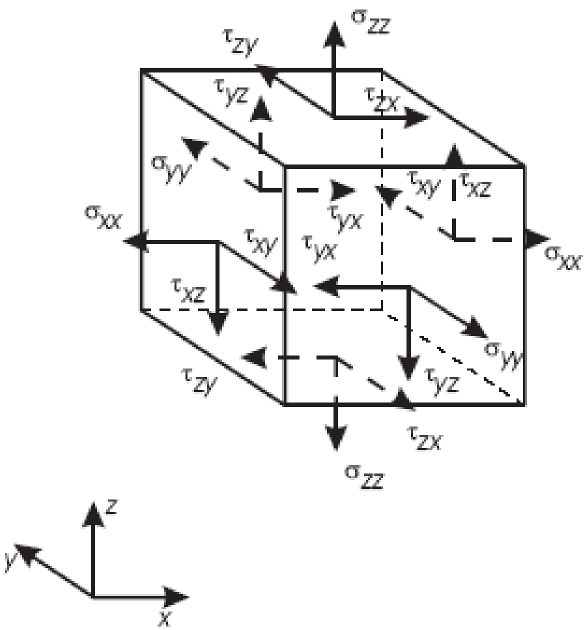
Opšte stanje napona

Tenzor napona

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{xz} = \tau_{zx}$$
$$\tau_{yz} = \tau_{zy}$$



Značenje indeksa

Normalni napon: indeksi pokazuju površinu na koju djeluje.

Tangencijalni napon: prvi indeks pokazuje površinu na koju napon djeluje, a drugi pravac u kojem djeluje.

Konvencija o predznaku napona

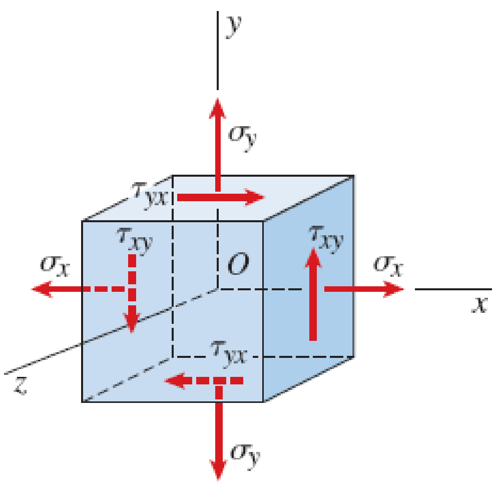
Normalni napon je pozitivan ako se njegov smjer poklapa sa smjerom vanjske normale na elementu površine

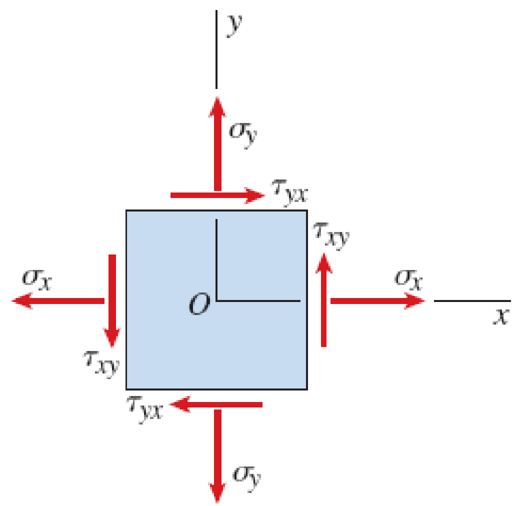
Tangencijalni napon je pozitivan ako je na gornjoj i desnoj površini elementa usmjeren ka pozitivnom smjeru ose.

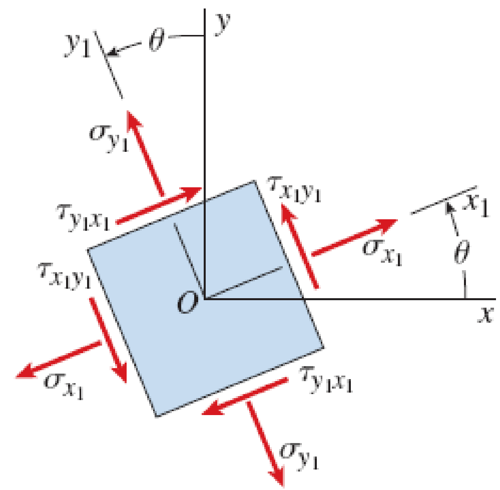
***Grupa autora, *Elastostatika I*, Tehnički fakultet, Bihać, 2003**

Ravno stanje napona

Ravno stanje napona – jedinstveno predstavljeno s dvije komponente normalnog napona i jednom komponentom tangencijalnog napona koji djeluju na element s određenim položajem u tački elementa.



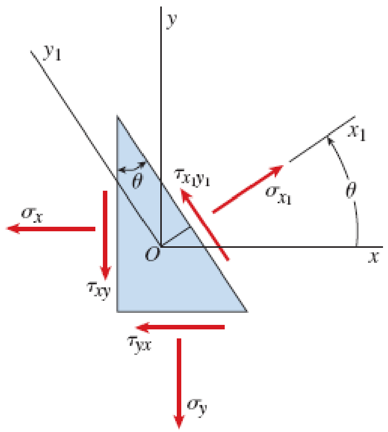




$$\tau_{xy} = \tau_{yx}$$

$$\tau_{x_1y_1} = \tau_{y_1x_1}$$

Naponi na kosoj ravni

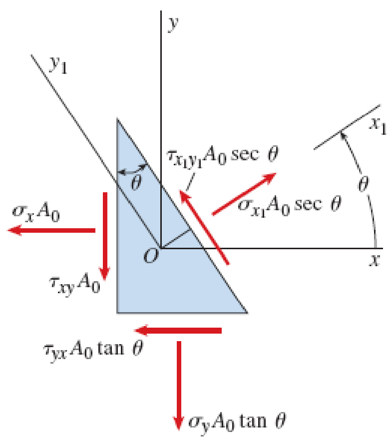


$$\sum_i F_{x_1 i} = 0$$

$$\sigma_{x_1} A_0 \sec(\theta) - \sigma_x A_0 \cos(\theta) - \tau_{xy} A_0 \sin(\theta) + \\ - \sigma_y A_0 \mathrm{tg}(\theta) \sin(\theta) - \tau_{yx} A_0 \mathrm{tg}(\theta) \cos(\theta) = 0 \quad (3.1)$$

$$\sum_i F_{y_1 i} = 0$$

$$\tau_{x_1 y_1} A_0 \sec(\theta) + \sigma_x A_0 \sin(\theta) - \tau_{xy} A_0 \cos(\theta) + \\ - \sigma_y A_0 \mathrm{tg}(\theta) \cos(\theta) + \tau_{yx} A_0 \mathrm{tg}(\theta) \sin(\theta) = 0 \quad (3.2)$$



$$\sigma_{x_1} = \sigma_x \cos^2(\theta) + \sigma_y \sin^2(\theta) + 2\tau_{yx} \sin(\theta)\cos(\theta) \quad (3.3)$$

$$\tau_{x_1 y_1} = -(\sigma_x - \sigma_y)\sin(\theta)\cos(\theta) + 2\tau_{yx}(\cos^2(\theta) - \sin^2(\theta)) \quad (3.4)$$

Naponi na kosoj ravni

$$\sigma_{x_1} = \sigma_x \cos^2(\theta) + \sigma_y \sin^2(\theta) + 2\tau_{yx} \sin(\theta)\cos(\theta) \qquad (3.3)$$

$$\tau_{x_1 y_1} = -(\sigma_x - \sigma_y)\sin(\theta)\cos(\theta) + 2\tau_{yx}(\cos^2(\theta) - \sin^2(\theta)) \qquad (3.4)$$

$$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta)) \qquad \sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta)) \qquad \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$$

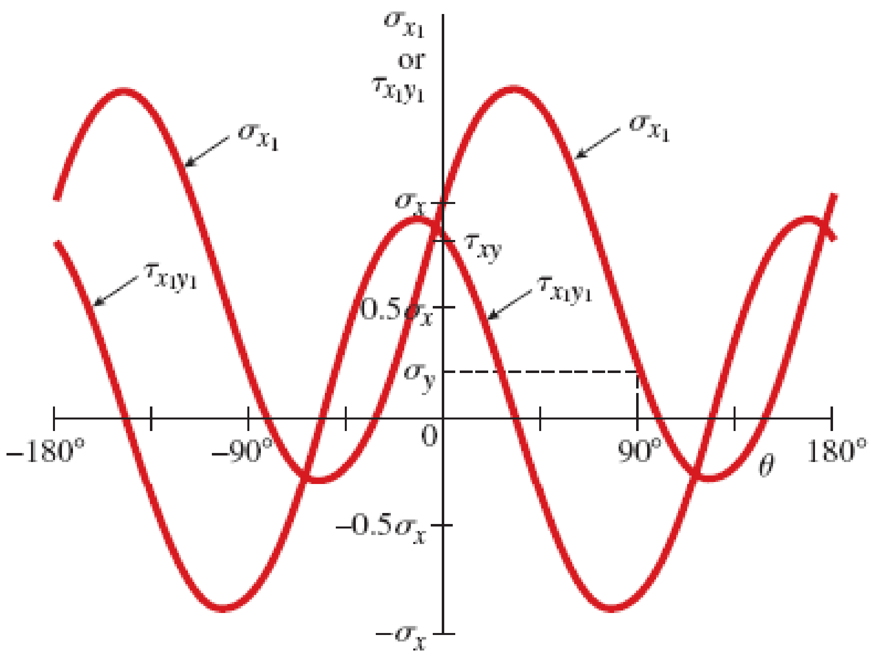
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \qquad (3.5)$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \qquad (3.6)$$

Naponi na kosoj ravni

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta) \quad (3.6)$$

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y \quad (3.7)$$



Glavni normalni naponi i najveći tangencijalni naponi

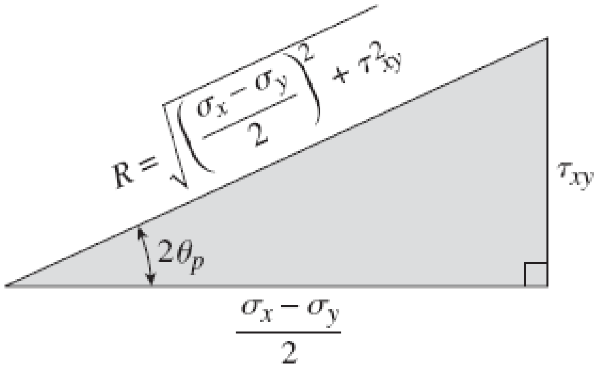
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \qquad (3.5)$$

$$\left.\frac{d\sigma_{x_1}}{d\theta}\right|_{\theta=\alpha} = \left.-(\sigma_x - \sigma_y)\sin(2\theta) + 2\tau_{xy}\cos(2\theta)\right|_{\theta=\alpha} = 0 \qquad (3.8)$$

(3.8) ➔ $$\mathrm{tg}(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (3.9)$$

- Ugao α određuje ravan maksimalnog/minimalnog normalnog napona
- Jednačina (3.9) ima dva korijena u domenu $(0,2\pi)$, a oni ovise od predznaka τ_{xy} i $(\sigma_x-\sigma_y)$ – ovi korijeni se nazivaju i **uglovi glavnih ravni**

Glavni normalni naponi i najveći tangencijalni naponi



$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\cos(2\alpha) = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin(2\alpha) = \frac{\tau_{xy}}{R}$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \qquad (3.5)$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \qquad (3.6)$$

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y = \sigma_1 + \sigma_2 \qquad (3.7)$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (3.10)$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (3.11)$$

U ravni glavnih normalnih napona ne djeluju tangencijalni naponi, tj. $\tau_{12}=0$

Glavni normalni naponi i najveći tangencijalni naponi

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \qquad (3.6)$$

$$\left.\frac{d\tau_{x_1 y_1}}{d\theta}\right|_{\theta=\beta} = -(\sigma_x - \sigma_y)\cos(2\theta) - 2\tau_{xy}\sin(2\theta)\Big|_{\theta=\alpha} = 0 \qquad (3.12)$$

(3.12) ➔ $$\mathrm{tg}(2\beta) = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{1}{\mathrm{tg}(2\alpha)} \qquad (3.13)$$

- Ugao β određuje ravan maksimalnog/minimalnog tamngencijalnog napona
- Jednačina (3.13) ima dva korijena u domenu $(0,2\pi)$, a oni ovise od predznaka τ_{xy} i $(\sigma_x$-$\sigma_y)$ – za ugao β prema (3.13) vrijedi

$$\beta = \alpha \pm 45°$$

Glavni normalni naponi i najveći tangencijalni naponi

$$\begin{cases} \sigma_{x_1} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) & (3.5) \\ \tau_{x_1 y_1} = -\dfrac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) & (3.6) \\ \sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y = \sigma_1 + \sigma_2 & (3.7) \end{cases}$$

$$\left. \begin{aligned} R &= \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \cos(2\alpha) &= \frac{\tau_{xy}}{2R} \\ \sin(2\alpha) &= -\frac{\sigma_x - \sigma_y}{R} \end{aligned} \right\}$$

$$\tau_{\max,\min} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\frac{\sigma_1 - \sigma_2}{2} \qquad (3.14)$$

U ravni najvećih tangencijalnih napona djeluju normalni naponi

$$\sigma = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_1 + \sigma_2}{2} \qquad (3.15)$$

Mohr-ov (Mor) krug napona

Jednačine za računanje napona mogu se predstaviti grafički pomoću *Mohr*-ovog kruga napona. Iz jednačina (3.5) i (3.6) se eliminiše ugao 2θ

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \tag{3.5}$$

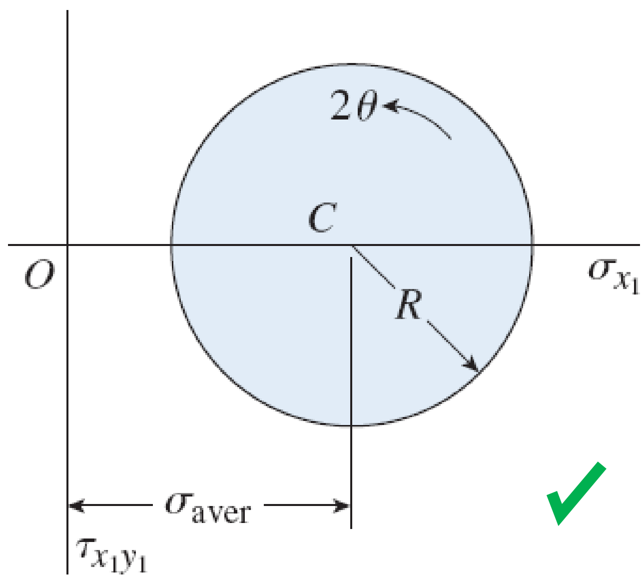
$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \tag{3.6}$$

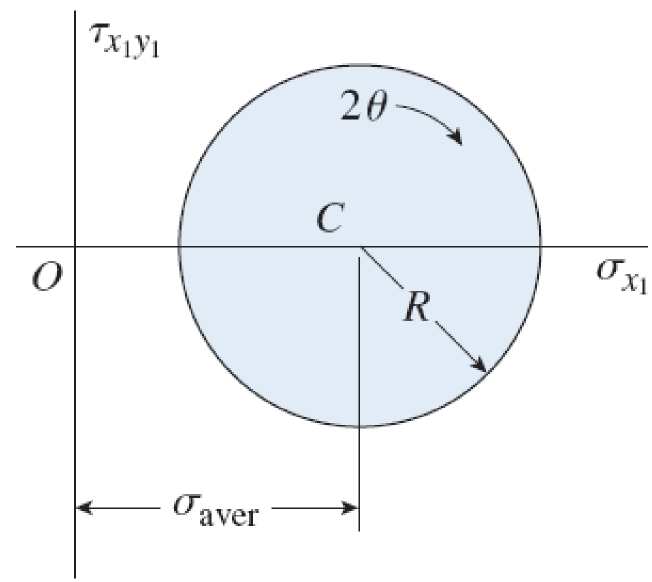
Jednačine za računanje napona mogu se predstaviti grafički pomoću *Mohr*-ovog kruga napona. Iz jednačina (3.5) i (3.6) se eliminiše ugao tako što se obje jednačine kvadriraju i saberu, pa se dobije:

$$\left(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x_1 y_1}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \tag{3.16}$$

$$\left(\sigma_{x_1} - \sigma_{\mathrm{sr}}\right)^2 + \tau_{x_1 y_1}^2 = R^2 \tag{3.16a}$$

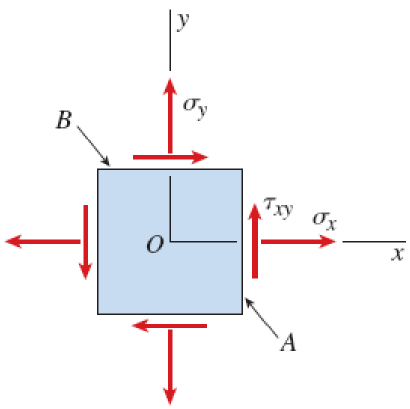
Mohr-ov (Mor) krug napona – konstrukcija

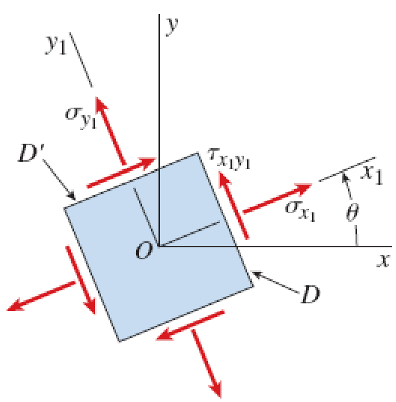




Dva načina crtanja *Mohr*-ovog kruga napona

Mohr-ov (Mor) krug napona – konstrukcija

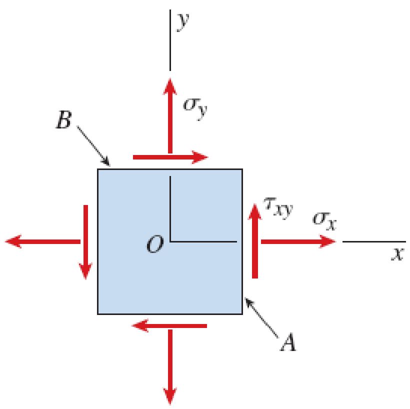


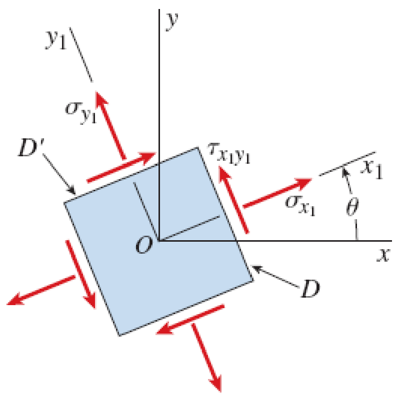


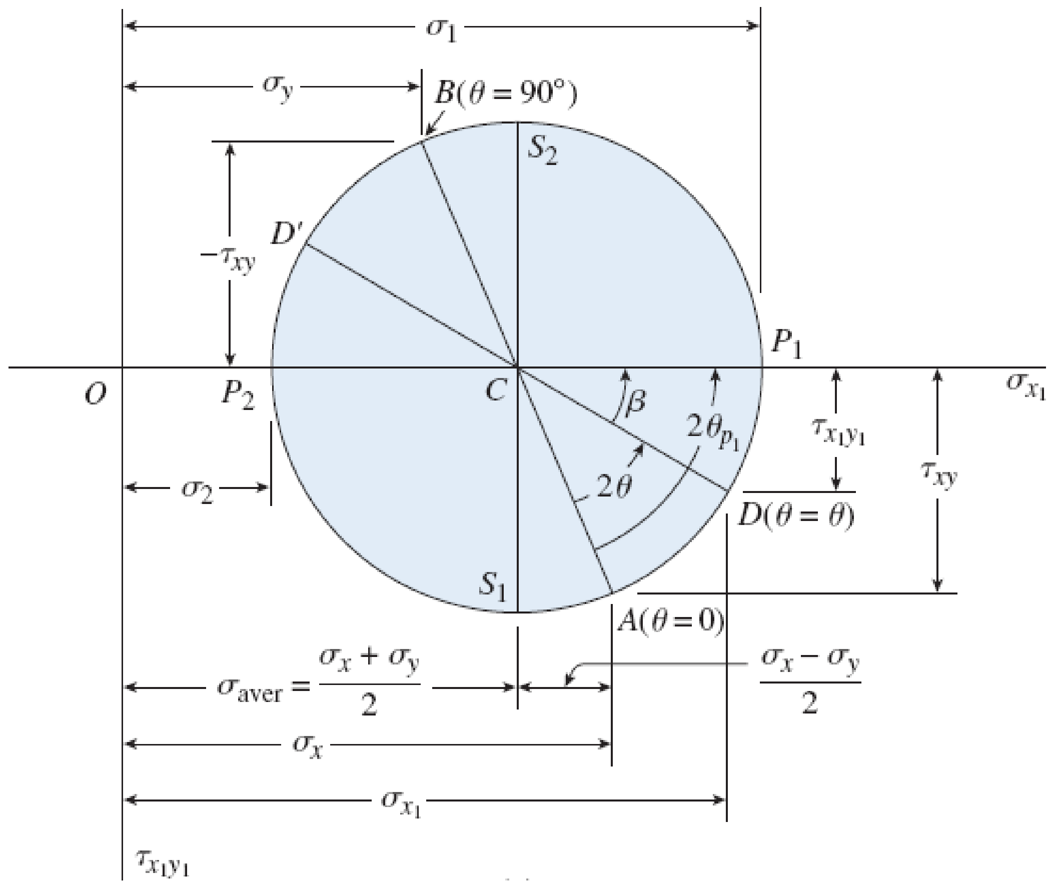
- Nacrta se koordinatni sistem s abscisom σ_{x1} (σ_n), pozitivna na desno, i ordinatom τ_{xy} (τ_n), pozitivna na dole
- U dijagramu se ucrta tačka *A* s koordinatama (σ_x, τ_{xy}) – predstavlja naponsko stanje na pozitivnoj *x* površi (površ *A*)
- U dijagramu se ucrta tačka *B* s koordinatama (σ_y,- τ_{xy}) – predstavlja naponsko stanje na pozitivnoj *y* površi (površ *B*)
- Povuče se duž od *AB* koja predstavlja prečnik kruga napona s centrom u tački *C*.
- Koristeći tačku *C* kao centar nacrta se kružnica koja prolaziu kroz tačke *A* i *B*.

- Ugao α koji određuje ravan normalnih napona određuje se na osnovu ugla 2α ($2\theta_p$)
- Ugao β koji određuje ravan maksimalnih tangencijalnih napona određuje se na osnovu ugla 2β ($2\theta_s$)

Mohr-ov (Mor) krug napona – konstrukcija

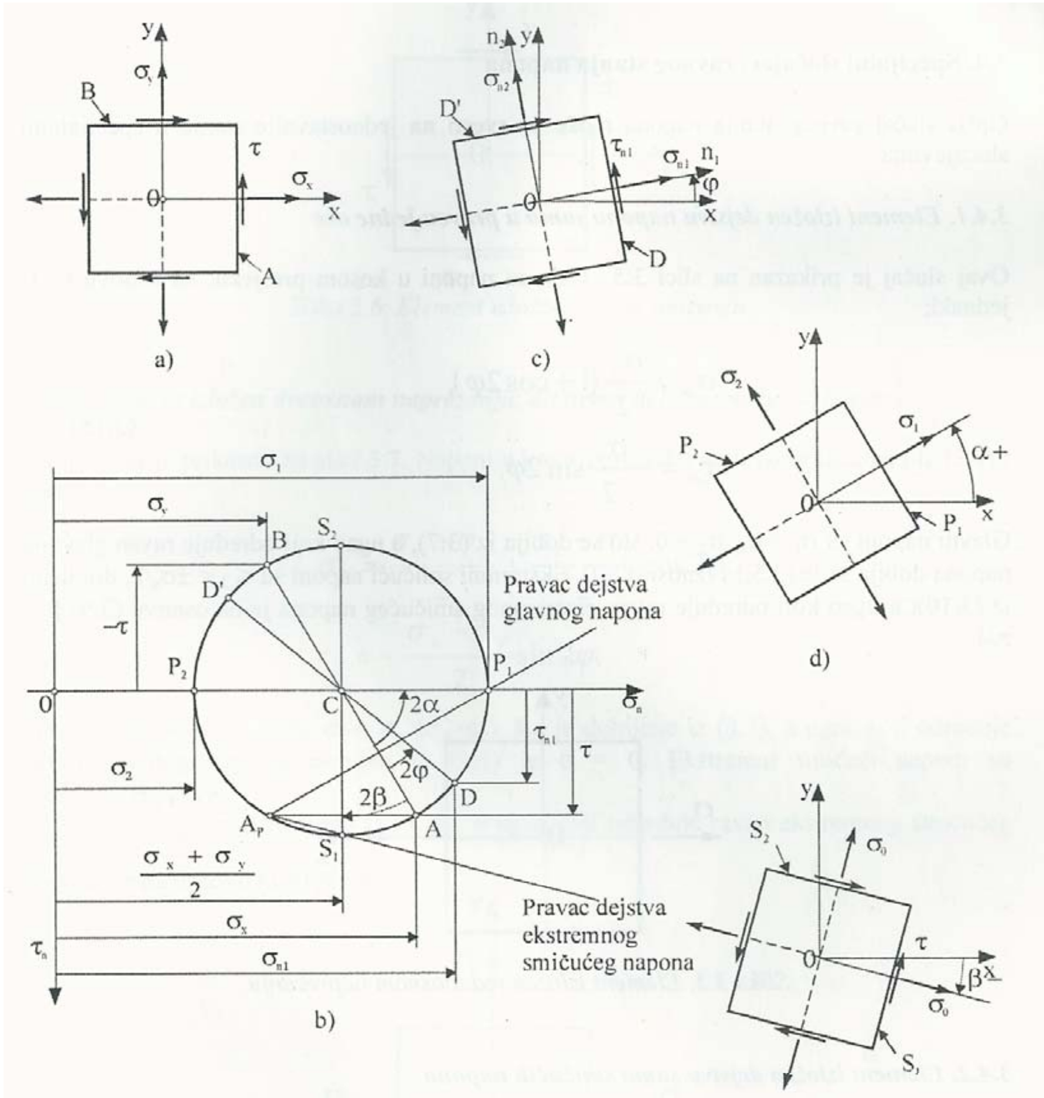






JM Gere, BJ Goodno, *Mechanics of Materials*, Cengage Learning, Seventh Edition, 2009.

Mohr-ov (Mor) krug napona – određivanje napona za proizvoljnu ravan



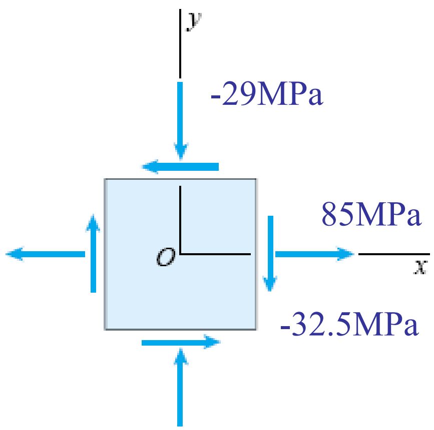
*Grupa autora, *Elastostatika I*, Tehnički fakultet, Bihać, 2003

Primjer 3.1: Element je izložen naponima kao na slici: σ_x=85MPa, σ_y=-29MPa, τ_{xy}=-32.5MPa.

a) Odrediti glavne napone i prikaži na skici elementa napona

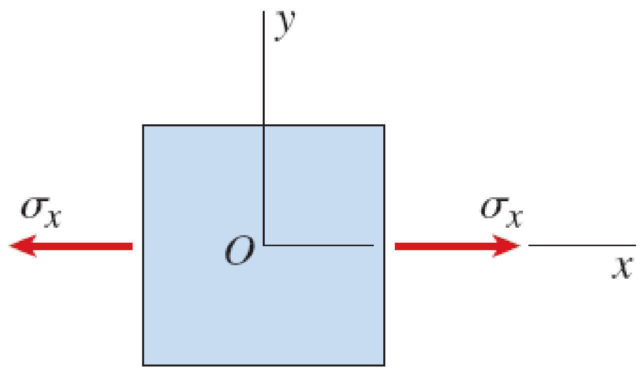
b) Odrediti maksimalni tangencijalni napon i pokaži na elementu napona

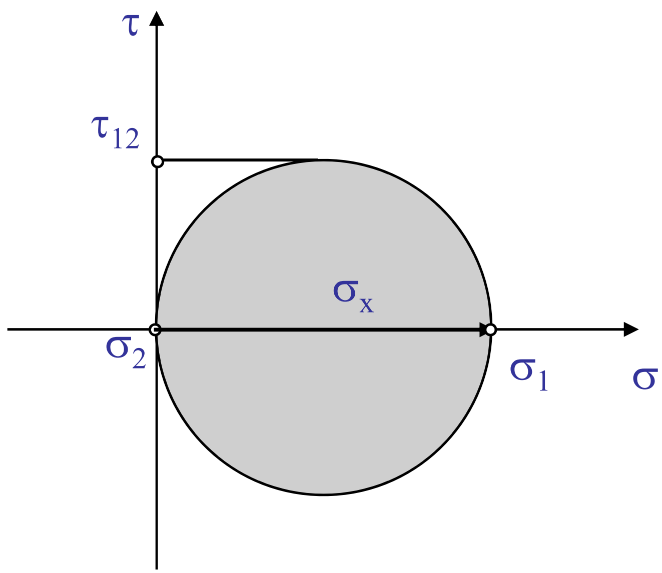
c) rezultate pod a) i b) potvrdi konstrukcijom *Mohr*-ovog kruga napona



Specijalni slučajevi dvoosnog naponskog stanja

Aksijalno naprezanje





$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

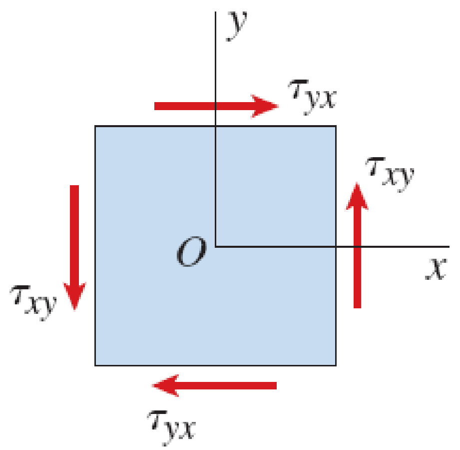
➔

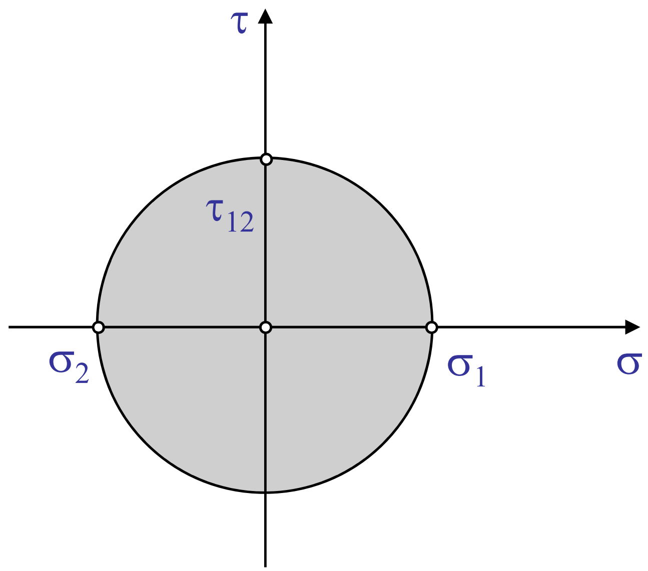
$$\sigma_{x_1} = \frac{\sigma_x}{2}(1 + \cos(2\theta))$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x}{2}\sin(2\theta)$$

Specijalni slučajevi dvoosnog naponskog stanja

Čisto smicanje (bez normalnih napona)





$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

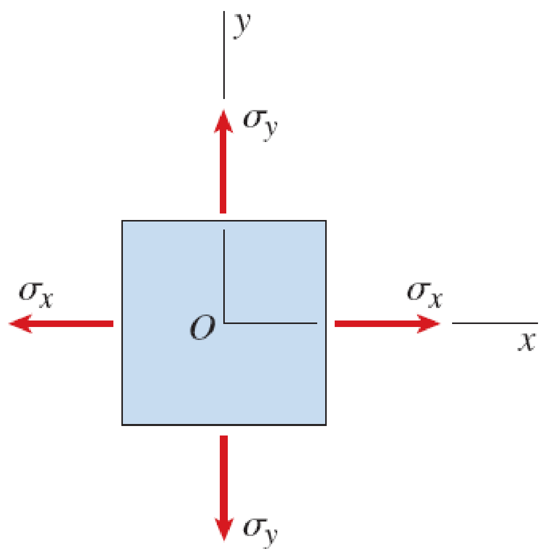
➔

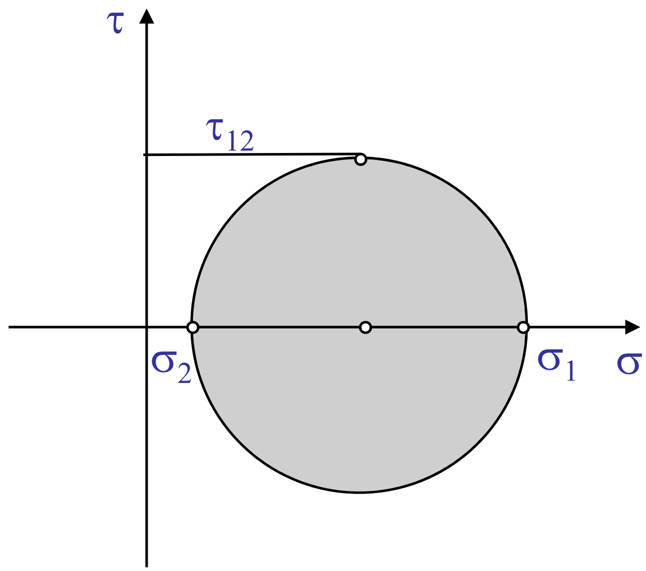
$$\sigma_{x_1} = \tau_{xy}\sin(2\theta)$$

$$\tau_{x_1 y_1} = \tau_{xy}\cos(2\theta)$$

Specijalni slučajevi dvoosnog naponskog stanja

Dvoosno naprezanje, bez tangencijalnih napona





$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

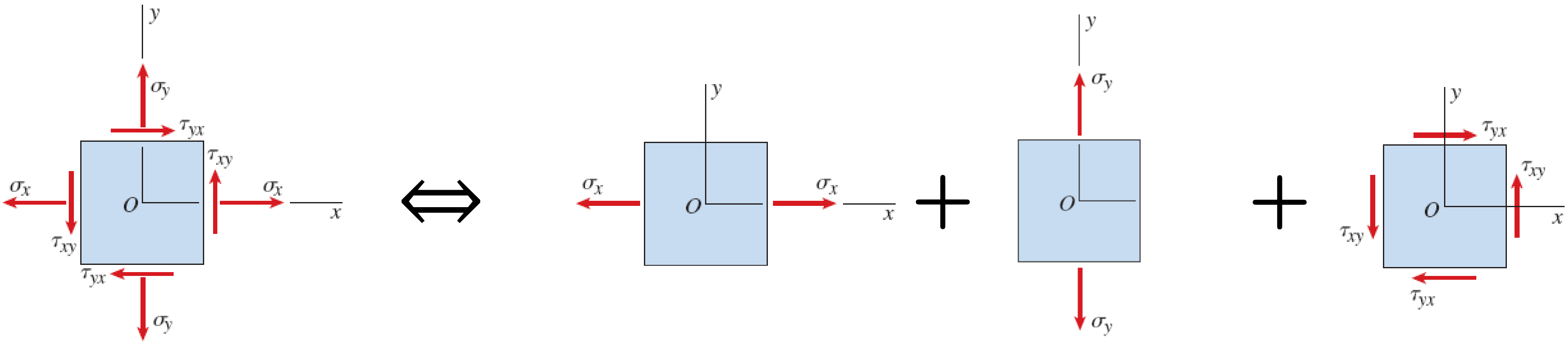
$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

➔

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta)$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta)$$

Hooke-ov zakon za ravno stanje napona



$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \varepsilon_x = -\frac{\sigma_y}{E} \qquad \gamma_{xy} = \gamma_{yx} = \gamma = \frac{\tau}{G}$$

$$\varepsilon_y = -\frac{\sigma_x}{E} \qquad \varepsilon_y = \frac{\sigma_y}{E} \qquad G = \frac{E}{2(1+\nu)}$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$+$ utjecaj temperature

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha\Delta T$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha\Delta T \qquad (3.17)$$

Hooke-ov zakon za ravno stanje napona

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

\+ utjecaj temperature

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha\Delta T \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha\Delta T \tag{3.17}$$

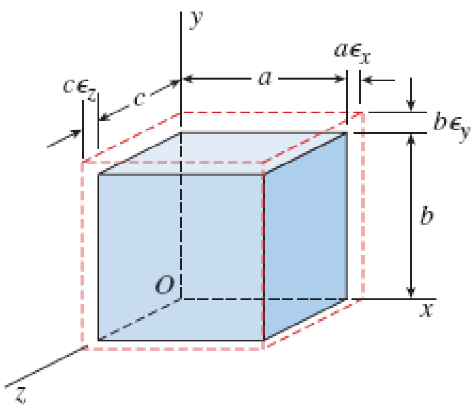
$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \qquad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

\+ utjecaj temperature

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) - \frac{E\alpha}{1-\nu}\Delta T \qquad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) - \frac{E\alpha}{1-\nu}\Delta T \tag{3.18}$$

Promjena zapremine i deformacioni rad

Dvoosno naprezanje, bez tangencijalnih napona



$$V_0 = abc$$

$$V_1 = (a + a\varepsilon_x)(b + b\varepsilon_y)(c + c\varepsilon_z) = abc(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$$

$$V_1 = V_0(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$$

$$V_1 \approx V_0(1+\varepsilon_x+\varepsilon_y+\varepsilon_z)$$

$$\Delta V = V_1 - V_0 \approx V_0(\varepsilon_x+\varepsilon_y+\varepsilon_z)$$

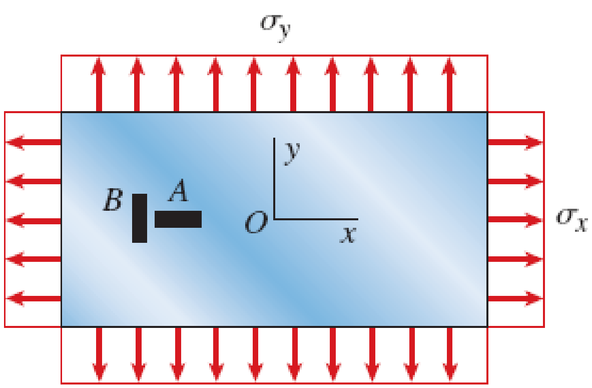
Specifična promjena zapremine (dilatacija): $$e = \frac{\Delta V}{V_0} = \varepsilon_x+\varepsilon_y+\varepsilon_z = \frac{1-2\nu}{E}(\sigma_x+\sigma_y)$$

Deformacioni rad

$$W = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})$$

$$W = \frac{1}{2E}(\sigma_x^2+\sigma_y^2-2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{2G}$$

Primjer 3.2: Pravougaona čelična ploča debljine t=6 mm napregnuta je na ravnomjerne napone σ_x i σ_y, kao na slici. Mjerne trake A i B, koje su postavljene u pravcima x i y su postavljene na ploču. Ako su čitanja mjernih traka ε_x=0.001 (izduženje) i ε_y=0.0007 (skraćenje) izračunati napone σ_x i σ_y, te promjenu debljine ploče.
Podaci: E= 200GPa.



Primjeri 3.1-3.14 (str. 74-88), Grupa autora, *Elastostatika I*, Tehnički fakultet, Bihać, 2003.