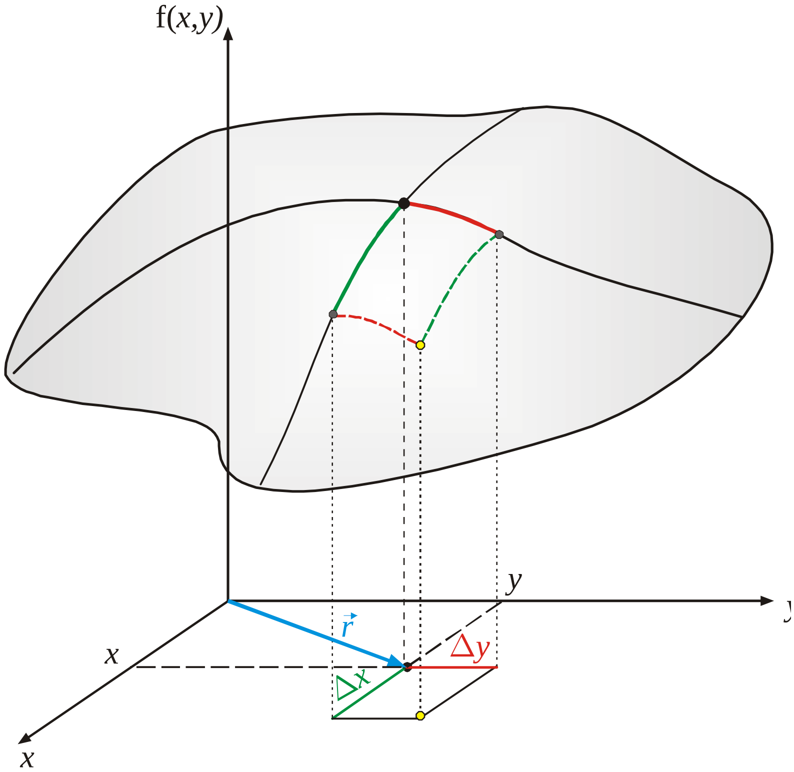
- Derivacija funkcije jedne varijable

$$\frac{\mathrm{d}f}{\mathrm{d}x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\mathrm{tg}\,\alpha$$

$$f(x+\Delta x)=f(x)+\underbrace{\frac{\mathrm{d}f}{\mathrm{d}x}\Delta x}_{\text{prirast funkcije}}$$




- Parcijalna derivacija




vektor položaja

$$\vec{r}=x\vec{i}+y\vec{j}$$

$$\frac{\partial f}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

- Totalni prirast funkcije $\Delta f=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$
- Gradijent $\mathrm{grad}\,f=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}$

$\Delta\vec{r}=\Delta x\vec{i}+\Delta y\vec{j}+\Delta z\vec{k}$ pomak u prostoru varijabli

$\Delta f=\mathrm{grad}\,f\cdot\Delta\vec{r}$ totalni prorast funkcije

- Prirast polja, $\mathrm{d}\varphi = \mathrm{grad}\varphi \cdot \mathrm{d}\vec{r}$

$\mathrm{d}\vec{r} = \mathrm{d}x\,\vec{i} + \mathrm{d}y\,\vec{j} + \mathrm{d}z\,\vec{k}$

$\varphi$ = skalar, vektor ili tenzor

$\mathrm{d}\varphi_{\max}$ za $\mathrm{grad}\varphi \uparrow\uparrow \mathrm{d}\vec{r}$ : $\varphi$ najbrže raste u smjeru gradijenta = gradijent pokazuje smjer najbrže promjene $\varphi$

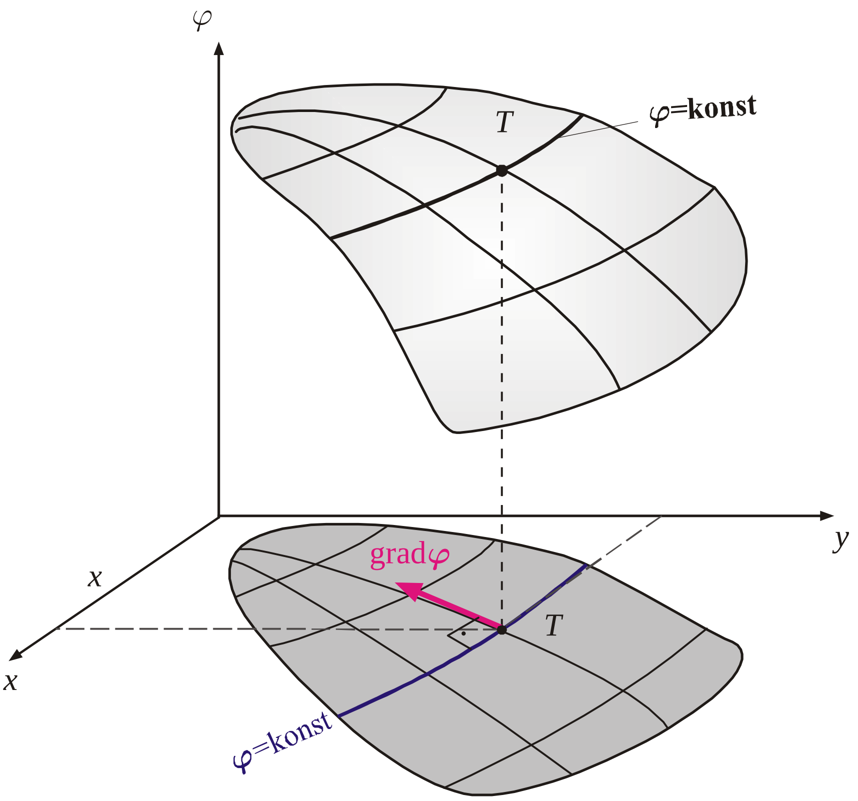
$\mathrm{d}\varphi_{\max}$ za $\mathrm{grad}\varphi \uparrow\downarrow \mathrm{d}\vec{r}$ : $\varphi$ najbrže opada u smjeru suprotnom od grad $\varphi$

$\mathrm{d}\varphi = 0$ za $\mathrm{grad}\varphi \perp \mathrm{d}\vec{r}$ : grad $\varphi$ je okomit na $\varphi$ = konst.

- Usmjerena derivacija u smjeru jediničnog vektora $\vec{s}_0$

$\mathrm{d}\varphi = \mathrm{grad}\varphi \cdot \underbrace{\mathrm{d}s \cdot \vec{s}_0}_{\mathrm{d}\vec{s}}$ = prirast u smjeru puta $\mathrm{d}\vec{s}$ /: $\mathrm{d}s$

$\boxed{\dfrac{\mathrm{d}\varphi}{\mathrm{d}s} = \mathrm{grad}\varphi \cdot \vec{s}_0}$ = projekcija vektora grad$\varphi$ na $\vec{s}_0$

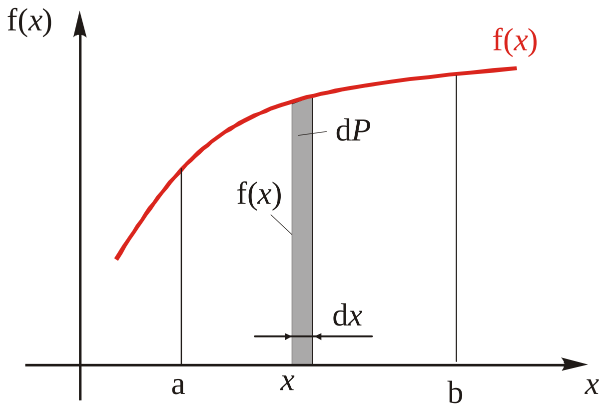


- Operator nabla $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ ...vektor

| operator | primjeri | zapis | raspis |
|---|---|---|---|
| grad $\nabla *$ | grad $f$ | $\nabla f$ | $\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ (vektor) |
| | grad $\vec{v}$ | $\nabla \vec{v}$ | $\begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}$ tenzor |
| | grad**T** | $\nabla$**T** | tenzor trećeg reda |
| div $\nabla \cdot *$ | div$\vec{v}$ | $\nabla \cdot \vec{v}$ | $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$ (skalar) |
| | div**T** | $\nabla \cdot$**T** | $\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{yz}}{\partial y}\right)\vec{i} + \left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial y}\right)\vec{j} + \left(\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial y}\right)\vec{k}$ (vektor) |
| rot $\nabla \times *$ | rot $\vec{v}$ | $\nabla \times \vec{v}$ | $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\vec{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{k}$ (vektor) |

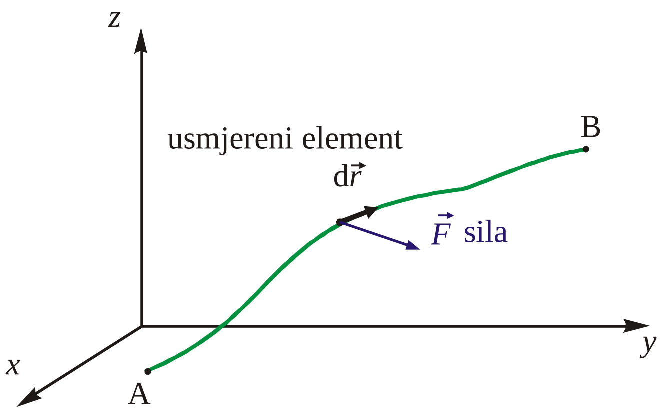
## 1.5 Integrali

jednostruki



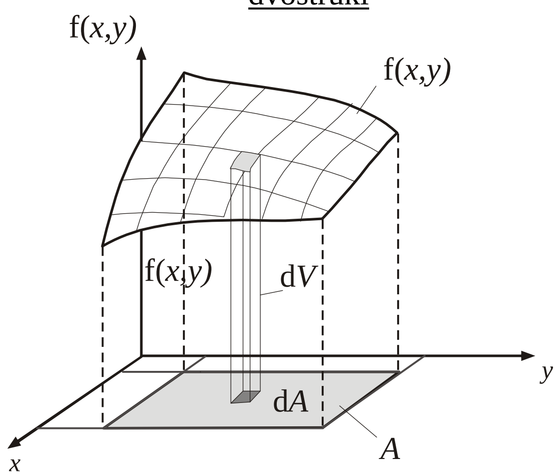

$$P = \int_a^b \underbrace{f(x)\,\mathrm{d}x}_{\mathrm{d}P}$$

krivuljni



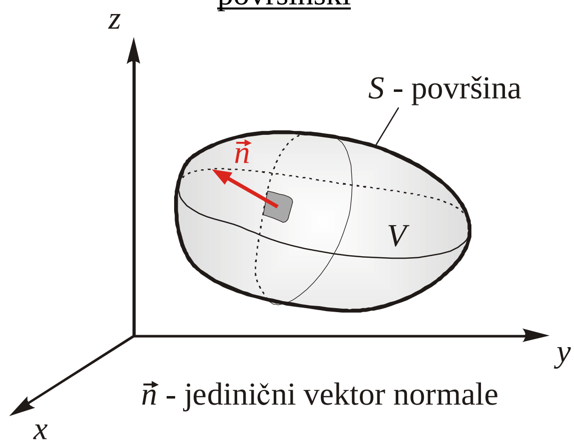

$$W = \int_A^B \vec{F}\cdot \mathrm{d}\vec{r} \;=\; \text{hod vektora po krivulji}$$

dvostruki




$$V = \int_A \underbrace{f(x,y)\,\mathrm{d}A}_{\mathrm{d}V}$$

površinski



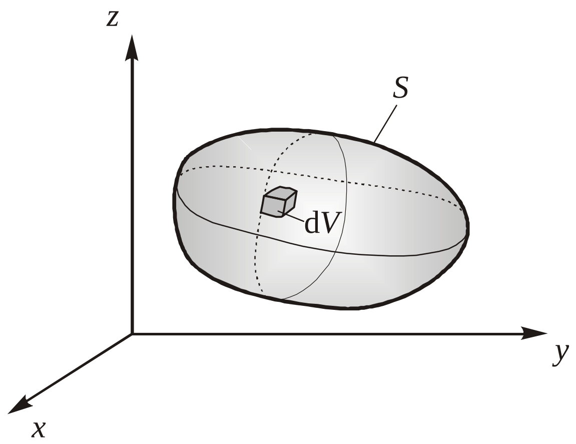

$\varphi$ = površinska gustoća fizikalnog svojstva

$$I = \int_S \varphi\,\mathrm{d}S$$

- Fluks vektora $\vec{v}$ kroz površinu S

$$F = \int_S \underbrace{\vec{v}\cdot\vec{n}}_{\substack{\text{projekcija}\\ \text{vektora}\\ \text{na smjer}\\ \text{normale}}}\,\mathrm{d}S$$

volumenski integral



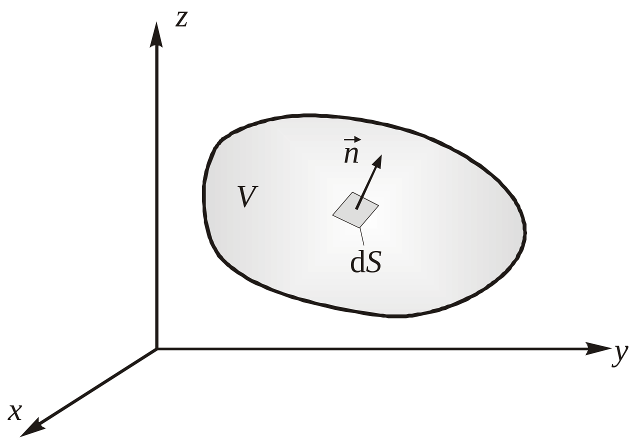

$$I = \int_V \Phi \, \mathrm{d}V$$

$\Phi$ = volumenska gustoća fizikalnog svojstva

npr. gustoća = volumenska gustoća mase

$$m = \int_V \rho \cdot \mathrm{d}V$$

- **Gaussova formula**




Primjeri:

$$\int_S p\vec{n} \, \mathrm{d}S = \int_V \overbrace{\nabla p}^{\mathrm{grad}p} \, \mathrm{d}V$$

$$\int_S \vec{v} \cdot \vec{n} \, \mathrm{d}S = \int_V \overbrace{\nabla \cdot \vec{v}}^{\mathrm{div}\vec{v}} \, \mathrm{d}V$$

$$\int_S \varphi \otimes \vec{n} \, \mathrm{d}S = \int_V \nabla \otimes \varphi \, \mathrm{d}V$$

$\otimes$ = operator

$\varphi$ = skalar, vektor ili tenzor