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Izvodi f-ja (deriviranje f-ja)

Definicija Neka je f f-ja definisana na otvorenom intervalu (a, b) ; pretpostavimo da je $c \in (a, b)$. Kazemo da f ima izvod (ili derivaciju) u tački c ako postoji limes $\lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c}$.

Vrijednost lineca obilježavamo sa $f'(c)$ i zovemo izvod f-je f u tački c .

Primeri:

1. Naci izvode f-ja koristenjem navedene definicije:

$$a) y = \sqrt{x} \quad c) y = \sin x \quad e) y = a^x$$

$$b) y = x^3 \quad d) y = x^\lambda, \lambda \in \mathbb{R}$$

Rj.

$$a) y = \sqrt{x}$$

$$\begin{aligned} y'(c) &= \lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c} = \lim_{x \rightarrow c} \frac{\sqrt{x}-\sqrt{c}}{x-c} \cdot \frac{(\sqrt{x}+\sqrt{c})}{(\sqrt{x}+\sqrt{c})} = \lim_{x \rightarrow c} \frac{x-c}{(x-c)(\sqrt{x}+\sqrt{c})} = \\ &= \lim_{x \rightarrow c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}} \Rightarrow (\sqrt{x})' = \frac{1}{2\sqrt{x}} \end{aligned}$$

$$b) y = x^3$$

$$\begin{aligned} y'(c) &= \lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c} = \lim_{x \rightarrow c} \frac{x^3-c^3}{x-c} = \lim_{x \rightarrow c} \frac{(x-c)(x^2+xc+c^2)}{x-c} = \\ &= \lim_{x \rightarrow c} (x^2+xc+c^2) = 3c^2 \Rightarrow (x^3)' = 3x^2 \end{aligned}$$

$$c) y = \sin x$$

$$\begin{aligned} y'(c) &= \lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c} = \lim_{x \rightarrow c} \frac{\sin x - \sin c}{x-c} \stackrel{(*)}{=} \lim_{x \rightarrow c} \frac{2 \sin \frac{x-c}{2} \cos \frac{x+c}{2}}{x-c} = \\ &= 2 \sin \frac{c-c}{2} \cos \frac{c+c}{2} = 2 \sin 0 \cos \frac{2c}{2} = 0 \quad \dots (*) \end{aligned}$$

$$= \lim_{x \rightarrow c} \frac{\sin \frac{x-c}{2} \cos \frac{x+c}{2}}{\frac{x-c}{2}} = \lim_{x \rightarrow c} \left(\frac{\sin \frac{x-c}{2}}{\frac{x-c}{2}} \right) \cdot \lim_{x \rightarrow c} \cos \frac{x+c}{2} = \cos c$$

$$(\sin x)' = \cos x$$

d) $y = x^{\lambda}$, $\lambda \in \mathbb{R}$

$$y'(c) = \lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c} = \lim_{x \rightarrow c} \frac{x^{\lambda} - c^{\lambda}}{x-c} = \lim_{x \rightarrow c} \frac{c^{\lambda} \left(\frac{x^{\lambda}}{c^{\lambda}} - 1 \right)}{x-c} = c^{\lambda} \lim_{x \rightarrow c} \frac{\left(\frac{x}{c} \right)^{\lambda} - 1}{x-c}$$

$$= \left| \begin{array}{l} \frac{x}{c} = t \\ x \rightarrow c \Rightarrow t \rightarrow 1 \\ x = ct \end{array} \right| = c^{\lambda} \lim_{t \rightarrow 1} \frac{t^{\lambda} - 1}{ct - c} = c^{\lambda-1} \lim_{t \rightarrow 1} \frac{(t-1)(t^{\lambda-1} + t^{\lambda-2} + \dots + t+1)}{t-1}$$

$$= \lambda c^{\lambda-1} \Rightarrow (x^{\lambda})' = \lambda x^{\lambda-1}$$

e) $y = a^x$, od ranije znamo $\lim_{x \rightarrow 0^{\pm}} (1+x)^{\frac{1}{x}} = \left| \begin{array}{l} \frac{1}{x} = 0 \\ x \rightarrow 0^{\pm} \\ t \rightarrow \pm \infty \end{array} \right| = \lim_{t \rightarrow \pm \infty} \left(1 + \frac{1}{t} \right)^t = e$

$$y'(c) = \lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c} = \lim_{x \rightarrow c} \frac{a^x - a^c}{x-c} = a^c \lim_{x \rightarrow c} \frac{a^{x-c} - 1}{x-c} = \left| \begin{array}{l} x-c = t \\ x \rightarrow c \Rightarrow t \rightarrow 0 \end{array} \right|$$

$$= a^c \lim_{t \rightarrow 0} \frac{a^t - 1}{t} = \left| \begin{array}{l} a^t - 1 = s \\ t \rightarrow 0 \Rightarrow s \rightarrow 0 \\ a^t = s+1 \\ t = \log_a(s+1) \end{array} \right| = a^c \lim_{s \rightarrow 0} \frac{s}{\log_a(s+1)} = \frac{a^c}{\lim_{s \rightarrow 0} \frac{\log_a(s+1)}{s}}$$

$$= \frac{a^c}{\lim_{s \rightarrow 0} \frac{1}{s} \log_a(s+1)} = \frac{a^c}{\log_a \lim_{s \rightarrow 0} (s+1)^{\frac{1}{s}}} = \frac{a^c}{\log_a e} = a^c / \ln a$$

$$\Rightarrow (a^x)' = a^x / \ln a.$$

2.) Uz pomoć definicije nadi izvode f-ja:

a) $y = c$, $c \in \mathbb{R}$

b) $y = x$,

c) $y = x^2$,

d) $y = \cos x$

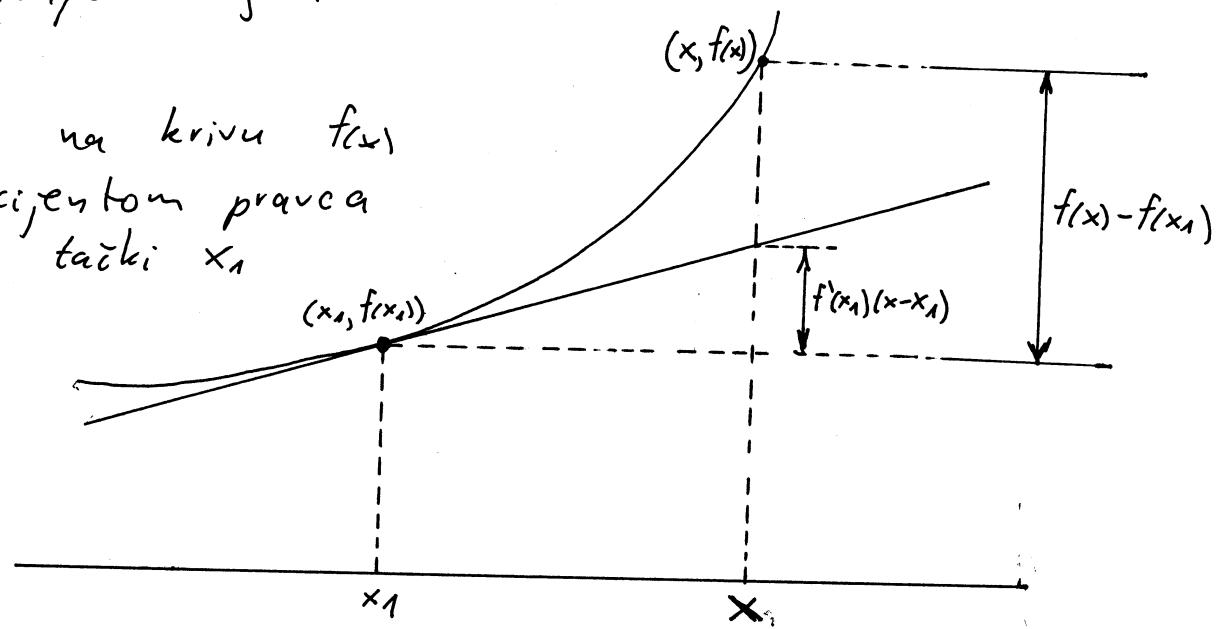
$$\left. \begin{aligned} \cos \alpha + \cos \beta &= 2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2} \\ \cos \alpha + \cos \beta &= 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \\ \sin \alpha + \sin \beta &= 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \end{aligned} \right|$$

Ako f-ja $f(x)$ ima izvod u tački c tada je $f(x)$ neprekidna u tački c .

Izvodi se upotrebljavaju u mnogim problemima, a najvažnije dvije skupine su:

1. određivanje brzine tačke koja se kreće pravolinijski
2. iznalaženje tangente na krivu

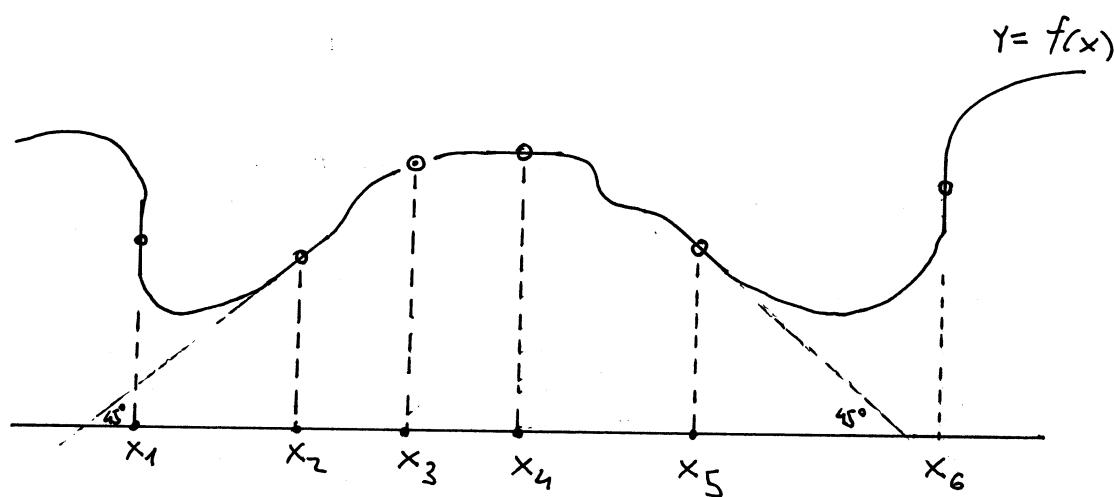
tangentna na krivu $f(x)$
sa koeficijentom pravca
 $f'(x_1)$ u tački x_1



$$y - y_1 = k(x - x_1)$$

$f(x) - f(x_1) = f'(x_1)(x - x_1)$ jednačina tangente na krivu
 $y = f(x)$ u nekoj tački $(x_1, f(x_1))$

$k_1 \cdot k_2 = -1$ uslov normalnosti dve prave



$$f'(x_1) = -\infty$$

$$f'(x_3) \text{ ne postoji}$$

$$f'(x_5) = -1$$

$$f'(x_2) = 1$$

$$f'(x_4) = 0$$

$$f'(x_6) = +\infty$$

Tablica izvoda

1. $c' = 0$, c - konst.	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $ x < 1$
2. $(x^\alpha)' = \alpha x^{\alpha-1}$, $\alpha \in \mathbb{R}$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $ x < 1$
$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, $x > 0$	
3. $(a^x)' = a^x \ln a$	$(\operatorname{arc tg} x)' = \frac{1}{1+x^2}$
$(e^x)' = e^x$	$(\operatorname{arcctg} x)' = -\frac{1}{1+x^2}$
4. $(\log_a x)' = \frac{1}{x \ln a}$	$\begin{aligned} \operatorname{sh} x &= \frac{e^x - e^{-x}}{2} & \operatorname{ch} x &= \frac{e^x + e^{-x}}{2} \end{aligned}$
$(\ln x)' = \frac{1}{x}$	
5. $(\sin x)' = \cos x$	$(\operatorname{sh} x)' = \operatorname{ch} x$
6. $(\cos x)' = -\sin x$	$(\operatorname{ch} x)' = \operatorname{sh} x$
7. $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$	$(\operatorname{th} x)' = \frac{1}{\operatorname{ch}^2 x}$
8. $(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$	$(\operatorname{cth} x)' = -\frac{1}{\operatorname{sh}^2 x}$

Pravila izvoda:

1. $(f \pm g)'(c) = f'(c) \pm g'(c)$
2. $(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$
3. $(\lambda f)'(c) = \lambda f'(c)$
4. $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$, $g(c) \neq 0$

Izračunati izvode f-ja:

$$1. \quad y = x^5 - 4x^3 + 2x - 3$$

$$R_j: \quad y' = 5x^4 - 12x^2 + 2$$

$$2. \quad y = ax^2 + bx + c$$

$$R_j: \quad y = 2ax + b$$

$$3. \quad y = -\frac{5x^3}{a}$$

$$R_j: \quad y' = -\frac{5}{a}(x^3)' = -\frac{15}{a}x^2$$

$$6. \quad y = x^2 \sqrt[3]{x^2}$$

$$R_j: \quad y = x^2 \cdot x^{\frac{2}{3}} = x^{\frac{8}{3}}$$

$$y' = \frac{8}{3}x^{\frac{5}{3}} = \frac{8}{3}\sqrt[3]{x^5} = \frac{8}{3}\sqrt[3]{x^2}$$

$$7. \quad y = \frac{a+bx}{c+dx}$$

$$R_j: \quad y' = \frac{b(c+dx) - (a+bx) \cdot d}{(c+dx)^2}$$

$$y' = \frac{bc + bdx - ad - bdx}{(c+dx)^2}$$

$$y' = \frac{bc - ad}{(c+dx)^2}$$

$$9. \quad y = \frac{2}{2x-1} - \frac{1}{x}$$

$$R_j: \quad y' = \frac{0(2x-1) - 2(2)}{(2x-1)^2} - (-1)x^{-2}$$

$$4. \quad y = \frac{ax^6 + b}{\sqrt{a^2 + b^2}}$$

$$R_j: \quad y = \frac{a}{\sqrt{a^2 + b^2}}x^6 + \frac{b}{\sqrt{a^2 + b^2}}$$

$$y' = \frac{6a}{\sqrt{a^2 + b^2}}x^5$$

$$5. \quad y = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$$

$$R_j: \quad y' = 3 \cdot \frac{2}{3}x^{-\frac{1}{3}} - 2 \cdot \frac{5}{2}x^{\frac{3}{2}} - 3x^{-4}$$

$$= 2x^{-\frac{1}{3}} - 5x^{\frac{3}{2}} - 3x^{-4}$$

$$8. \quad y = \frac{2x+3}{x^2-5x+5}$$

$$R_j: \quad y' = \frac{2(x^2-5x+5) - \overbrace{(2x+3)(2x-5)}^{4x^2-10x+6x-15}}{(x^2-5x+5)^2}$$

$$y' = \frac{2x^2 - 10x + 10 - 4x^2 + 4x + 15}{(x^2-5x+5)^2}$$

$$y' = \frac{-2x^2 - 6x + 25}{(x^2-5x+5)^2}$$

znamo:

$$\frac{1}{x} = x^{-1}$$

$$y' = \frac{-4}{(2x-1)^2} + \frac{1}{x^2}$$

$$y' = \frac{1-4x}{x^2(2x-1)^2}$$

$$10. \quad y = at^m + bt^{m+n} \quad R_j: \quad y' = ma t^{m-1} + b(m+n) t^{m+n-1}$$

$$11. \quad y = \frac{\pi}{x} + \ln 2, \quad R_j: \quad y' = -\frac{\pi}{x^2}$$

$$12. \quad y = \frac{a}{\sqrt[3]{x^2}} - \frac{b}{x\sqrt[3]{x}}, \quad R_j: \quad y' = \frac{4b}{3x^2\sqrt[3]{x}} - \frac{2a}{3x\sqrt[3]{x^2}}$$

$$13. \quad y = \frac{1+\sqrt{z}}{1-\sqrt{z}}, \quad (\sqrt{z})' = \frac{1}{2\sqrt{z}}$$

$$R_j: \quad y' = \frac{\frac{1}{2\sqrt{z}}(1-\sqrt{z}) - (1+\sqrt{z})(-\frac{1}{2\sqrt{z}})}{(1-\sqrt{z})^2} = \frac{\frac{1-\sqrt{z}}{2\sqrt{z}} + \frac{1+\sqrt{z}}{2\sqrt{z}}}{(1-\sqrt{z})^2} = \frac{1}{(1-\sqrt{z})^2\sqrt{z}}$$

$$14. \quad y = \operatorname{tg} x - \operatorname{ctg} x$$

$$R_j: \quad y' = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x} = \frac{4}{(2 \sin x \cos x)^2}$$

$$y' = \frac{4}{\sin^2 2x}$$

$$15. \quad y = \frac{\sin x + \cos x}{\sin x - \cos x}$$

$$R_j: \quad y' = \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$y' = \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} = \frac{-(\sin^2 x - 2\sin x \cos x + \cos^2 x) - (\sin^2 x + 2\sin x \cos x + \cos^2 x)}{(\sin x - \cos x)^2}$$

$$y' = \frac{-2}{(\sin x - \cos x)^2}$$

$$= 2\sin t + t^2 \sin t - 2\sin t$$

$$y' = t^2 \sin t$$

$$16. \quad y = 2t \sin t - (t^2 - 2) \cos t$$

$$R_j: \quad y' = 2(\sin t + t \cos t) - [2t \cos t + (t^2 - 2)(-\sin t)] = \\ = 2\sin t + 2t \cos t - 2t \cos t + (t^2 - 2)\sin t = 2\sin t + (t^2 - 2)\sin t =$$

$$17_0 \quad y = x \arcsin x$$

$$R_j: \quad y' = \arcsin x + x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$18_0 \quad y = (x-1)e^x$$

$$R_j: \quad y' = e^x + (x-1)e^x$$

$$y' = e^x(1+x-1) = xe^x$$

$$19_0 \quad y = \frac{x^5}{e^x}$$

$$R_j: \quad y' = \frac{5x^4e^x - x^5e^x}{e^{2x}} = \frac{x^4e^x(5-x)}{(e^x)^2}$$

$$y' = \frac{x^4(5-x)}{e^x}$$

$$22_0 \quad y = x \operatorname{ctg} x$$

$$R_j: \quad y' = \operatorname{ctg} x - \frac{x}{\sin^2 x}$$

$$23_0 \quad y = \frac{(1+x^2) \operatorname{arctg} x - x}{2}$$

$$20_0 \quad y = \frac{x^2}{\ln x}$$

$$R_j: \quad y' = \frac{2x \cdot \ln x - x^2 \cdot \frac{1}{x}}{\ln^2 x} = \frac{2x \ln x - x}{\ln^2 x}$$

$$\log_B A = \frac{\ln A}{\ln B}$$

$$y' = \frac{x(2 \ln x - 1)}{\ln^2 x}$$

$$21_0 \quad y = \ln x (\log_{10} x - \log_a x)$$

$$R_j: \quad y' = \frac{1}{x} \log_{10} x + \frac{\ln x}{x \ln 10} - \frac{1}{\ln a} \frac{1}{x \ln a}$$

$$y' = \frac{1}{x} \frac{\ln x}{\ln 10} + \frac{\ln x}{x \ln 10} - \frac{1}{x}$$

$$y' = \frac{2 \ln x}{x \ln 10} - \frac{1}{x}$$

$$24_0 \quad y = \frac{1}{x} + 2 \ln x - \frac{\ln x}{x}$$

$$R_j: \quad y' = \frac{2}{x} + \frac{\ln x}{x^2} - \frac{2}{x^2}$$

$$\log_B A = \frac{\log_a A}{\log_a B},$$

$$\ln x = \log_e x, \quad \log_a B = \frac{1}{\log_B a}$$

Izvodi složenih f-ja

$$Y = f(g(x)), \quad Y'_x = f'_s \cdot g'_x \quad \text{ili} \quad \left. \begin{array}{l} Y = \psi(u) \\ u = \varphi(x) \end{array} \right\} \quad Y = \psi(\varphi(x))$$

Naci izvode sljedećih f-ja:

$$\textcircled{1.}_0 \quad Y = (1+3x-5x^2)^{30}$$

$$\text{Rj: } Y = u^{30}, \quad \text{gdje je } u = 1+3x-5x^2$$

$$Y' = 30u^{29} \cdot u', \quad u' = 3-10x$$

$$Y' = 30(1+3x-5x^2)^{29} \cdot (3-10x)$$

$$\textcircled{2.}_0 \quad Y = (3+2x^2)^4$$

$$\text{Rj: } Y' = 4(3+2x^2)^3 \cdot (3+2x^2)'$$

$$Y' = 4(3+2x^2)^3 \cdot 4x = 16x(3+2x^2)^3$$

$$\textcircled{3.}_0 \quad Y = \sqrt[3]{a+bx^3}$$

$$\text{Rj: } Y = \sqrt[3]{u}, \quad u = a+bx^3$$

$$Y' = \frac{1}{3} u^{-\frac{2}{3}} \cdot u', \quad u' = 3bx^2$$

$$Y' = \frac{1}{\sqrt[3]{u^{\frac{2}{3}}}} \cdot 3bx^2$$

$$Y' = \frac{bx^2}{\sqrt[3]{(a+bx^3)^2}}$$

$$\textcircled{7.}_0 \quad F(Y) = (2a+3by)^2$$

$$\text{Rj: } F'(Y) = 12ab + 18b^2Y$$

$$\textcircled{4.}_0 \quad Y = \sqrt{\operatorname{ctg} x} - \sqrt{\operatorname{ctg} 2}$$

$$\text{Rj: } Y = \sqrt{u} - \sqrt{\operatorname{ctg} 2}, \quad u = \operatorname{ctg} x$$

$$Y' = \frac{1}{2\sqrt{u}} \cdot u', \quad u' = -\frac{1}{\sin^2 x}$$

$$Y' = \frac{-1}{2\sin^2 x \sqrt{\operatorname{ctg} x}}$$

$$\textcircled{5.}_0 \quad Y = 2x + 5\cos^3 x$$

$$\text{Rj: } Y' = 2 + 15\cos^2 x \cdot (-\sin x)$$

$$Y' = 2 - 15\cos^2 x \sin x$$

$$\textcircled{6.}_0 \quad f(x) = -\frac{1}{6(1-3\cos x)^2}$$

$$\text{Rj: } Y' = \frac{\sin x}{(1-3\cos x)^3}$$

Izvodí složených - f-jí - nastavak:

1. Nači izvode složených f-jí:

$$Y = x^4(a-2x^3)^2$$

$$R_j: Y' = 4x^3(a-2x^3)^2 + x^4 \cdot 2(a-2x^3) \cdot \frac{(-2)(3)}{(-6)}x^2$$

$$Y' = 4x^3(a-2x^3) \cdot [a-2x^3 + x \cdot (-1) \cdot 3x^2] \\ a-2x^3-3x^3$$

$$Y' = 4x^3(a-2x^3)(a-5x^3)$$

$$2. Y = (a+x)\sqrt{a-x}$$

$$R_j: Y' = 1 \cdot \sqrt{a-x} + (a+x) \frac{1}{2\sqrt{a-x}} \cdot (-1)$$

$$Y' = \sqrt{a-x} - \frac{a+x}{2\sqrt{a-x}} = \frac{2(a-x)-(a+x)}{2\sqrt{a-x}}$$

$$Y' = \frac{a-3x}{2\sqrt{a-x}}$$

$$3. Z = \sqrt[3]{Y+\sqrt{Y}}$$

$$R_j: (\sqrt[3]{x})' = (x^{\frac{1}{3}})' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$Z' = \frac{1}{3\sqrt[3]{(Y+\sqrt{Y})^2}} \cdot (Y+\sqrt{Y})'$$

$$Z' = \frac{1}{3\sqrt[3]{(Y+\sqrt{Y})^2}} \cdot \left(1 + \frac{1}{2\sqrt{Y}}\right)$$

$$Z' = \frac{1}{3\sqrt[3]{(Y+\sqrt{Y})^2}} \cdot \frac{2\sqrt{Y}+1}{2\sqrt{Y}}$$

$$Z' = \frac{2\sqrt{Y}+1}{6\sqrt{Y}\sqrt[3]{(Y+\sqrt{Y})^2}}$$

$$4. Y = \operatorname{tg}^2 5x$$

$$R_j: Y' = 2 \operatorname{tg} 5x \cdot (\operatorname{tg} 5x)'$$

$$Y' = 2 \operatorname{tg} 5x \cdot \frac{1}{\cos^2 x} \cdot (5x)'$$

$$Y' = \frac{10 \operatorname{tg} 5x}{\cos^2 x}$$

$$5. Y = \sqrt{\cos x} \cdot a^{\sqrt{\cos x}}$$

$$R_j: Y' = \frac{1}{2\sqrt{\cos x}} \cdot (\cos x)' \cdot a^{\sqrt{\cos x}}$$

$$+ \sqrt{\cos x} \cdot a^{\sqrt{\cos x}} \ln a \cdot (\sqrt{\cos x})'$$

$$Y' = -\frac{\sin x}{2\sqrt{\cos x}} \cdot a^{\sqrt{\cos x}} + \ln a \sqrt{\cos x} \cdot a^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot (\cos x)'$$

$$Y' = -\frac{\sin x a^{\sqrt{\cos x}}}{2\sqrt{\cos x}} - \frac{\ln a \cdot \sin x \cdot \sqrt{\cos x} \cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x}}$$

$$Y' = -\frac{\sin x a^{\sqrt{\cos x}}}{2\sqrt{\cos x}} \left[1 + \ln a \cdot \sqrt{\cos x} \right]$$

$$Y' = -\frac{\sin x \sqrt{\cos x} \cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x} \cdot \sqrt{\cos x}} \left[1 + \ln a \cdot \sqrt{\cos x} \right]$$

$$Y' = -\frac{1}{2} \operatorname{tg} x \cdot y \cdot \left[1 + \ln a \sqrt{\cos x} \right]$$

$$6. Y = 3^{\operatorname{ctg} \frac{1}{x}} \quad R_j: Y' = \frac{3^{\operatorname{ctg} \frac{1}{x}} \cdot \ln 3}{\left(x \sin \frac{1}{x}\right)^2}$$

$$7. Y = \ln \left(x + \sqrt{a^2 + x^2} \right) \quad R_j: Y' = \frac{1}{\sqrt{a^2 + x^2}}$$

$$8_0 \quad y = \ln \frac{(x-2)^5}{(x+1)^3}$$

Rj.

$$y = \ln(x-2)^5 - \ln(x+1)^3$$

$$y' = \frac{1}{(x-2)^5} \cdot ((x-2)^5)' - \frac{1}{(x+1)^3} \cdot [(x+1)^3]'$$

$$y' = \frac{5(x-2)^4}{(x-2)^5} - \frac{3(x+1)^2}{(x+1)^3} \quad \leftarrow \text{komplikovano}$$

Y mogu napisati i kao

$$y = 5\ln(x-2) - 3\ln(x+1)$$

$$y' = 5 \cdot \frac{1}{x-2} - 3 \cdot \frac{1}{x+1}$$

$$y' = \frac{5(x+1) - 3(x-2)}{(x-2)(x+1)}$$

$$y' = \frac{2x+11}{x^2-x-2}$$

$$9_0 \quad y = \ln \ln(3-2x^3)$$

Rj.

$$y' = \frac{1}{\ln(3-2x^3)} \cdot (\ln(3-2x^3))'$$

$$y' = \frac{1}{\ln(3-2x^3)} \cdot \frac{1}{3-2x^3} \cdot (3-2x^3)'$$

$$y' = \frac{-6x^2}{(3-2x^3)\ln(3-2x^3)}$$

$$12_0 \quad y = \ln \frac{(x-1)^3(x-2)}{x-3}$$

Rj.

$$y' = \frac{3x^2-16x+19}{(x-1)(x-2)(x-3)}$$

$$13_0 \quad f(x) = \sqrt{x^2+1} - \ln \frac{1+\sqrt{x^2+1}}{x}$$

$$10_0 \quad y = \ln \frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2} - x}$$

Rj. provojednostavimo izvaz

$$\frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2} - x} \cdot \frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2} + x} =$$

$$= \frac{(\sqrt{x^2+a^2} + x)^2}{x^2+a^2 - x^2} = \frac{(\sqrt{x^2+a^2} + x)^2}{a^2}$$

$$y = \ln \frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2} - x} = 2 \ln \frac{\sqrt{x^2+a^2} + x}{a^2}$$

$$y' = 2 \cdot \frac{1}{\frac{\sqrt{x^2+a^2} + x}{a^2}} \cdot \left(\frac{\sqrt{x^2+a^2} + x}{a^2} \right)'$$

$$y' = \frac{2a^2}{\sqrt{x^2+a^2} + x} \cdot \frac{1}{a^2} \cdot \left[\frac{1}{2\sqrt{x^2+a^2}} \cdot \frac{2x}{(x^2+a^2)} + 1 \right]$$

$$y' = \frac{2}{\sqrt{x^2+a^2} + x} \cdot \frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2}}$$

$$y' = \frac{2}{\sqrt{x^2+a^2}}$$

$$11_0 \quad y = \arctg \ln x$$

Rj.

$$y' = \frac{1}{1+\ln^2 x} \cdot (\ln x)'$$

$$y' = \frac{1}{x(1+\ln^2 x)}$$

Rj.

$$y' = \frac{\sqrt{1+x^2}}{x}$$

Izvodi f-ja koje nisu eksplicitno zadane

$y = f(x)$ je eksplicitni oblik f-je. Pored eksplicitnog oblika postoje: $\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$ parametarski oblik f-je

i $F(x, y) = 0$ implicitan oblik f-je

1.) Izračunati $y' = \frac{dy}{dx}$ ako je f-ja y zadana parametarski:

$$\begin{cases} x = a \cos t \\ y = a \sin t \end{cases}$$

$$\frac{dy}{dx} = \frac{a \cos t}{-a \sin t} = -ctg t$$

Rj: $\frac{dx}{dt} = -a \sin t \quad \frac{dy}{dt} = a \cos t$

tj. $y' = -ctg t$

2.) Izračunati $y' = \frac{dy}{dx}$ ako je f-ja y zadana $\begin{cases} x = \sqrt{t} \\ y = \sqrt[3]{t} \end{cases}$

Rj: $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt} = \frac{1}{3} t^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{t^2}}$

$$\frac{dy}{dx} = \frac{\frac{1}{3\sqrt[3]{t^2}}}{\frac{1}{2\sqrt{t}}} = \frac{2\sqrt{t}}{3\sqrt[3]{t^2}} = \frac{2}{3} \sqrt[6]{\frac{t^2}{t^4}} = \frac{2}{3\sqrt[6]{t^2}}$$

3.) Izračunati $y' = \frac{dy}{dx}$ ako je f-ja y zadana par. $\begin{cases} x = a \cos^3 t \\ y = b \sin^3 t \end{cases}$

Rj. $y' = -\frac{b}{a} \operatorname{tg} t$

4.) Izračunati izvod y'_x ako je f-ja zadana implic. $x^3 + y^3 - 3axy = 0$.

Rj. $x^3 + y^3 - 3axy = 0 \quad (3y^2 - 3ax)y' = 3ay - 3x^2 \quad | : 3$

$$3x^2 + 3y^2 \cdot y' - 3ay - 3axy' = 0 \quad y' = \frac{ay - x^2}{y^2 - ax}$$

5.) Izračunati izvod y'_x ako je f-ja zadana implicitno $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Rj. $\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot y' = 0 \quad y' = -\frac{x^2}{ya^2}$

$$\frac{2y}{b^2} y' = -\frac{2x}{a^2} \quad | : 2$$

6.) Izračunati izvod y'_x ako je f-ja zadana implicitno $\sqrt{x^2 + y^2} = c \cdot \arctg \frac{y}{x}$.

Rj. $y' = \frac{cy + x\sqrt{x^2 + y^2}}{cx - y\sqrt{x^2 + y^2}}$

Logaritamski izvod

Logaritamskim izvodom f-je $y = f(x)$ nazivamo izvodom logaritma te f-je tj. $(\ln y)' = \frac{y'}{y} = \frac{f'(x)}{f(x)}$.

1.) Naći izvod složene eksplicitno zadane f-je $y = u^v$ ako je $u = \varphi(x)$ i $v = \psi(x)$.

$$\text{Rj. } y = u^v \quad / \ln \quad \frac{1}{y} \cdot y' = v' \ln u + v \cdot \frac{1}{u} \cdot u' \quad / \cdot y \\ \ln y = \ln u^v \quad \nearrow \quad y' = y \left(v' \ln u + v \cdot \frac{1}{u} \cdot u' \right) \\ \ln y = v \ln u \quad /'$$

2.) Izračunati y' ako je $y = (\sin x)^x$.

$$\text{Rj. } y = (\sin x)^x \quad / \ln \quad \frac{1}{y} \cdot y' = \ln \sin x + x \cdot \frac{1}{\sin x} \cdot (\sin x)' \\ \ln y = \ln (\sin x)^x \quad \nearrow \quad y' = y \left(\ln \sin x + x \cdot \frac{\cos x}{\sin x} \right) \\ \ln y = x \ln \sin x \quad /' \quad y' = (\sin x)^x \left(\ln \sin x + x \operatorname{ctg} x \right)$$

3.) Izračunati y' ako je $y = \sqrt[3]{x^2} \cdot \frac{1-x}{1+x^2} \cdot \sin^3 x \cdot \cos^2 x$.

$$\text{Rj. } \ln y = \ln \sqrt[3]{x^2} + \ln \frac{1-x}{1+x^2} + \ln \sin^3 x + \ln \cos^2 x \quad \begin{aligned} \sqrt{\left(\frac{1-x}{1+x^2} \right)} &= \frac{(-1)(1+x^2)-(1-x) \cdot 2x}{(1+x^2)^2} \\ \ln y &= \frac{2}{3} \ln x + \ln \frac{1-x}{1+x^2} + \ln \sin^3 x + \ln \cos^2 x \quad /' \\ \frac{1}{y} \cdot y' &= \frac{2}{3} \cdot \frac{1}{x} + \frac{1+x^2}{1-x} \cdot \frac{x^2-2x-1}{(1+x^2)^2} + \frac{3 \sin^2 x}{\sin^3 x} \cdot \cos x + \frac{2 \cos x}{\cos^3 x} \cdot (-\sin x) \end{aligned} \\ y' = y \left(\frac{2}{3x} \cdot \frac{x^2-2x-1}{(1-x)(1+x^2)} + 3 \operatorname{ctg} x - 2 \operatorname{tg} x \right)$$

4.) $y = x^x$, Rj. $y' = x^x (1 + \ln x)$

5.) $y = x^{x^2}$, Rj. $y' = x^{x^2+1} (1 + 2 \ln x)$

6.) $y = \sqrt[x]{x}$, Rj. $y' = \sqrt[x]{x} \frac{1-\ln x}{x^2}$

Primjena izvoda u geometriji

Ako je $y=f(x)$ data kriva i (x_0, y_0) data tačka tad
 $y-y_0 = f'(x_0)(x-x_0)$ je jednačina tangente u tački (x_0, y_0) .

$$x-x_0 + f'(x_0)(y-y_0) = 0 \quad \begin{matrix} \text{je jednačina normale na krivu u} \\ \text{(ili } y-y_0 = \frac{-1}{f'(x_0)}(x-x_0) \text{)} \end{matrix} \quad \text{tački } (x_0, y_0).$$

Ako su $y_1 = k_1 x + n_1$; $y_2 = k_2 x + n_2$ dvije date prave tad je
 $\operatorname{tg} \varphi = \frac{k_2 - k_1}{1 + k_1 \cdot k_2}$ tangens ugla između dvije prave.

Podugom između dvije krive $y=f_1(x)$ i $y=f_2(x)$ u njihovoj zajedničkoj tački podrazumijevamo ugao φ između njihovih zajedničkih tangentnih u tački (x_0, y_0) .

$$\operatorname{tg} \varphi = \frac{f'_2(x_0) - f'_1(x_0)}{1 + f'_1(x_0) \cdot f'_2(x_0)}$$

Neka je $y=f(x)$ data f -ja.

Ako je $y'(x) > 0$ za $\forall x \in (a, b)$ tad f -ja y raste u intervalu (a, b) .
Ako je $y'(x) < 0$ za $\forall x \in (a, b)$ tad f -ja y opada (\searrow) u (a, b) ,

1.) Naći jednačinu tangente i normale na krivu
 $y=\sqrt{x}$ u tački s apscisom $x=4$.

Rj: $y=\sqrt{x}$ $y'(4) = \frac{1}{4}, \quad y(4)=2$

$$y-y_0 = f'(x_0)(x-x_0) \quad \text{jedn. tang.}$$

$$y-2 = \frac{1}{4}(x-4) \quad | \cdot 4$$

$$4y-8 = x-4$$

$$x-4y+4=0 \quad \text{jedn. tangente}$$

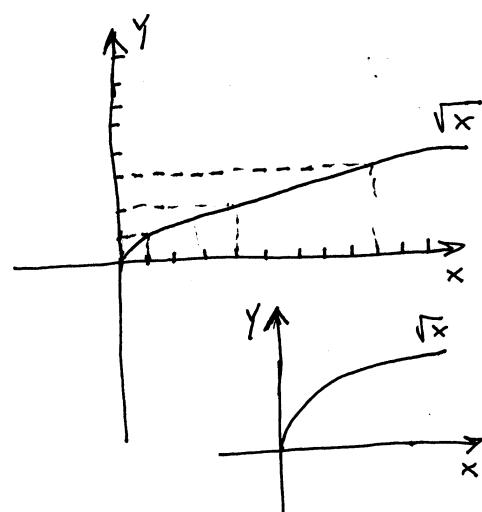
$$-\frac{1}{f'(x_0)} = -4$$

$$y-y_0 = \frac{-1}{f'(x_0)}(x-x_0) \quad \text{jedn. norm.}$$

$$y-2 = -\frac{1}{4}(x-4)$$

$$4x+y-18=0 \quad \text{jednačina normale}$$

$$y-2 = -\frac{1}{4}x + 16$$



(2.) Napišite jednačinu tangente i normale na krivu

$$y = x^3 + 2x^2 - 4x - 3 \text{ u tački } (-2, 5).$$

Rj. $y' = 3x^2 + 4x - 4$

$$x - x_0 + Y_0'(Y - Y_0) = 0$$

jedn. norm.

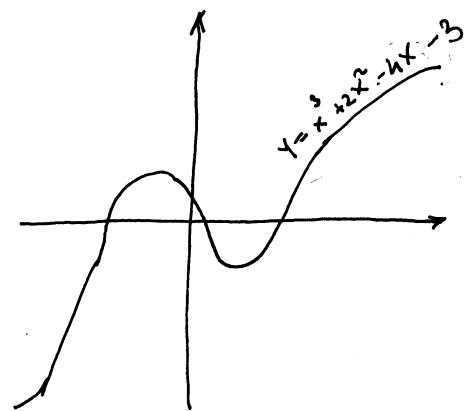
$$Y - Y_0 = f'(x_0)(x - x_0)$$

$$x + 2 = 0$$

jedn. normale

$$Y - 5 = 0(x + 2)$$

$Y - 5 = 0$ jednačina
tangente



(3.) Nadi jednačinu tangente i normale na krivu $y = \sqrt[3]{x-1}$ u tački $(1, 0)$. Rj. $x-1=0, Y=0$

(4.) Odrediti ugao pod kojim se sijeku krive $y = x^2$; $x = y^2$!

Rj. Prvo nadimo tačke presjeka krivih.

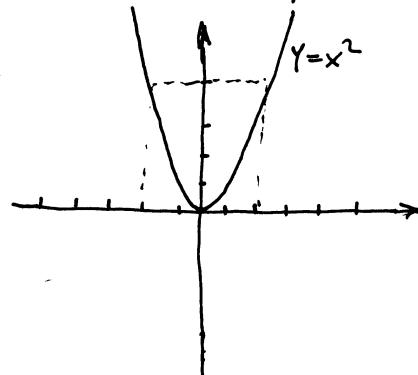
$$y = x^2 \quad Y(Y^3 - 1) = 0$$

$$x = y^2 \quad Y(Y-1)(Y^2 + Y + 1) = 0$$

$$Y = Y^4 \quad Y_1 = 0 \text{ ili } Y_2 = 1$$

$$Y - Y^4 = 0 \quad Y_1 = 0 \Rightarrow x_1 = 0$$

$$Y^4 - Y = 0 \quad Y_2 = 1 \Rightarrow x_2 = 1$$



Pošto je duže tačke presjeka $(0, 0)$ i $(1, 1)$

f: $y = x^2$

$f_2: x = y^2$

$$y' = 2x$$

$$1 = 2Y Y'$$

$$f'_1(0) = 0$$

$$Y' = \frac{1}{2Y}$$

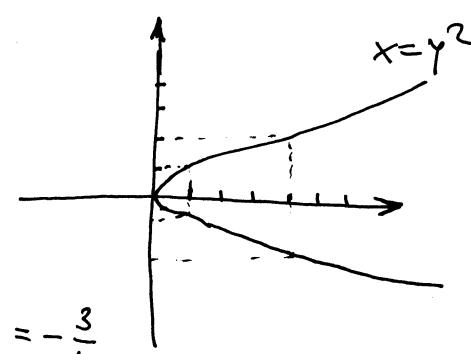
$$f'_1(1) = 2$$

$f'_2(0)$ nije def.

$$f'_2(1) = \frac{1}{2}$$

$$\operatorname{tg} \varphi = \frac{f'_1(x_0) - f'_2(x_0)}{1 - f'_1(x_0) \cdot f'_2(x_0)}$$

$$\operatorname{tg} \varphi = \frac{\frac{1}{2} - 2}{1 + \frac{1}{2} \cdot 2} = \frac{-\frac{3}{2}}{2} = -\frac{3}{4}$$



$\varphi = \arctg(-\frac{3}{4})$ ugao pod kojim se

sijeku date krive u tački $(1, 1)$.

(5.) Nadi ugao pod kojim se sijeku parabole

$$y = (x-2)^2 \text{ i } y = -4 + 6x - x^2.$$

Rj. $\varphi = 40^\circ 36'$

Izvodi višeg reda

$y = f(x)$ - data f-ja

$y' = f'(x)$ prvi izvod

$y'' = (f'(x))' = f''(x)$ drugi izvod

$y''' = [f''(x)]' = f'''(x)$ treci izvod

\vdots

$y^{(n)} = [y^{(n-1)}]' = f^{(n)}(x)$ n-ti izvod f-je $y = f(x)$

① Nadi y''' f-je $y = x e^x$

$$R_j: \quad y = x e^x$$

$$y'' = e^x + (x+1) e^x = (x+2) e^x$$

$$y' = e^x + x e^x = (x+1) e^x$$

$$y''' = e^x + (x+2) e^x = (x+3) e^x$$

② Nadi $y^{(5)}$ f-je $y = 2x^3 + 3x^2 - 4x + 5$

$$R_j: \quad y' = 6x^2 + 6x - 4$$

$$y^{(4)} = 0$$

$$y'' = 12x + 6$$

$$y^{(5)} = 0$$

③ Nadi y'' f-je $y = \ln \frac{x^2+3}{x^2+1}$.

$$R_j: \quad y' = \frac{1}{\frac{x^2+3}{x^2+1}} \cdot \left(\frac{x^2+3}{x^2+1} \right)' = \frac{x^2+1}{x^2+3} \cdot \frac{2x(x^2+1) - (x^2+3) \cdot 2x}{(x^2+1)^2}$$

$$y' = \frac{2x^3 + 2x - 2x^3 - 6x}{(x^2+3)(x^2+1)} = \frac{-4x}{(x^2+3)(x^2+1)} = \frac{-4}{x^4 + 4x^2 + 3}$$

$$y'' = \frac{(-4)(x^4 + 4x^2 + 3) - (-4x)(4x^3 + 8x)}{(x^2+3)^2 (x^2+1)^2} = \frac{-4x^4 - 16x^2 - 12 + 16x^4 + 32x^2}{(x^2+3)^2 (x^2+1)^2} = \frac{12x^4 + 16x^2 - 12}{(x^2+3)^2 (x^2+1)^2}$$

$$y''' = \frac{4(3x^4 + 4x^2 - 3)}{(x^2+3)^2 (x^2+1)^2}$$

$$4.) \text{ Nádi } Y'' \text{ f-je } Y = (x-1) e^{-\frac{1}{x+1}}$$

$$\text{Rj: } Y' = \left((x-1) e^{-\frac{1}{x+1}} \right)' = e^{-\frac{1}{x+1}} + (x-1) e^{-\frac{1}{x+1}} \cdot \left(-\frac{1}{x+1} \right)' = \\ = e^{-\frac{1}{x+1}} + (x-1) \cdot \frac{1}{(x+1)^2} e^{-\frac{1}{x+1}} = \left(1 + \frac{x-1}{(x+1)^2} \right) e^{-\frac{1}{x+1}}$$

$$\left[\left(-\frac{1}{(x+1)} \right)' = \left[-(x+1)^{-1} \right]' = (x+1)^{-2} \right] \quad Y' = \frac{(x+1)^2 + x-1}{(x+1)^2} e^{-\frac{1}{x+1}}$$

$$Y' = \frac{x^2 + 2x + 1 + x-1}{(x+1)^2} e^{-\frac{1}{x+1}} = \frac{x(x+3)}{(x+1)^2} e^{-\frac{1}{x+1}} = \frac{(x^2 + 3x)}{x^2 + 2x + 1} e^{-\frac{1}{x+1}}$$

$$Y'' = \left[\frac{x(x+3) e^{-\frac{1}{x+1}}}{(x+1)^2} \right]' = \frac{[(2x+3)e^{-\frac{1}{x+1}} + (x^2 + 3x)e^{-\frac{1}{x+1}} \cdot \frac{1}{(x+1)^2}] \cdot (x+1)^2 - (x^2 + 3x)e^{-\frac{1}{x+1}} \cdot 2(x+1)}{(x+1)^4}$$

$$Y'' = \frac{[(2x+3)(x+1)^2 + x^2 + 3x - 2(x^2 + 3x)(x+1)]}{(x+1)^4} e^{-\frac{1}{x+1}}$$

$$Y'' = \frac{\cancel{2x^3 + 4x^2} + \cancel{2x + 3x^2} + \cancel{6x + 3 + x^2} + \cancel{3x} - \cancel{2x^3} - \cancel{8x^2} - \cancel{6x}}{(x+1)^4} e^{-\frac{1}{x+1}}$$

$$Y'' = \frac{5x + 3}{(x+1)^4} e^{-\frac{1}{x+1}}$$

$$5.) \text{ Nádi } Y'' \text{ f-ja:}$$

$$\text{a) } Y = \frac{x^3}{x^2 - 2x - 8}$$

$$\text{Rj: } Y'' = \frac{24x(x^2 + 4x + 16)}{(x^2 - 3x - 8)^3}$$

$$\text{b) } Y = \frac{16}{x^2 \cdot (x-4)} \quad \text{Rj: } Y'' = \frac{64(3x^2 - 16x + 24)}{x^4 (x-4)^3}$$

$$\text{c) } Y = (2x-1) e^{-\frac{x}{x-1}} \quad \text{Rj: } Y'' = \frac{e^{-\frac{x}{x-1}}}{(x-1)^4}$$

L'Hospital-Bernoulliјeve pravilo

Ako su obe f-je $f(x)$ i $g(x)$ beskonačno male ili beskonačno velike kad $x \rightarrow a$ tj. ako razlomak predstavlja u tački $x=a$ neodređen oblik tipa $\frac{0}{0}$ ili $\frac{\infty}{\infty}$ tada je $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.

Pomoću Lopitalovog pravila rješavamo limese u obliku:
 $\frac{0}{0}, \frac{\infty}{\infty}, \underbrace{0 \cdot \infty, \infty - \infty, 1^\infty, 0^0, \infty^0}$
 ove oblike prvo svedeno na neki od oblika $\frac{0}{0}$ ili $\frac{\infty}{\infty}$.

Izračunati:

$$\textcircled{1} \lim_{x \rightarrow 0} \frac{\ln x}{\operatorname{ctg} x} \left(\frac{-\infty}{\infty} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 0} \frac{(\ln x)'}{(\operatorname{ctg} x)'} = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = -\lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \\ = -\lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \sin x = -1 \cdot 0 = 0$$

$$\textcircled{2} \lim_{x \rightarrow 1} \frac{x^3 - 2x^2 - x + 2}{x^3 - 7x + 6} \left(\frac{0}{0} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 1} \frac{3x^2 - 4x - 1}{3x^2 - 7} = \frac{-2}{-4} = \frac{1}{2}$$

$$\textcircled{3} \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3} \left(\frac{0}{0} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 0} \frac{\frac{-x \sin x}{\cos x + x(-\sin x) - \cos x}}{3x^2} \left(\frac{0}{0} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 0} \frac{-\sin x + (-x)\cos x}{6x} \\ \left(\frac{0}{0} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 0} \frac{-\cos x - \cos x - x(-\sin x)}{6} = \frac{-2}{6} = -\frac{1}{3}$$

$$\textcircled{4} \lim_{x \rightarrow 1} \frac{1-x}{1-\sin \frac{\pi x}{2}} \left(\frac{0}{0} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 1} \frac{-1}{-\cos \frac{\pi x}{2} \cdot \frac{\pi}{2}} = \frac{-1}{-\frac{\pi}{2}} = +\infty$$

$$\textcircled{5} \lim_{x \rightarrow 0} \frac{\operatorname{tg} x - \sin x}{x - \sin x} \left(\frac{0}{0} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{\cos^2 x} - \cos x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos^3 x}{\cos^2 x}}{1 - \cos x} \\ = \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x + \cos^2 x)}{\cos^2 x(1 - \cos x)} = 3$$

$$\textcircled{6} \lim_{x \rightarrow 0} \frac{\sin 5x}{x} \left(\frac{0}{0} \right) \stackrel{\text{Lop.}}{=} \lim_{x \rightarrow 0} \frac{\cos 5x \cdot 5}{1} = 5$$

$$\textcircled{7.0} \lim_{x \rightarrow \infty} \frac{e^x}{x^5} \left(\frac{\infty}{\infty} \right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{5x^4} \left(\frac{\infty}{\infty} \right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{20x^3} \left(\frac{\infty}{\infty} \right) \stackrel{L'H}{=} \dots = \frac{\infty}{120} = \infty$$

$$\textcircled{8.0} \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}} \left(\frac{\infty}{\infty} \right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{3\sqrt[3]{x^2}}} = \lim_{x \rightarrow \infty} \frac{3\sqrt[3]{x^2}}{x} = 3 \lim_{x \rightarrow \infty} \frac{x^{\frac{2}{3}}}{x} : x = 3 \lim_{x \rightarrow \infty} x^{-\frac{1}{3}} = 0$$

$$\textcircled{9.0} \checkmark \lim_{x \rightarrow 0} \frac{\ln(\sin mx)}{\ln \sin x} \quad R_j. \quad 1$$

$$\textcircled{10.0} \lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) (\infty - \infty) = \lim_{x \rightarrow 1} \frac{\ln x - (x-1)}{(x-1)\ln x} \left(\frac{0}{0} \right) \stackrel{L'H}{=}$$

$$= \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{\ln x + (x-1)\frac{1}{x}} = \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{\ln x - \frac{1}{x} + 1} \left(\frac{0}{0} \right) \stackrel{L'H}{=} \lim_{x \rightarrow 1} \frac{-\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = -\frac{1}{2}$$

$$\textcircled{11.0} \lim_{x \rightarrow 0} (1 - \cos x) \operatorname{ctg} x (0 \cdot \infty) = \lim_{x \rightarrow 0} \frac{(1 - \cos x) \cos x}{\sin x} \left(\frac{0}{0} \right) \stackrel{L'H}{=}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} \cdot \lim_{x \rightarrow 0} \cos x \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} \cdot \lim_{x \rightarrow 0} \cos x = 0 \cdot 1 = 0$$

$$\textcircled{12.0} \lim_{x \rightarrow \infty} [x \cdot (e^{-\frac{2}{x}} - 1)] (\infty \cdot 0) = \lim_{x \rightarrow \infty} \frac{e^{-\frac{2}{x}} - 1}{\frac{1}{x}} \left(\frac{0}{0} \right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{e^{-\frac{2}{x}} \cdot (-2) \cdot (-1) \cdot x^{-2}}{(-1) \cdot x^{-2}}$$

$$= e^0 \cdot (-2) = -2$$

$$\textcircled{13.0} \checkmark \lim_{x \rightarrow \infty} x \cdot \sin \frac{a}{x} \quad R_j. \quad a$$

$$\textcircled{14.0} \lim_{x \rightarrow 1} x^{\frac{1}{1-x}} (1^\infty) = \lim_{x \rightarrow 1} e^{\ln x^{\frac{1}{1-x}}} = \lim_{x \rightarrow 1} e^{\frac{1}{1-x} \cdot \ln x} = e^{\lim_{x \rightarrow 1} \frac{\ln x}{1-x}} \left(\frac{0}{0} \right) \stackrel{L'H}{=}$$

$$\stackrel{L'H}{=} e^{\lim_{x \rightarrow 1} \frac{\frac{1}{x}}{-1}} = e^{-1} = \frac{1}{e}$$

$$\textcircled{15.0} \lim_{x \rightarrow 0} (\operatorname{ctg} x)^{\frac{1}{\ln x}} (\infty^\infty) = \lim_{x \rightarrow 0} e^{\frac{1}{\ln x} \cdot \frac{1}{\operatorname{ctg} x}} = e^{\lim_{x \rightarrow 0} \frac{\ln(\operatorname{ctg} x)}{\ln x}} \stackrel{L'H}{=} e^{\lim_{x \rightarrow 0} \frac{\frac{1}{\operatorname{ctg} x} \cdot -\frac{1}{\sin^2 x}}{\frac{1}{x}}} = e^{\lim_{x \rightarrow 0} \frac{-x}{\sin x \cos x}} \stackrel{L'H}{=} e^{\lim_{x \rightarrow 0} \frac{-1}{\cos^2 x - \sin^2 x}} = e^{-1} = \frac{1}{e}$$

$$\textcircled{16.0} \checkmark \lim_{x \rightarrow 0} x^{\sin x} \quad R_j. \quad 1$$

$$\textcircled{17.0} \checkmark \lim_{x \rightarrow \infty} [(x-1)e^{\frac{-1}{x+1}} - x] \quad R_j. \quad -2$$

Ispitivanje f-ja

Ispitati f-ju znači odrediti:

1. oblast definisanosti
2. parnost (neparnost) i periodičnost
3. nule, presjek grafa sa y -osom, znak f-je
4. ponašanje na krajevima intervala definisanosti i asymptote
5. rast; opadanje f-je (intervale u kojima f-ja raste ili opada)
6. ekstremi; f-je (minimum; maksimum ako ih ima)
7. prevojne tačke i intervale konveksnosti i konkavnosti
8. na osnovu svega ovoga nacrtati grafik

Definicione područje označavamo sa D ; to je područje u kojem je f-ja definisana (ima konačnu ili beskonačnu vrijednost). Npr.

a) $y = \frac{1}{x}$, $D: \mathbb{R} \setminus \{0\}$, $D: x \in (-\infty, 0) \cup (0, +\infty)$

b) $y = \sqrt{x}$, $D: x \in \mathbb{R}_0^+$, $D: x \in [0, +\infty)$, $D: x \geq 0$

c) $y = \log x$, $D: x \in \mathbb{R}^+$, $D: x \in (0, +\infty)$, $D: x > 0$

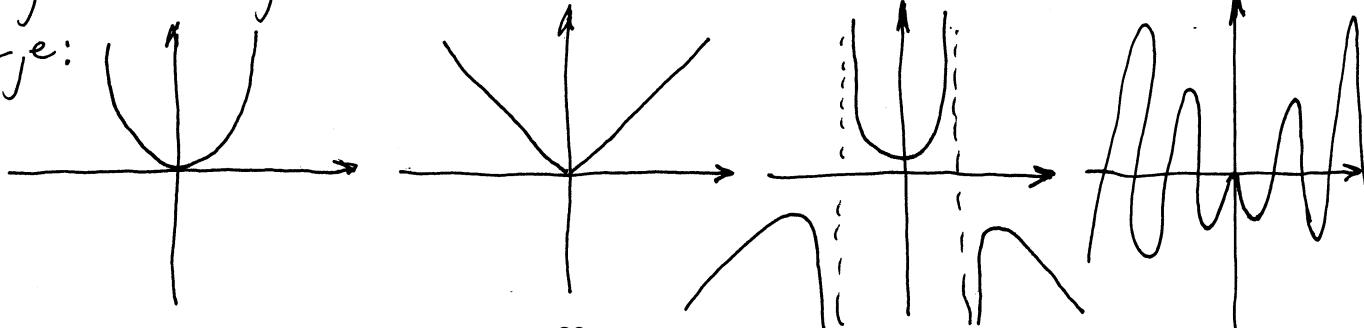
d) $y = \frac{1}{\sqrt{x}}$, $D: x \in \mathbb{R}^+$, $D: x \in (0, +\infty)$, $D: x > 0$

e) $y = \frac{\log x}{x-2}$, $x > 0$, $x \neq 2$ $D: x \in \mathbb{R}^+ \setminus \{2\}$, $D: x \in (0, 2) \cup (2, +\infty)$

F-ja je parna ako je $\forall x \in D$ $f(-x) = f(x)$

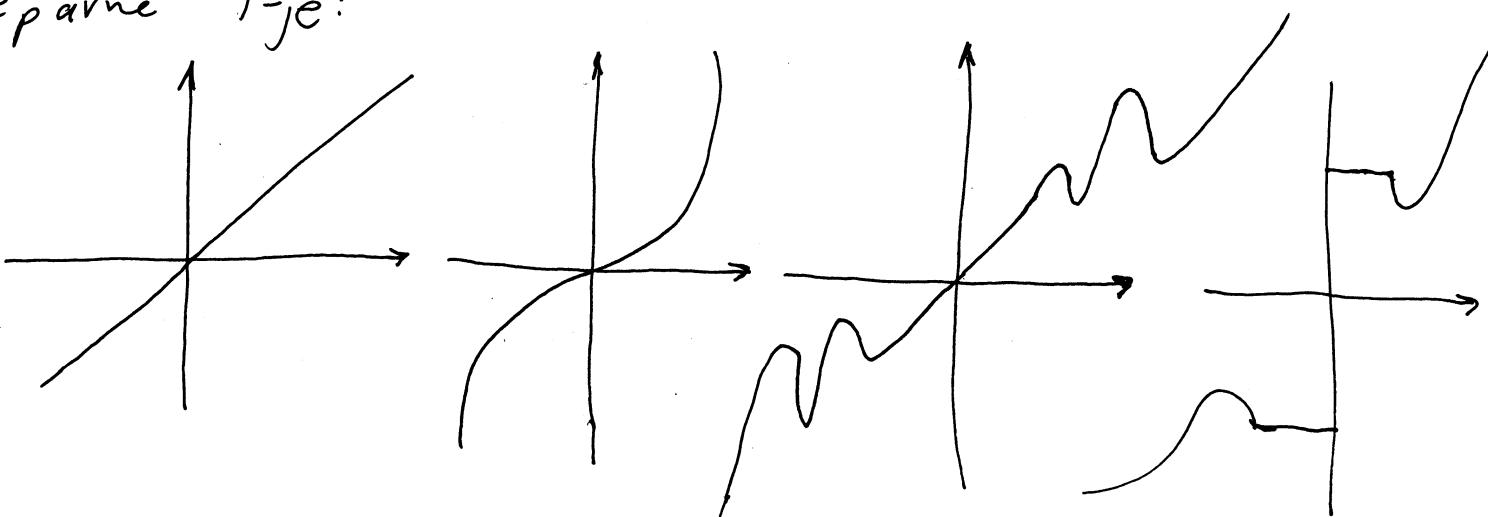
Grafik parne f-je je simetričan u odnosu na y -osu; f-ju je dovoljno ispitati za $x \geq 0$.

Parne f-je:



Ako $\forall x \in D$, $f(-x) = -f(x)$ f-ja $f(x)$ je neparna f-ja.
 Grafik neparne f-je je simetričan u odnosu na
 koordinatni početak $(0,0)$, pa je f-ju dovoljno ispitati
 za $x \geq 0$.

Neparne f-je:



Parnost i neparnost ima slična ispitati samo ako je D simetrično.

Periodične su samo trigonometričke f-je.

(npr. $y = \cos x$, $y = \sin(\frac{1}{x^2})$, $y = \operatorname{tg}(x^2 \log x)$...)

① Ispitati i grafički predstaviti f-je:

$$a) y = \frac{x}{x-3} \quad b) y = \frac{x^2}{x+2} \quad c) y = \frac{x^2-9x+18}{x-2}$$

$$a) y = \frac{x}{x-3}$$

Rj. 1° definiciono područje

$$\frac{x}{x-3} \neq 0 \quad D: x \in \mathbb{R} \setminus \{3\}, \quad D: x \in (-\infty, 3) \cup (3, +\infty)$$

2° parnost (neparost): periodičnost

D nije simetrično \Rightarrow f-ja nije ni parna ni neparna
 f-ja nije simetrična

3° nule, presek grafa sa y -osom, znak f-je

$$y = \frac{x}{x-3}$$

$$y=0 \text{ ažda je } x=0$$

$(0,0)$ je nula f-je
 i presek sa y -osom

tačka oblike $(A, 0)$ je nula f-je, a tačka oblike $(0, B)$ je tačka preseka grafa sa Y-osiom

$$y = \frac{x}{x-3}$$

$$\begin{array}{l} x=0 \\ x-3=0 \\ x=0 \\ x=3 \end{array}$$

x	$(-\infty, 0)$	$(0, 3)$	$(3, +\infty)$
x	-	+	+
$x-3$	-	-	+
y	+	-	+

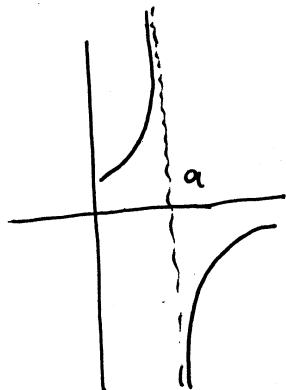
4° ponarjanje na krajevima intervala definisaneosti i asimptote znak f-je

a je tačka u kojoj f-ja nije definisana

$\lim_{x \rightarrow a^-} f(x) = -\infty$ (ili $+\infty$) $\Rightarrow x=a$ vertikalna asimptota

$\lim_{x \rightarrow a^+} f(x) = +\infty$ (ili $-\infty$) $\Rightarrow x=a$ vertikalna asimptota

npr.



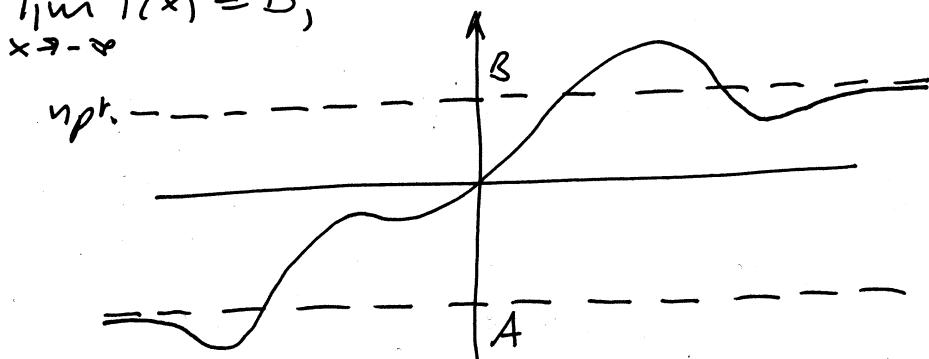
$$\lim_{x \rightarrow a^-} f(x) = +\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

$\Rightarrow x=a$ je vertikalna asimptota

$\lim_{x \rightarrow \infty} f(x) = A$, $A \neq +\infty$; $A \neq -\infty \Rightarrow y=A$ je horizontalna asimptota

$\lim_{x \rightarrow -\infty} f(x) = B$, $B \neq +\infty$; $B \neq -\infty \Rightarrow y=B$ je H.o.A.



$$\lim_{x \rightarrow \infty} f(x) = B \Rightarrow y=B \text{ je H.o.B.}$$

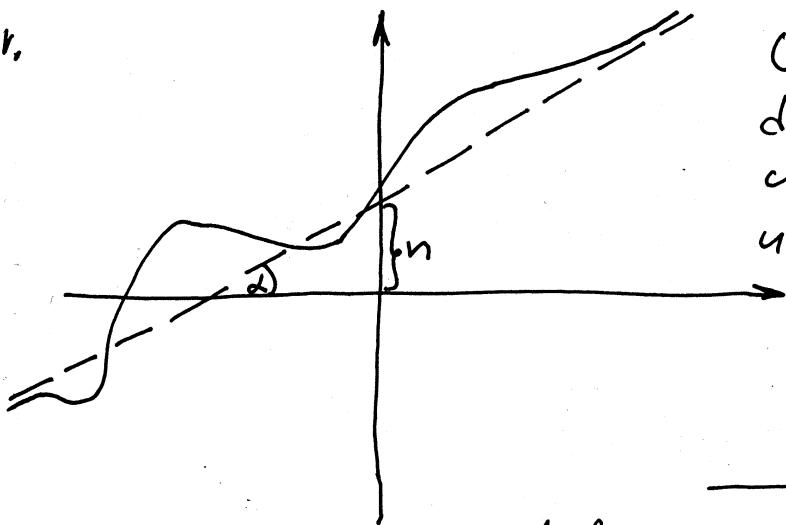
$$\lim_{x \rightarrow -\infty} f(x) = A \Rightarrow y=A \text{ je H.o.A.}$$

Ako f-ja nema horizontalnu asimptotu onda tražimo kosu asimptotu u obliku $y=kx+n$.

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}, \quad n = \lim_{x \rightarrow \infty} [f(x) - kx]$$

Ako je $k=\pm\infty$ ili $k=0$ f-ja nema kosu asimptotu

npr.



U beskonačnosti f , je ne dodiruje asymptotu ali je u "normalnom položaju" u nekoj tački može sijeci.

Za $x=3$ f -ja nije definisana.

$$\lim_{x \rightarrow 3^-} \frac{x}{x-3} = \frac{3-0}{3-0-3} = \frac{3-0}{-0} = -\infty \Rightarrow x=3 \text{ je } V_o A.$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{x}{x-3} = \frac{3+0}{3+0-3} = \frac{3+0}{+0} = +\infty \Rightarrow x=3 \text{ je } V_o A.$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{x}{x-3} : x = \lim_{x \rightarrow +\infty} \frac{1}{1-\frac{3}{x}} = 1 \Rightarrow y=1 \text{ je } H_o A.$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x}{x-3} : x = \lim_{x \rightarrow -\infty} \frac{1}{1-\frac{3}{x}} = 1 \Rightarrow y=1 \text{ je } H_o A.$$

Pošlije 4 koraka počinjem sa crtanjem grafika f , e.

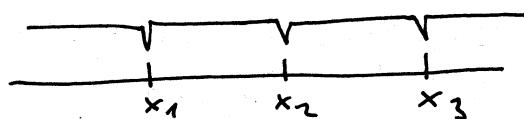
5° intervali rastu i opadaju

$\forall x \in (a, b) \quad y'(x) < 0 \Rightarrow f$ -ja y na (a, b)

$\forall x \in (a, b) \quad y'(x) > 0 \Rightarrow f$ -ja y raste na intervalu (a, b)

Intervalne formirane pomoću prekida f , e y ;

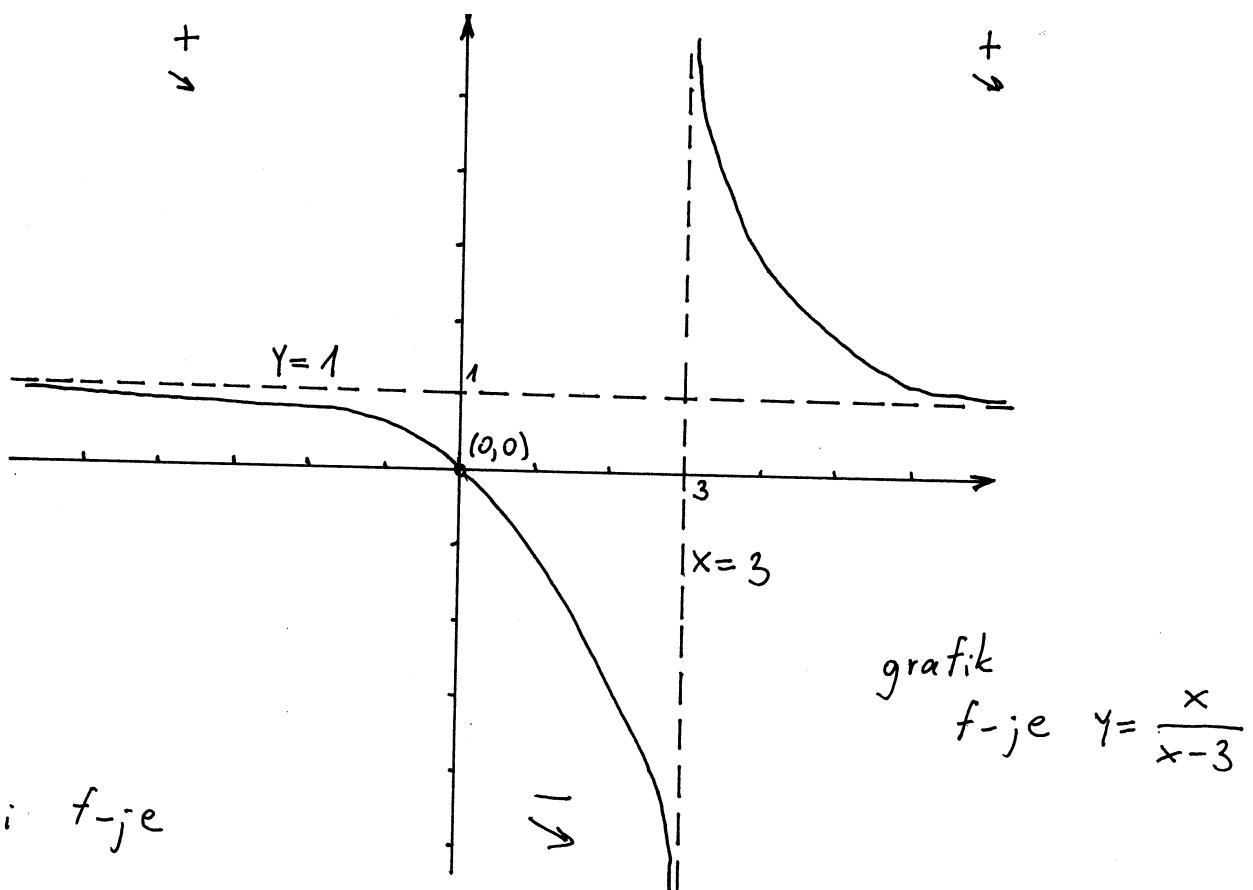
nula f , e y'



← prekidi y
+ nula y'

$$y' = \left(\frac{x}{x-3} \right)' = \frac{1(x-3) - x \cdot 1}{(x-3)^2} = \frac{-3}{(x-3)^2} < 0 \quad \forall x \in \mathcal{D}$$

f , e $y \downarrow$ za $\forall (x \in \mathcal{D})$



6° ekstremi: f-je

za $y' = 0$ dobijemo stacionarne tačke x_1, x_2, \dots koje konkuruju za ekstrem. Stacionarne tačke x_1, x_2, \dots mogu (ali i ne moraju) da budu tačke u kojima f-je poprima ekstrem. To zaključujemo na osnovu drugog izvoda.

Ako je $y''(x_1) < 0 \Rightarrow (x_1, f(x_1))$ tačka u kojoj f-je y ima maksimalnu vrijednost

Ako je $y''(x_1) > 0 \Rightarrow (x_1, f(x_1))$ tačka u kojoj f-je y ima minimalnu vrijednost

* Ekstreme f-je možemo odrediti i na drugi način: na osnovu tabele rasta i opadanja koju dobijemo u 5 koraka.

$$y' = \frac{-3}{(x-3)^2} \neq 0 \quad \forall x \in \mathbb{D} \Rightarrow f\text{-je nema ekstrema.}$$

7° prevojne tačke; intervali konveksnosti i konkavnosti

Konveksnost (\cup) i konkavnost (\cap) f-je određujemo na osnovu znaka $f\text{-je } y''$.

Ako je $\forall x \in (a, b) \quad y''(x) < 0 \Rightarrow f\text{-je } y \text{ je } \cap \text{ na } (a, b)$.

Ako je $\forall x \in (a, b) \quad y''(x) > 0 \Rightarrow f\text{-je } y \text{ je } \cup \text{ na } (a, b)$.

Za $y''=0$ dobijemo tačke x_1, x_2, \dots koje konkuruju za prevojne tačke. Tačka x_1 je prevojna tačka ako u njoj f-je y prelazi iz \cup u \cap ; obratno.

$$Y'' = \left(\frac{-3}{(x-3)^2} \right)' = (-3) \cdot (-2) \cdot (x-3)^{-3} = \frac{6}{(x-3)^2} \neq 0 \quad f\text{-ja nema prevojnih tački}$$

u tabelu stavljamo prekide $f\text{-je } Y$ i nule Y''

x	$(-\infty, 3)$	$(3, +\infty)$
Y''	-	+
Y	\cap	\cup

konveksnost
i konkavnost

8° grafik $f\text{-je}$

$$b) \quad Y = \frac{x^2}{x+2}$$

Rj. 1° definicijom područje

$$\begin{array}{l} x+2 \neq 0 \\ x \neq -2 \end{array}$$

$$D: x \in \mathbb{R} \setminus \{-2\}$$

$$D: x \in (-\infty, -2) \cup (-2, +\infty)$$

2° parnost (neparnost) i periodičnost

kako D nije simetrično $f\text{-ja nije ni parna ni neparna}$
nije periodična

3° nule, presjek sa y -osom, znak $f\text{-je}$

$$x+2 > 0$$

$$Y=0 \text{ akko } x^2=0 \text{ tj. } x=0$$

$$x^2 > 0 \quad \forall x \in D$$

$$\exists x > -2$$

$(0, 0)$ je nula $f\text{-je i}$
presjek sa y -osom

$$x+2 = 0 \quad \exists x = -2$$

x	$(-\infty, -2)$	$(-2, +\infty)$
Y	-	+

znak
 $f\text{-je}$

4° ponajanje na krajevinama intervala definisavanosti i asymptote

$x=-2$ je tačka u kojoj $f\text{-ja nije definisana}$

$$\lim_{x \rightarrow -2^-} \frac{x^2}{x+2} = \frac{(-2-0)^2}{-2-0+2} = \frac{(-2-0)^2}{-0} = -\infty \Rightarrow x=-2 \text{ je } V_o A_o$$

$$\lim_{x \rightarrow -2^+} \frac{x^2}{x+2} = \frac{(-2+0)^2}{-2+0+2} = \frac{(-2+0)^2}{+0} = +\infty \Rightarrow x=-2 \text{ je } V_o A_o$$

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x+2} : x^2 = \lim_{x \rightarrow -\infty} \frac{1}{\frac{1}{x} + \frac{2}{x^2}} = -\infty \Rightarrow \text{kad } x \rightarrow -\infty \text{ nema } H_o A_o$$

$$\lim_{x \rightarrow +\infty} \frac{x^2}{x+2} = \lim_{x \rightarrow +\infty} \frac{1}{\frac{1}{x} + \frac{2}{x^2}} = +\infty \Rightarrow f\text{-ja nema } H_o A_o$$

Tražimo koju asymptotu u obliku $Y=kx+n$

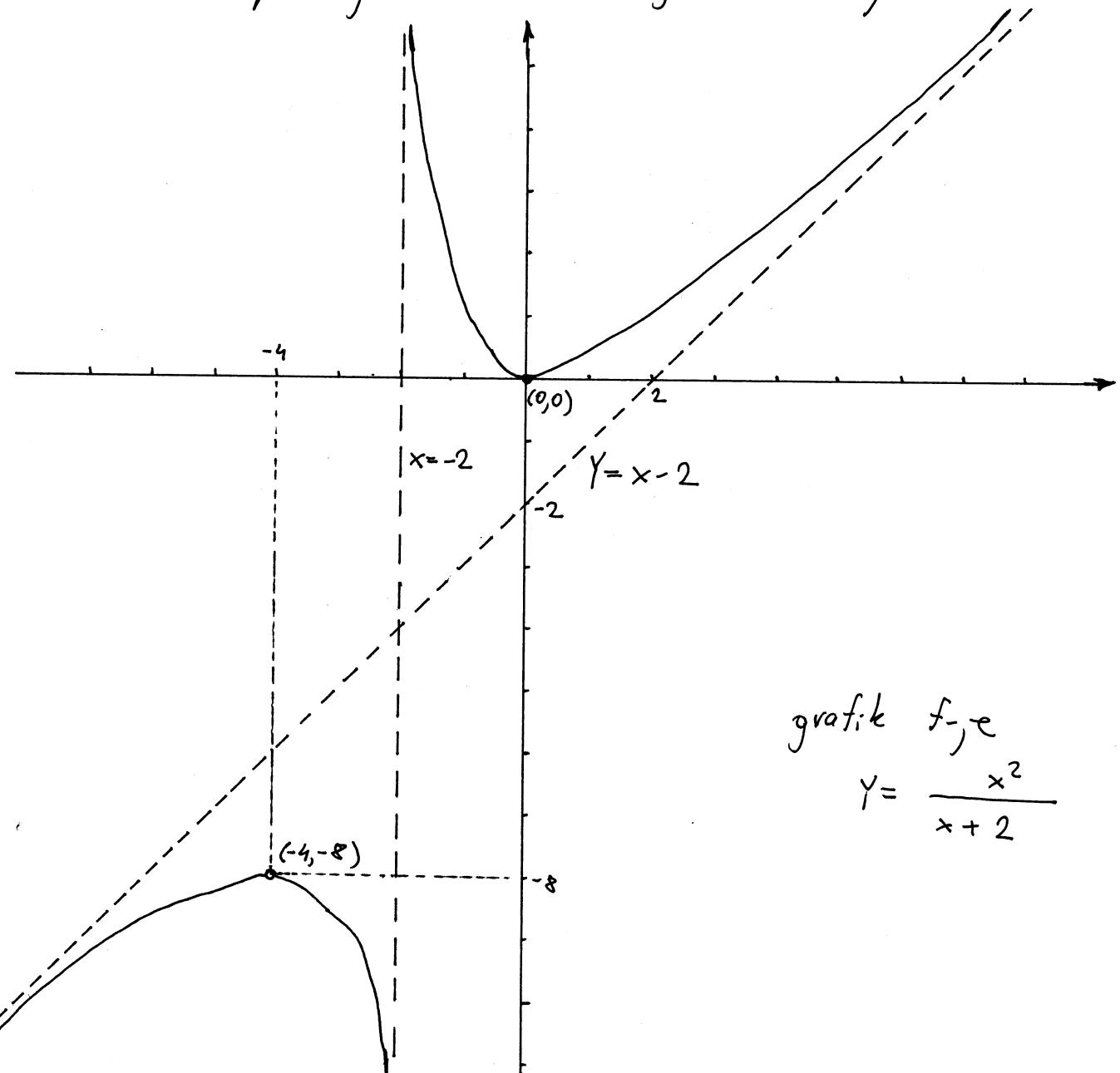
$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x^2}{x^2+2x} : x^2 = \lim_{x \rightarrow \infty} \frac{1}{1+\frac{2}{x}} = 1 \Rightarrow k=1$$

$$n = \lim_{x \rightarrow \infty} [f(x) - k \cdot x] = \lim_{x \rightarrow \infty} \left(\frac{x^2}{x+2} - x \right) = \lim_{x \rightarrow \infty} \frac{x^2-x^2-2x}{x+2} = \lim_{x \rightarrow \infty} \frac{-2x}{x+2} = -2$$

$Y=x-2$ je $K_o A_o$

NASTAVAK ZADATKA NA SLJEDEĆIM VJEŽBAMA

poslije 4 koraka počinjeno crtati grafik f -je



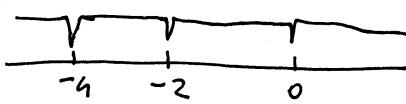
5° intervali rasta i opadanja f -je

$$y' = \left(\frac{x^2}{x+2} \right)' = \frac{2x(x+2) - x^2}{(x+2)^2} = \frac{2x^2 + 4x - x^2}{(x+2)^2} = \frac{x^2 + 4x}{(x+2)^2}$$

$$y' = 0 \text{ ažd.} \\ x^2 + 4x = 0 \quad t.j. \quad x(x+4) = 0$$

prekidi f -je

+ nule f -je y'



x	$(-\infty, -4)$	$(-4, -2)$	$(-2, 0)$	$(0, +\infty)$
y'	+	-	-	+
y	\nearrow	\searrow	\searrow	\nearrow

rast; opadanje

6° ekstremi f -je

$y' = 0 \Rightarrow$ stacionarne tačke f -je su $x_1 = -4$ i $x_2 = 0$.

Na osnovu tabele u 5° f -ja u tački $-4 < 0 \rightarrow$ a u tački $-4 + 0 \rightarrow \Rightarrow f_{\max}(-4) = \frac{16}{-2} = -8$, t.j. $(-4, -8)$ je tačka maksimuma f -je

Na osnovu tabele u 5° f-ja u tački $0 \rightarrow$ a u $0+1 \Rightarrow f_{\min}(0)=0$ tj. $(0,0)$ je tačka minimum f-je.

7° prevojuje tačke i intervali konveksnosti i konkavnosti.

$$Y'' = \left(\frac{x^2+4x}{(x+2)^2} \right)' = \frac{(2x+4)(x+2)^2 - (x^2+4x) \cdot 2(x+2)}{(x+2)^4} = \frac{2x^2+4x+4x+8 - 2(x^2+4x)}{(x+2)^3}$$

$$Y'' = \frac{2x^2+8x+8-2x^2-8x}{(x+2)^3} = \frac{8}{(x+2)^3}$$

$Y'' \neq 0 \Rightarrow f$ -je nema $f(x \in D)$ prevojnih tački.

prekid f-je
+ nule f-je Y''



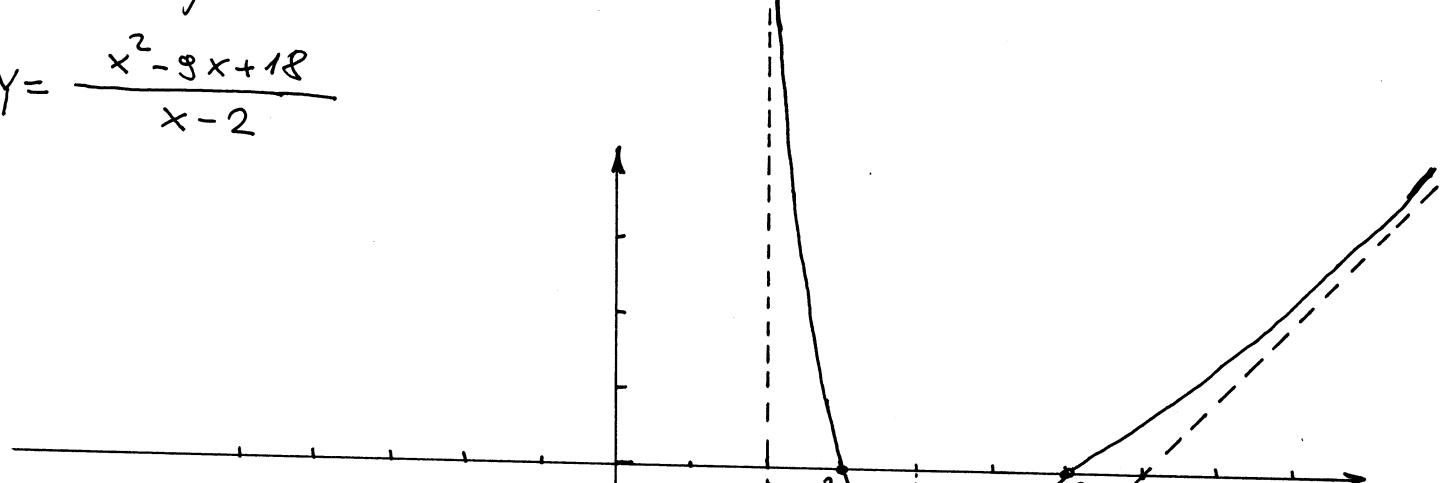
x	$(-\infty, -2)$	$(-2, +\infty)$
Y''	-	+
Y	\(\sim\)	\(\cup\)

konveksnost
i konkavnost

8° grafik f-je

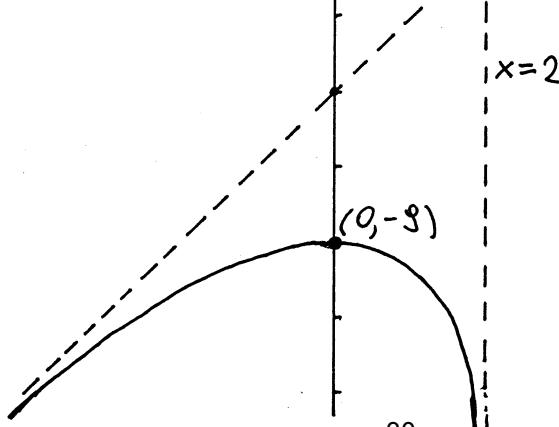
c) $y = \frac{x^2-9x+18}{x-2}$

R.j.



$$Y' = 1 - \frac{4}{(x-2)^2}$$

$$Y'' = \frac{8}{(x-2)^3}$$



grafik f-je

$$y = \frac{x^2-9x+18}{x-2}$$

(2.) Ispitati i grafički predstaviti f-je:

$$a) y = \frac{x^3}{x^2 - 4}$$

$$b) y = \frac{x^2 + 1}{\sqrt{x^2 - 1}}$$

$$c) y = \frac{(x+1)^3}{(x-1)^2}$$

$$d) y = \frac{x^3}{x^2 - 4}$$

Rj. 1° definicijom područje $\begin{matrix} x^2 - 4 \neq 0 \\ (x-2)(x+2) \neq 0 \\ x \neq 2 \quad x \neq -2 \end{matrix}$ $D: x \in (-\infty, -2) \cup (2, +\infty)$

2° parnost (neparnost): periodičnost

$$\text{D simetrično, } f(-x) = \frac{(-x)^3}{(-x)^2 - 4} = \frac{-x^3}{x^2 - 4} = -\frac{x^3}{x^2 - 4} = -f(x)$$

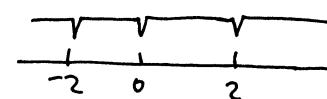
f-ja je neparna, nije periodična

3° nule, presek grata sa y-osi, znak f-je

$$y=0 \text{ ažd } x^3=0 \text{ tj. } x=0$$

$(0, 0)$ je nula f-je
i presek sa y-osi

nule f-je
+ prekidi f-je



x	$(0, 2)$	$(2, +\infty)$
x^3	+	+
$x^2 - 4$	-	+
Y	-	+

znak f-je

Kako je f-ja neparna dovoljno je ispitati za $x > 0$.

4° poniranje na krajevima intervala definisivosti; asymptote u tačkama $x=-2$ i $x=2$ f-ja ima prekid

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x^3}{x^2 - 4} = \frac{(2-0)^3}{(2-0)^2 - 4} = \frac{(2-0)^3}{4-0-4} = \frac{(2-0)^3}{-0} = -\infty$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{x^3}{x^2 - 4} = \frac{(2+0)^3}{(2+0)^2 - 4} = \frac{(2+0)^3}{+0} = +\infty \Rightarrow x=2 \text{ je V}_0 A.$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{x^3}{x^2 - 4} : x^3 = \lim_{x \rightarrow +\infty} \frac{1}{\frac{1}{x^2} - \frac{4}{x^3}} = +\infty, \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

f-ja nemao horizontalnu asymptotu

$$y = kx + b$$

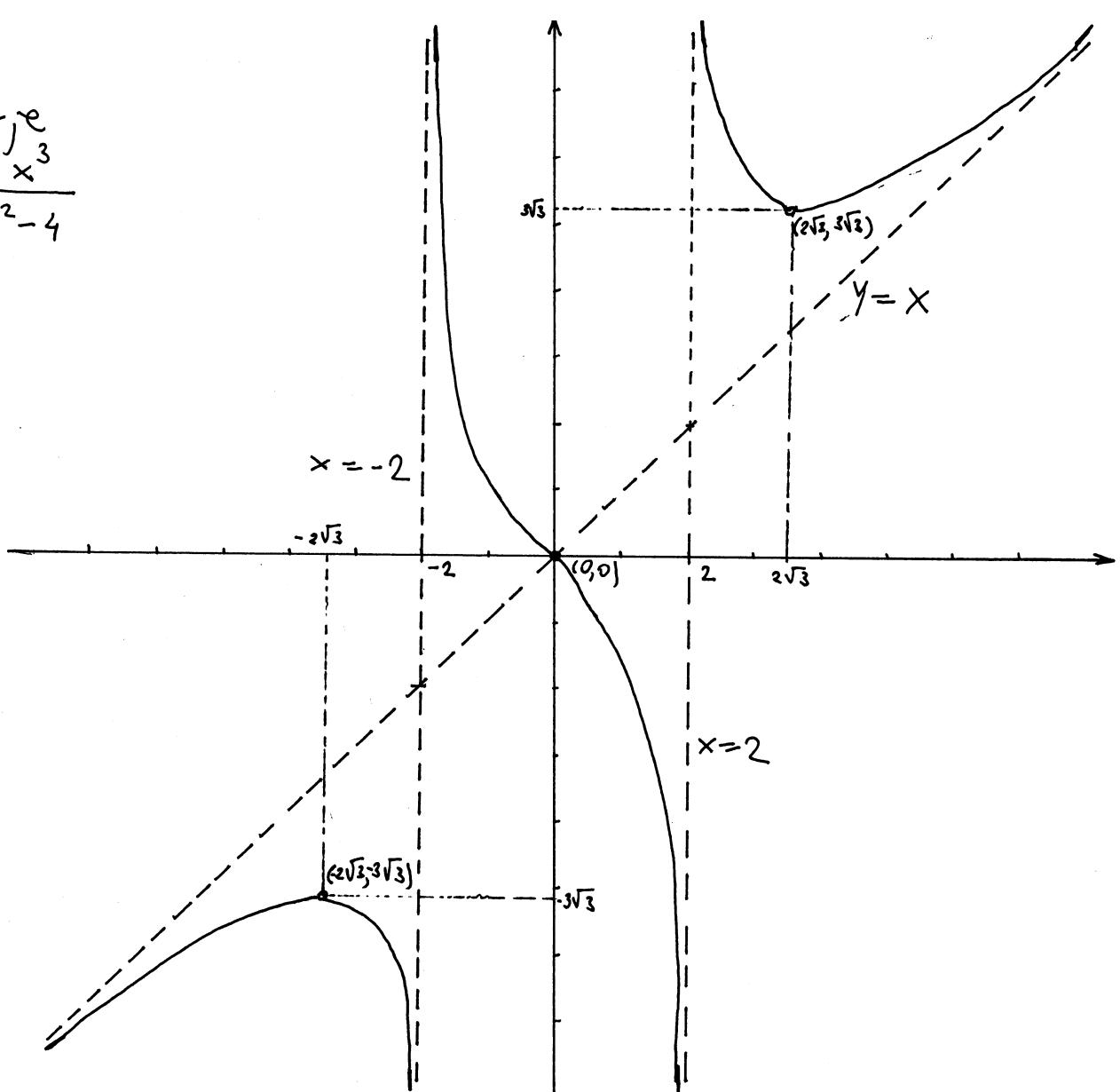
$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{x^3}{x^2 - 4} : x^3}{x^3 : x^3} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{4}{x^2}} = 1 \Rightarrow k=1$$

$$b = \lim_{x \rightarrow \infty} [f(x) - kx] = \lim_{x \rightarrow \infty} \left[\frac{x^3}{x^2 - 4} - x \right] = \lim_{x \rightarrow \infty} \frac{x^3 - x^2 + 4x}{x^2 - 4} = \lim_{x \rightarrow \infty} \frac{4x}{x^2 - 4} = 0$$

$y = x$ je kosa asymptota. Počinjem sa crtanjem grafa

grafik $f_j e$

$$y = \frac{x^3}{x^2 - 4}$$



5° intervali rasta i opadanja $f_j e$

$$y' = \left(\frac{x^3}{x^2 - 4} \right)' = \frac{3x^2(x^2 - 4) - x^3 \cdot 2x}{(x^2 - 4)^2} = \frac{3x^4 - 12x^2 - 2x^4}{(x^2 - 4)^2} = \frac{x^4 - 12x^2}{(x^2 - 4)^2}$$

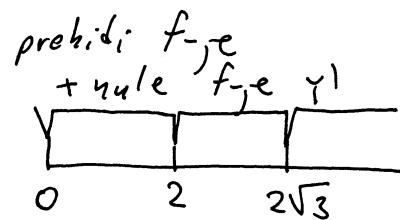
$$y' = \frac{x^2(x^2 - 12)}{(x^2 - 4)^2}$$

$$y' = 0$$

$$\text{akko } x_1 = 0$$

$$x_2 = -\sqrt{12}$$

$$x_3 = \sqrt{12} = 2\sqrt{3}$$



x	(0, 2)	(2, 2\sqrt{3})	(2\sqrt{3}, +\infty)
y'	-	-	+
y	↓	↓	↗

6° ekstremini $f_j e$

$y' = 0 \Rightarrow$ stacionarne točke $f_j e$ sa $x_1 = -2\sqrt{3}, x_2 = 0;$

Na osnovu tabele u 5° možemo zaključiti $x_3 = 2\sqrt{3}$

$y'(0-) < 0 \wedge y'(0+) < 0 \Rightarrow$ za $x=0$ $f_j e$ nema ekstrem

$y'(2\sqrt{3}-0) < 0 \wedge y'(2\sqrt{3}+0) > 0 \Rightarrow f_{\min}(2\sqrt{3}) = 3\sqrt{3}$ tj. $(2\sqrt{3}, 3\sqrt{3})$ je točka minimuma

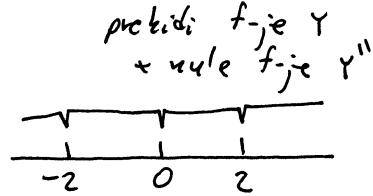
7° prevojne tačke i intervali konveknosti i konkavnosti

$$y'' = \left(\frac{x^4 - 12x^2}{(x^2 - 4)^2} \right)' = \frac{(4x^3 - 24x)(x^2 - 4)^2 - (x^4 - 12x^2) \cdot 2(x^2 - 4) \cdot 2x}{(x^2 - 4)^3} = \frac{4x^5 - 24x^3 - 16x^3 + 96x^5}{(x^2 - 4)^3}$$

$$y'' = \frac{8x^3 + 96x}{(x^2 - 4)^3} = \frac{8x(x^2 + 12)}{(x^2 - 4)^3}$$

$y'' = 0 \Rightarrow x = 0$

kandidat za prevojnu tačku



x	(-\infty, -2)	(-2, 0)	(0, 2)	(2, +\infty)
Y''	-	+	-	+
Y	\nearrow	\downarrow	\nearrow	\downarrow

f(0) = 0 $\Rightarrow (0, 0)$ je prevojna tačka P.T.

8° grafik f-je

b) $y = \frac{x^2 + 1}{\sqrt{x^2 - 1}}$

Napomena: (zadatak nije detaljno raspisan)

Rj: D: $x \in (-\infty, -1) \cup (1, +\infty)$

f-ja je parna (nije periodična)

$\lim_{x \rightarrow 1+0} \frac{x^2 + 1}{\sqrt{x^2 - 1}} = +\infty \Rightarrow x=1 \text{ je ver. asimptota}$

$\lim_{x \rightarrow \pm\infty} \frac{x^2 + 1}{\sqrt{x^2 - 1}} = +\infty \Rightarrow f-ja \text{ never horizontalna asimptota}$

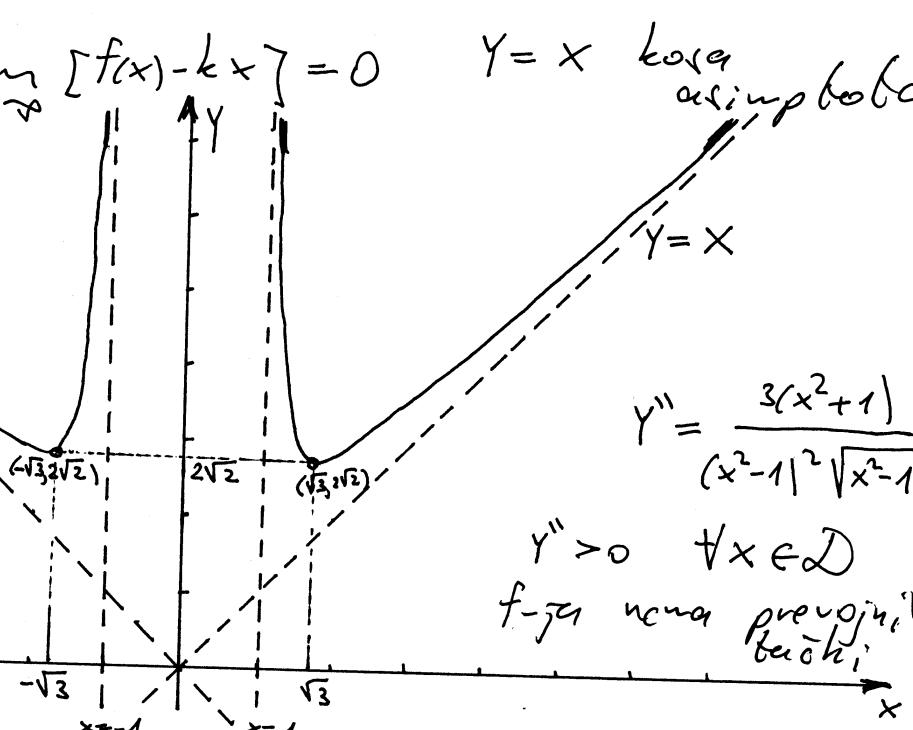
$k - \lim_{x \rightarrow \infty} \frac{f(x)}{x} = 1, \quad n = \lim_{x \rightarrow \infty} [f(x) - kx] = 0 \quad Y = x \text{ kora asimptote}$

$y' = \frac{x(x^2 - 3)}{(x^2 - 1)\sqrt{x^2 - 1}}$

$y' = 0 \text{ akko}$

$x_1 = -\sqrt{3}, \quad x_2 = 0, \quad x_3 = \sqrt{3}$

$y'' = \frac{3(x^2 + 1)}{(x^2 - 1)^2 \sqrt{x^2 - 1}}$



x	(1, \sqrt{3})	(\sqrt{3}, +\infty)
y'	-	+
Y	\searrow	\nearrow

$\Rightarrow (\sqrt{3}, 2\sqrt{2})$
je tačka
minimum
f-je

x	(1, +\infty)
y''	+
Y	\downarrow

konveknost
i konkavnost

$$c) \gamma = \frac{(x+1)^3}{(x-1)^2}$$

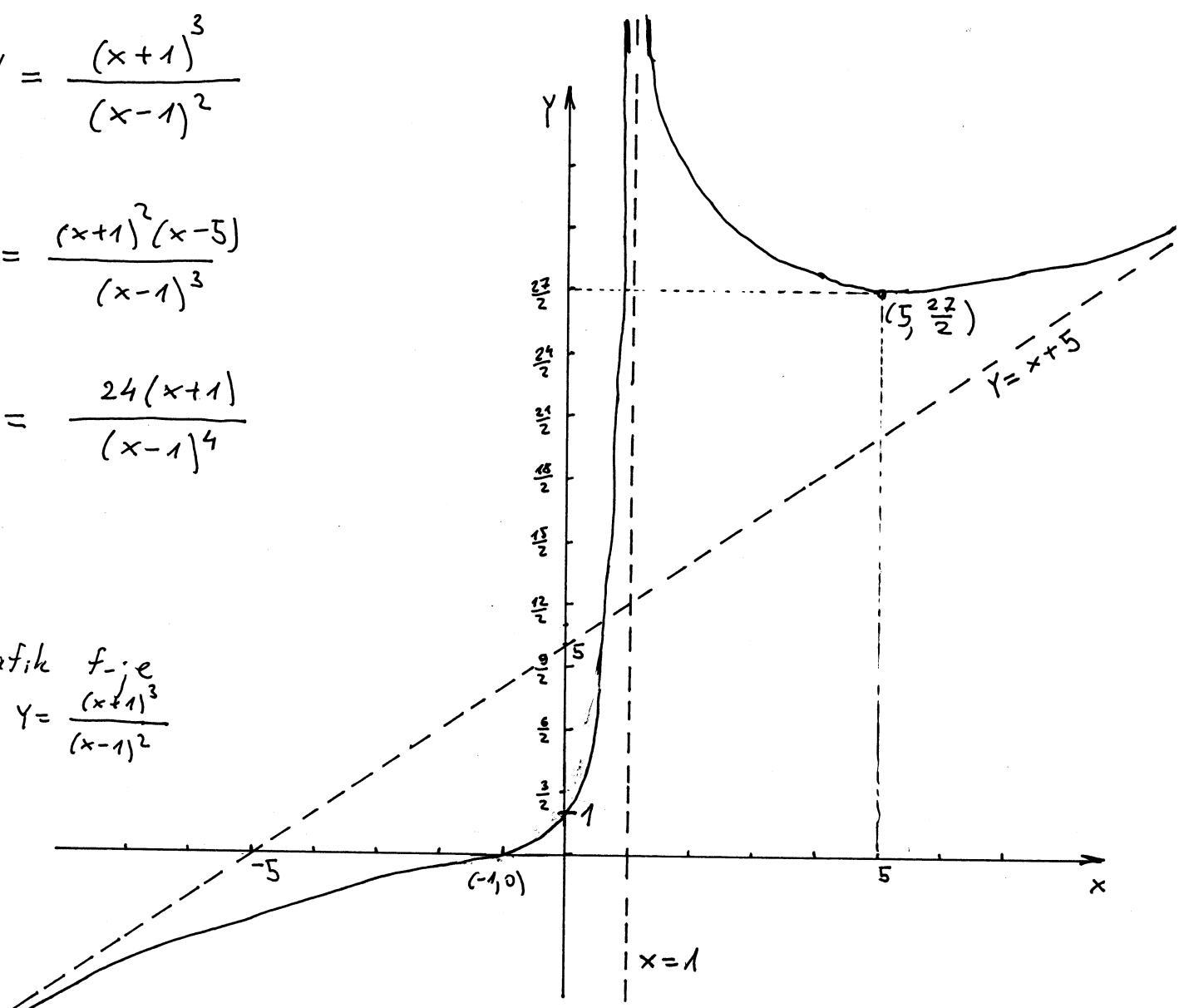
Rj.

$$\gamma' = \frac{(x+1)^2(x-5)}{(x-1)^3}$$

$$\gamma'' = \frac{24(x+1)}{(x-1)^4}$$

grafik $f_j e$

$$\gamma = \frac{(x+1)^3}{(x-1)^2}$$



Eksponencijalne f-je

1. Ispitati i grafički predstaviti f-je:

$$a) \gamma = (x-1) e^{\frac{1}{x-3}} \quad b) \gamma = \frac{e^{x-2}}{x-1} \quad c) \gamma = \frac{x^2+x+1}{e^x}$$

$$a) \gamma = (x-1) e^{\frac{1}{x-3}}$$

Rj.
1° definiciono područje $D: x \in (-\infty, 3) \cup (3, +\infty)$
 $x-3 \neq 0 \Rightarrow x \neq 3$

2° parnost (neparnost), periodičnost

D nije simetrično $\Rightarrow f_j e$ nije ni parna ni neparna
 $f_j e$ nije periodična

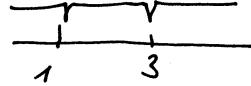
3° nule, presjek grada sa Y-oxom, znak f-je

$$Y=0 \Leftrightarrow x=1 \quad (1, 0) \text{ je nula } f_j e$$

$$f(0) = (0-1) e^{\frac{1}{0-3}} = -e^{-\frac{1}{3}} = -\frac{1}{\sqrt[3]{e}}$$

prekida f-je
+ nula f-je

$(0, -\frac{1}{\sqrt[3]{e}})$ je presek
grafa sa
 $y = 0$ ozn.



X	$(-\infty, 1)$	$(1, 2)$	$(2, +\infty)$
Y	-	+	+

4° ponavlja se krajevina intervala definisanih i asimptote

$x=3$ je tačka prekida f-je

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x-1) e^{\frac{1}{x-3}} = (3-0-1) e^{\frac{1}{3-0-3}} = (2-0) e^{\infty} = (2-0) \cdot \frac{1}{e^{\infty}} = 0$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x-1) e^{\frac{1}{x-3}} = (3+0-1) e^{\frac{1}{3+0-3}} = (2+0) e^{+\infty} = \infty \Rightarrow \\ \Rightarrow x=3 \text{ je V.A.}$$

$$\lim_{x \rightarrow \infty} (x-1) e^{\frac{1}{x-3}} = \infty \cdot e^0 = \infty \cdot 1 = \infty$$

$$\lim_{x \rightarrow -\infty} (x-1) e^{\frac{1}{x-3}} = -\infty \cdot 1 = -\infty \Rightarrow f-je \text{ nema horizontale asimptote}$$

$y = kx + n$ kosa asimptota

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x-1}{x} e^{\frac{1}{x-3}} = \lim_{x \rightarrow \infty} \left(1 - \left(\frac{1}{x}\right)\right) e^{\frac{1}{x-3}} = 1 \cdot e^0 = 1 \text{ t.j. } k=1$$

$$n = \lim_{x \rightarrow \infty} [f(x) - kx] = \lim_{x \rightarrow \infty} \left[(x-1) e^{\frac{1}{x-3}} - x \right] = \lim_{x \rightarrow \infty} \left[x e^{\frac{1}{x-3}} - e^{\frac{1}{x-3}} - x \right] \\ = \lim_{x \rightarrow \infty} \left[x \left(e^{\frac{1}{x-3}} - 1 \right) - e^{\frac{1}{x-3}} \right] = \lim_{x \rightarrow \infty} \frac{e^{\frac{1}{x-3}} - 1}{\frac{1}{x}} - \underbrace{\lim_{x \rightarrow \infty} e^{\frac{1}{x-3}}}_{\stackrel{\text{L'H.}}{=}}$$

$$\begin{aligned} (x^{-1})' &= -\frac{1}{x^2} & (x^{-3})' &= \frac{(x^{-3})'}{x} \\ \left(e^{\frac{1}{x-3}}\right)' &= e^{\frac{1}{x-3}} \cdot \left(\frac{1}{x-3}\right)' & & \\ &= e^{\frac{1}{x-3}} \cdot (-1)(x-3)^{-2} & & \end{aligned}$$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{-\frac{e^{\frac{1}{x-3}}}{(x-3)^2}}{-\frac{1}{x^2}} - 1 = \\ &= \lim_{x \rightarrow \infty} \frac{x^2 \cdot e^{\frac{1}{x-3}}}{x^2 - 6x + 9} : x^2 - 1 = \end{aligned}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{6}{x} + \frac{9}{x^2}} \cdot e^{\frac{1}{x-3}} - 1 = 1 - 1 = 0$$

$y=x$ je kosa asimptota

počinjemo sa crtanjem graf-a f -je

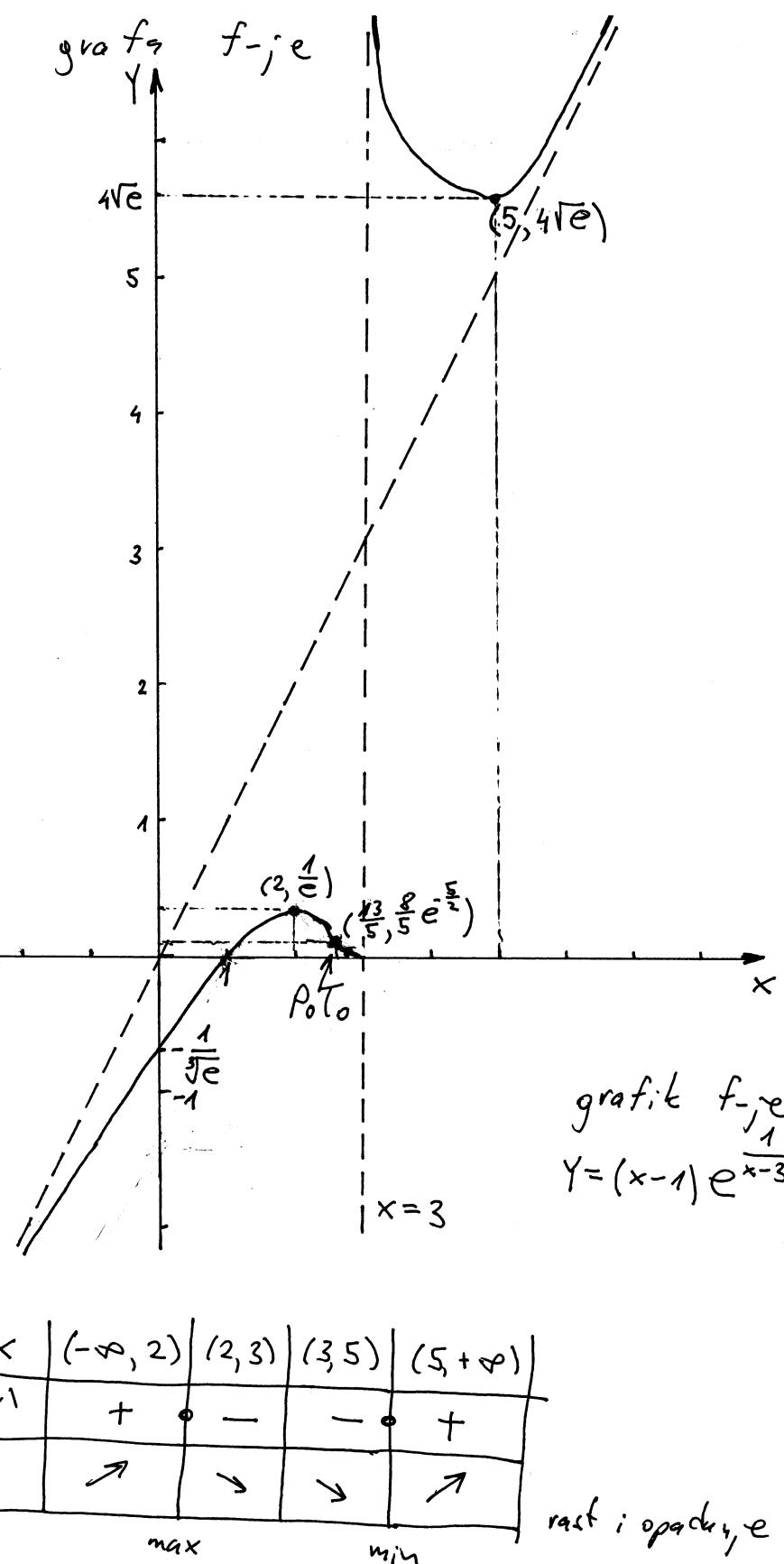
5° intervali rasta i opadanja

$$\begin{aligned} y' &= \left((x-1) e^{\frac{1}{x-3}} \right)' = \\ &= e^{\frac{1}{x-3}} + (x-1) e^{\frac{1}{x-3}} \cdot \frac{-1}{(x-3)^2} = \\ &= e^{\frac{1}{x-3}} \left[1 - \frac{x-1}{(x-3)^2} \right] = \\ &= e^{\frac{1}{x-3}} \cdot \frac{x^2 - 6x + 9 - x + 1}{(x-3)^2} \\ &= e^{\frac{1}{x-3}} \cdot \frac{x^2 - 7x + 10}{(x-3)^2} \end{aligned}$$

$$y' = e^{\frac{1}{x-3}} \cdot \frac{(x-2)(x-5)}{(x-3)^2}$$

prekidi f -e y
+ nule f -e y'

$$\begin{array}{c} \diagdown \quad \diagdown \quad \diagdown \\ \frac{1}{2} \quad 3 \quad 5 \end{array}$$



6° ekstremi f -je

$y' = 0 \Rightarrow$ stacionarne tačke f -je su $x_1 = 2$ i $x_2 = 5$

Na osnovu tabele u 5° možemo zaključiti:

$$y'(2-0) > 0 ; y'(2+0) < 0 \Rightarrow f_{\max}(2) = (2-1) e^{\frac{1}{2-3}} = \frac{1}{e}$$

tj. $(2, \frac{1}{e})$ je tačka maksimuma

$$y'(5-0) < 0 ; y'(5+0) > 0 \Rightarrow f_{\min}(5) = (5-1) e^{\frac{1}{5-3}} = 4\sqrt{e}$$

NASTAVAK NA SLJEDEĆIM VJEŽB. t.; $(5, 4\sqrt{e})$ je tačka minimuma f -je

7° prevojne tačke i intervali konveksnosti i konkavnosti

$$Y'' = \left(e^{\frac{1}{x-3}} \cdot \frac{x^2 - 7x + 10}{(x-3)^2} \right)' = - \frac{e^{\frac{1}{x-3}}}{(x-3)^2} \cdot \frac{x^2 - 7x + 10}{(x-3)^2} + e^{\frac{1}{x-3}} \cdot \frac{(2x-7) \cdot (x-3)}{(x-3)^4} -$$

$$\frac{-(x^2 - 7x + 10) \cdot 2(x-3)}{(x-3)^4} = - \frac{e^{\frac{1}{x-3}} (x^2 - 7x + 10)}{(x-3)^4} + e^{\frac{1}{x-3}} \cdot \frac{(2x^2 - 6x - 2x + 21 - 2x^2 + 14x - 20)}{(x-3)^3}$$

$$= - e^{\frac{1}{x-3}} \left[\frac{x^2 - 7x + 10}{(x-3)^4} - \frac{x+1}{(x-3)^3} \right] = - e^{\frac{1}{x-3}} \frac{x^2 - 7x + 10 - (x^2 - 3x + x - 3)}{(x-3)^4}$$

$$= - e^{\frac{1}{x-3}} \frac{x^2 - 7x + 10 - x^2 + 2x + 3}{(x-3)^4} = - e^{\frac{1}{x-3}} \frac{-5x + 13}{(x-3)^4}$$

$$Y'' = e^{\frac{1}{x-3}} \cdot \frac{-5x + 13}{(x-3)^4} \quad , \quad Y'' = 0 \Leftrightarrow -5x + 13 = 0$$

$$Y'' = 0 \Rightarrow x = \frac{13}{5} \text{ kandidat za prevojnu tačku}$$

prekida: $f_j \in Y$
+ nula $f_j \in Y''$

$\frac{13}{5}$	3

x	$(-\infty, \frac{13}{5})$	$(\frac{13}{5}, 3)$	$(3, +\infty)$
Y''	-	+	+
Y	\cap	\cup	\cup

$P_0 T_0$

konveksnost
i konkavnost

$$Y\left(\frac{13}{5}\right) = \left(\frac{13}{5} - 1\right) e^{\frac{1}{\frac{13}{5}-3}} = \frac{8}{5} e^{-\frac{5}{2}}$$

$\left(\frac{13}{5}, \frac{8}{5} e^{-\frac{5}{2}}\right)$ je prevojna tačka

8° graf $f_j \in$

b) $Y = \frac{e^{x-2}}{x-1}$

Rj. 1° definicijeno područje $D: x \in (-\infty, 1) \cup (1, +\infty)$
 $x \neq 1$ $x \in \mathbb{R} \setminus \{1\}$

2° parnost (neparnost), periodičnost

D nije simetrično \Rightarrow nije ni parna, ni neparna
 f_j nije periodična

3^o nula, presjek grafu sa y -osom, znak f -je

$$y=0 \Rightarrow \frac{e^{x-2}}{x-1} = 0 ? \quad e^{x-2} > 0 \quad \forall x \in \mathbb{D}$$

$y \neq 0 \quad \forall x \in \mathbb{D}$

f -ja nema nulu

$$f(0) = \frac{e^{-2}}{-1} = -\frac{1}{e^2}$$

$(0, -\frac{1}{e^2})$ je tačka

presjeka grafu
sa y -osom

x	$(-\infty, 1)$	$(1, +\infty)$
y	-	+

znak f -je

4^o ponašanje na krajevima intervala definisanosti i asimptote
 $x=1$ je tačka prekida f -je

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{e^{x-2}}{x-1} = \frac{e^{1-0-2}}{1-0-1} = \frac{e^{-1-0}}{-0} = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{e^{x-2}}{x-1} = \frac{e^{1+0-2}}{1+0-1} = \frac{e^{-1+0}}{+0} = +\infty$$

$\Rightarrow x=1$ je vertikalna asimptota

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^{x-2}}{x-1} \left(\frac{\infty}{\infty} \right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{e^{x-2}}{1} = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{e^{x-2}}{x-1} \left(\frac{e^{-\infty}}{-\infty} \right) = \frac{e^{-\infty}}{-\infty} = \frac{1}{-\infty \cdot e^{\infty}} = \frac{1}{-\infty \cdot \infty} = 0$$

$\Rightarrow y=0$ je horizontalna asimptota

F -ja nema kose asimptote

Počinjemo sa crtanjem grafu f -je

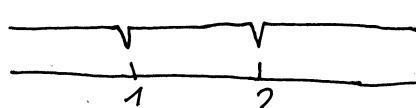
5^o intervali rasta i opadanja

$$y' = \left(\frac{e^{x-2}}{x-1} \right)' = \frac{e^{x-2} \cdot (x-1) - e^{x-2}}{(x-1)^2} = \frac{e^{x-2}(x-1-1)}{(x-1)^2} = \frac{e^{x-2}(x-2)}{(x-1)^2}$$

$$y'=0 \text{ akko } e^{x-2}(x-2)=0 \Rightarrow x-2=0 \Rightarrow x=2$$

prekidi f -je y

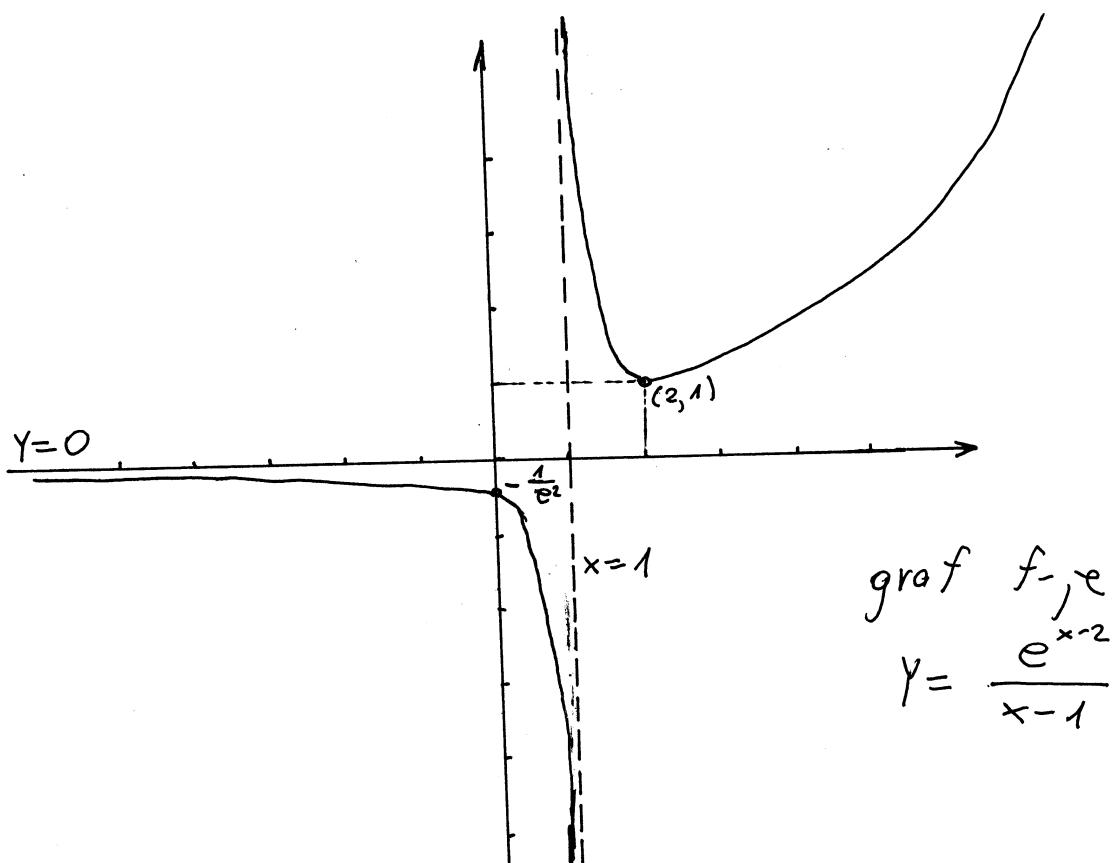
+ nule f -je y'



x	$(-\infty, 1)$	$(1, 2)$	$(2, +\infty)$
y'	-	-	+
y	\searrow	\searrow	\nearrow

\min

raet: opadajće



graf $f_j(e)$
 $y = \frac{e^{x-2}}{x-1}$

6° ekstremi $f_j(e)$

$y' = 0 \Rightarrow$ stacionarne tačke $f_j(e)$ su $x=2$

Na osnovu tabele u 5° možemo zaključiti:

$$y'(2-0) < 0 \quad f_{j,e} \downarrow \quad ; \quad y'(2+0) \quad f_{j,e} \uparrow \quad \Rightarrow \quad f_{\min}(2) = \frac{e^0}{2-1} = 1$$

$(2, 1)$ je tačka minimuma $f_j(e)$

7° prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{e^{x-2}(x-2)}{(x-1)^2} \right)' = \frac{[e^{x-2}(x-2) + e^{x-2}] (x-1)^2 - e^{x-2}(x-2) 2(x-1)}{(x-1)^4}$$

$$y'' = \frac{e^{x-2} [(x-2+1)(x-1) - 2(x-2)]}{(x-1)^3} = \frac{e^{x-2} (x^2 - 2x + 1 - 2x + 4)}{(x-1)^3}$$

$$y'' = \frac{e^{x-2} (x^2 - 4x + 5)}{(x-1)^3}, \quad y'' = 0 \text{ a k t o } x^2 - 4x + 5 = 0$$

$$D = 16 - 20 < 0, \quad a > 0 \Rightarrow x^2 - 4x + 5 > 0 \quad \forall x$$

$y'' \neq 0 \quad \forall x \in D$, $f_j(e)$ nema prevojnih tački

prekidi f'_j e y
+ nule f''_j e y''

x	(-\infty, 1)	(1, +\infty)
y''	-	+
y'	\wedge	\vee

8° grafik f'_j e

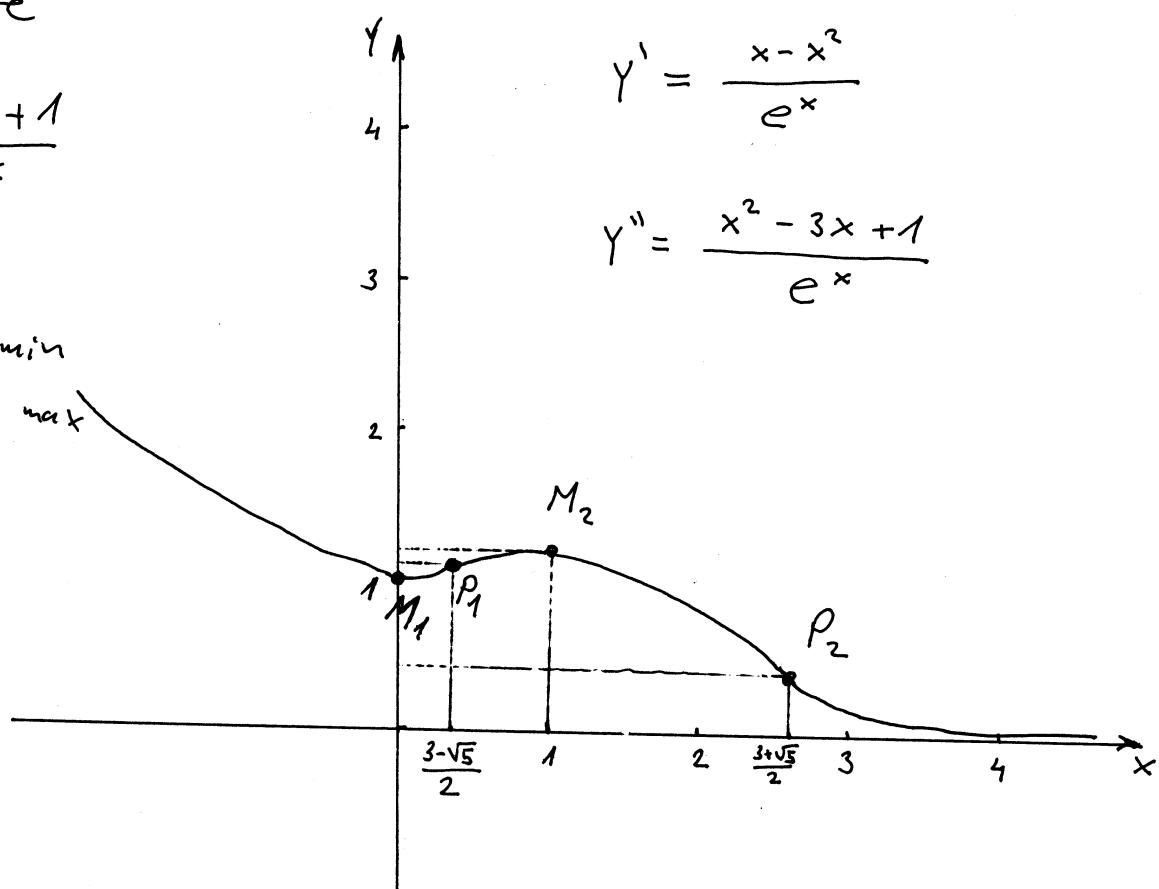
c) $y = \frac{x^2+x+1}{e^x}$

Rj.

$M_1(0, 1)$ tačka min

$M_2(1, \frac{3}{e})$ tačka max

P_1, P_2 su
prevojne
tačke



Logaritamske f-j-e

1) Ispitati i grafički predstaviti f'_j e

a) $y = \frac{\ln^2 x}{x}$ b) $y = \ln(x^2 - 4x + 5)$ c) $y = \frac{x^2}{\ln x - 2}$

a) $y = \frac{\ln^2 x}{x}$

Rj. 1° definicijeno područje

$x \neq 0$ $D: x \in \mathbb{R}^+$ $D: x \in (0, +\infty)$

2° parnost/neparnost, periodičnost

D nije simetrično \Rightarrow nije ni parna, ni neparna

3° f'_j e nije periodična

nula, presek grafa sa y -osom, znak f'_j e

$y=0 \Leftrightarrow \ln^2 x = 0$. Kako je $\ln x = 0$ za $x=1$ to $(1, 0)$ je nula f'_j e

$f(0)$ nije definisano $\Rightarrow f_{-}$ ne riječe y -osa

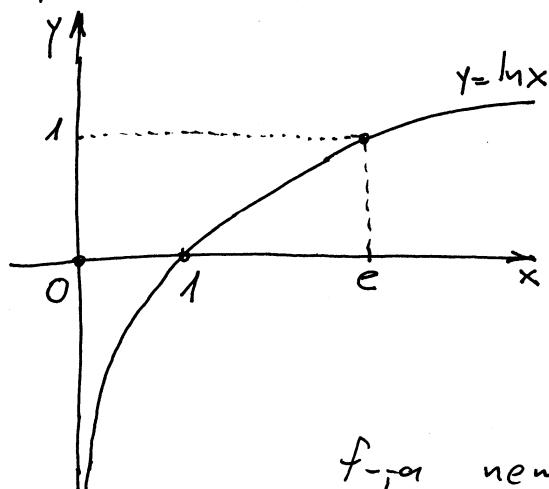
$\ln^2 x > 0$ za $\forall x \in D$

$$\text{pa } \begin{array}{|c|c|} \hline x & (0, +\infty) \\ \hline y & + \\ \hline \end{array}$$

znak f_{-}

4° ponaranje na krajevima intervala definisanosti i asymptote
 $x=0$ je tačka u kojoj f_{-} nije definisana

$$\lim_{x \rightarrow 0^+} \frac{\ln^2 x}{x} = \frac{(-\infty)^2}{0^+} = +\infty \Rightarrow x=0 \text{ VoA.}$$



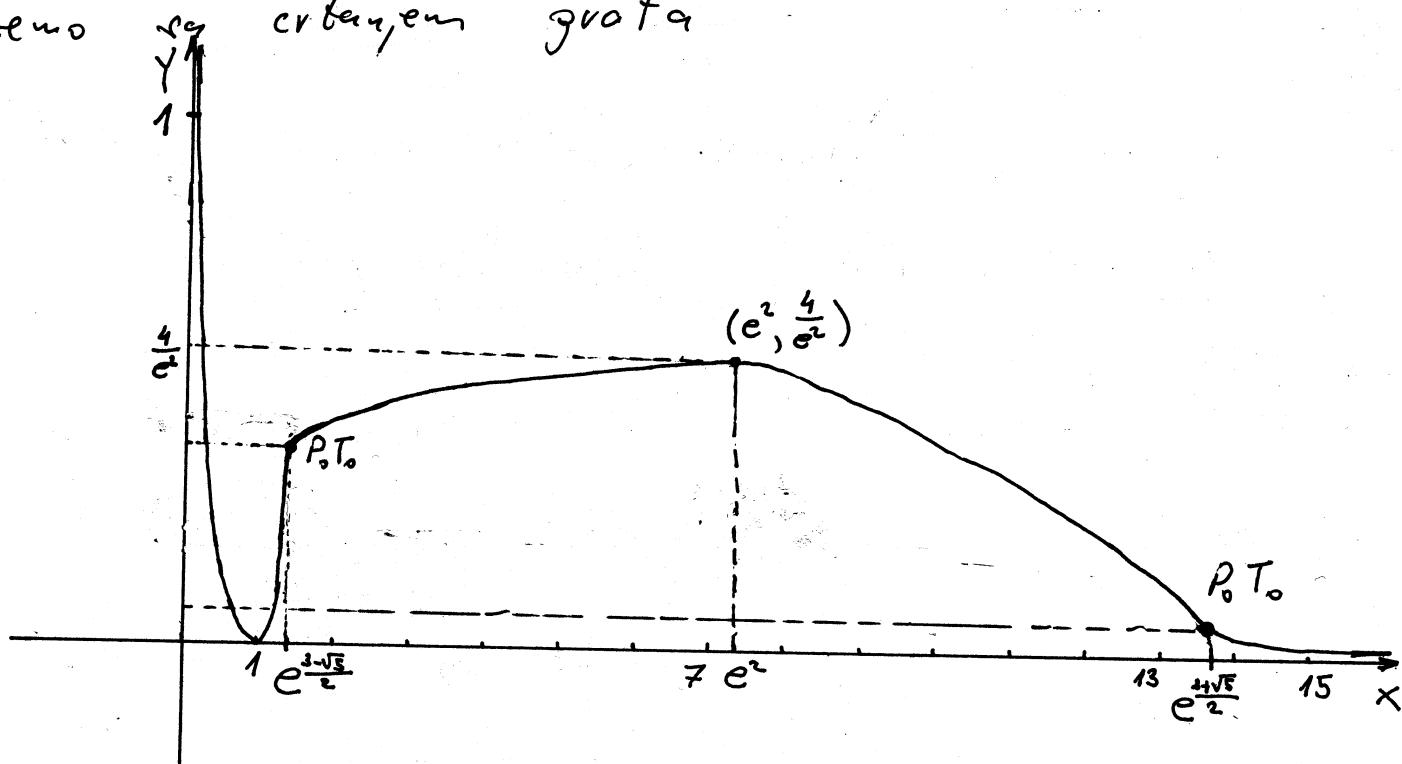
$$\lim_{x \rightarrow \infty} \frac{\ln^2 x}{x} \left(\frac{\infty}{\infty}\right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2 \ln x \cdot \frac{1}{x}}{1}$$

$$= \lim_{x \rightarrow \infty} \frac{2 \ln x}{x} \left(\frac{\infty}{\infty}\right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2 \cdot \frac{1}{x}}{1} = 0$$

$\Rightarrow x=0$ je HoA.

f_{-} ne može krenuti asymptotu

Počinjemo sa crtežem grafa



5° intervali raste i opadaju

$$y' = \left(\frac{\ln^2 x}{x} \right)' = \frac{2 \ln x \cdot \frac{1}{x} \cdot x - \ln^2 x}{x^2} = \frac{2 \ln x - \ln^2 x}{x^2}$$

$$y' = \frac{\ln x \cdot (2 - \ln x)}{x^2}, \quad y' = 0 \text{ akko } \ln x = 0 \vee 2 - \ln x = 0$$

$$\text{pričini } f_{-} \text{ je } y' \quad x_1 = 1, \quad x_2 = e^2$$

+ nule f_{-} je y'

$$(2 - \ln e^2) = 2 - 2 = 0$$

x	$(0, 1)$	$(1, e^2)$	$(e^2, +\infty)$
y'	-	+	-
y	\searrow	\nearrow	\searrow

min max rast
opadaje

$$e^{-2} \in (0, 1) \quad f'(e^{-2}) = \frac{(-2)}{(e^{-2})^2}$$

$$e \in (1, e^2) \quad f'(e) = \frac{1 \cdot (2-1)}{e^2} > 0$$

$$e^4 \in (e^2, +\infty) \quad f'(e^4) = \frac{4 \cdot (2-4)}{(e^4)^2} < 0$$

6° ekstreumi $f_{-j} e$

$y' = 0 \Rightarrow x_1 = 1 ; x_2 = e^2$ su stacionarne tačke $f_{-j} y$

Na osnovu tabele u 5°

$y'(1-0) < 0 \Rightarrow y'(1+0) \nearrow \Rightarrow f_{\min}(1) = 0$

$(1, 0)$ je tačka minimuma

$y'(e^2-0) > 0 \nearrow, y'(e+0) \searrow \Rightarrow f_{\max}(e^2) = \frac{4}{e^2}$

$(e^2, \frac{4}{e^2})$ je tačka maximuma

7° prevojne tačke: intervali konveknosti i konkavnosti

$$y'' = \left(\frac{\ln(2-\ln x)}{x^2} \right)' = \frac{\frac{2-2\ln x}{x} - \ln x(2-\ln x) \cdot 2x}{x^4} =$$

$$\boxed{(\ln x \cdot (2-\ln x))' = \frac{1}{x}(2-\ln x) + \ln x \cdot (-\frac{1}{x}) = \frac{1}{x}(2-\ln x - \ln x) = \frac{2-2\ln x}{x}}$$

$$= 2 \cdot \frac{1-\ln x - 2\ln x + \ln^2 x}{x^3} = 2 \cdot \frac{\ln^2 x - 3\ln x + 1}{x^3}$$

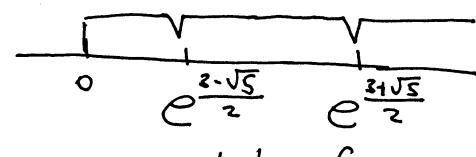
$$y'' = \frac{2(\ln^2 x - 3\ln x + 1)}{x^3}, \quad y'' = 0 \text{ akko } \ln^2 x - 3\ln x + 1 = 0$$

$$\ln x = t, \quad t^2 - 3t + 1 = 0$$

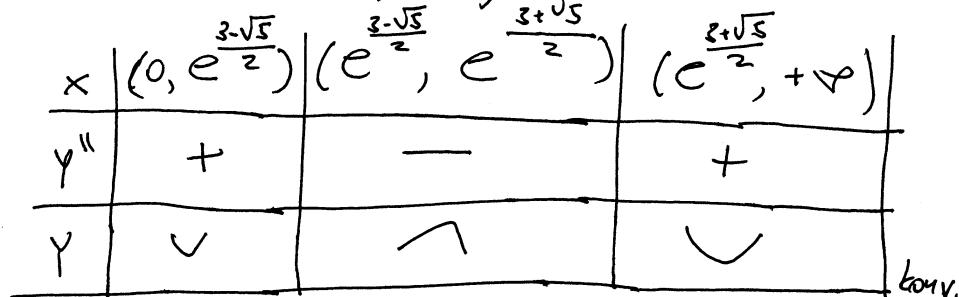
$$D = 9 - 4 = 5 \quad t_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

$$\ln x = \frac{3 - \sqrt{5}}{2} \Rightarrow x_1 = e^{\frac{3 - \sqrt{5}}{2}}$$

Tačke x_1 i x_2 su kandidati za prevojne tačke.



prekidi $f_{-j} e$
+ nula $f_e y''$



8° graf $f_{-j} e$

$$1 \in (0, e^{\frac{3-\sqrt{5}}{2}}) \quad y''(1) = \frac{2 \cdot (0-0+1)}{42-1^3} > 0, \quad e^2 \in (e^{\frac{3-\sqrt{5}}{2}}, e^{\frac{3+\sqrt{5}}{2}}) \quad \text{i konk.}$$

$$b) y = \ln(x^2 - 4x + 5)$$

Lj. 1° definicija područje

$$\mathcal{D}: x \in \mathbb{R}, \quad x \in (-\infty, +\infty)$$

2° parnost (neparnost), periodicitet

$$y = \ln(x^2 - 4x + 5)$$

$$y = \ln[(x-2)^2 + 1]$$

$$f(-x) = \ln[-(x-2)^2 + 1]$$

f_{-x} nije ni parna ni neparna
 f_{-x} nije periodična

3° nule, presek grafa sa y -osom, znak f_{-x}

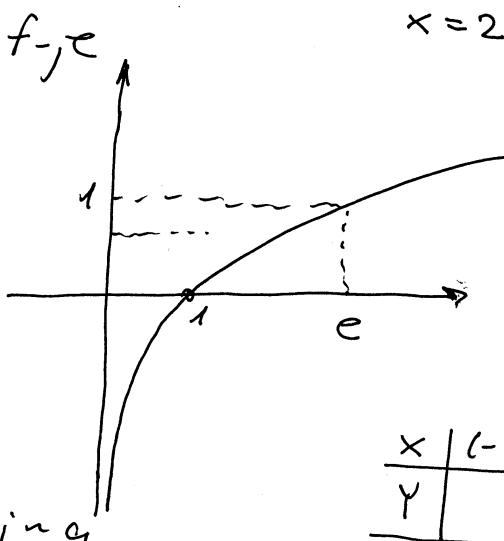
$$y=0 \text{ akko } \ln[(x-2)^2 + 1] = \underbrace{\ln 1}_{=0} \text{ tj. } (x-2)^2 + 1 = 1$$

(2, 0) je nula f_{-x}

$$f(0) = \ln(0 - 0 + 5)$$

$$f(0) = \ln 5$$

(0, $\ln 5$) presek
grafa sa
 y -osom



$$\begin{array}{l|l} x & (-\infty, 2) \\ \hline y & + \end{array}$$

$$\ln[(x-2)^2 + 1] > \underbrace{\ln 1}_{=0}$$

$$(x-2)^2 + 1 > 1$$

$$(x-2)^2 > 0$$

$\forall x$

$$\begin{array}{l|l} x & (-\infty, 2) \\ \hline y & + \end{array}$$

znak f_{-x}

4° ponarađenje na krajevinu
intervala definicije i asymptote

f_{-x} je definisana u svim tačkama \Rightarrow nema V_oA.

$\lim_{x \rightarrow \infty} \ln[(x-2)^2 + 1] = \infty \Rightarrow$ nema horizontalne asymptote

Tracimo kuru asymptotu u obliku $y = kx + b$

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\ln[(x-2)^2 + 1]}{x} \left(\frac{\infty}{\infty}\right) \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{(x-2)^2 + 1} \cdot 2(x-2) \cdot 1}{1}$$

$$= \lim_{x \rightarrow \infty} \frac{2(x-2)}{(x-2)^2 + 1} = \lim_{x \rightarrow \infty} \frac{2x-4}{x^2-4x+5} : x = 0 \quad \text{nema ko A.}$$

Počinjemmo sa crtanjem grafa

5° intervali rasta i opadanja

$$y' = (\ln(x^2 - 4x + 5))' = \frac{1}{x^2 - 4x + 5} \cdot (2x-4)$$

$$y' = \frac{2(x-2)}{(x^2 - 4x + 5)}, \quad y' = 0 \text{ akko } x=2$$

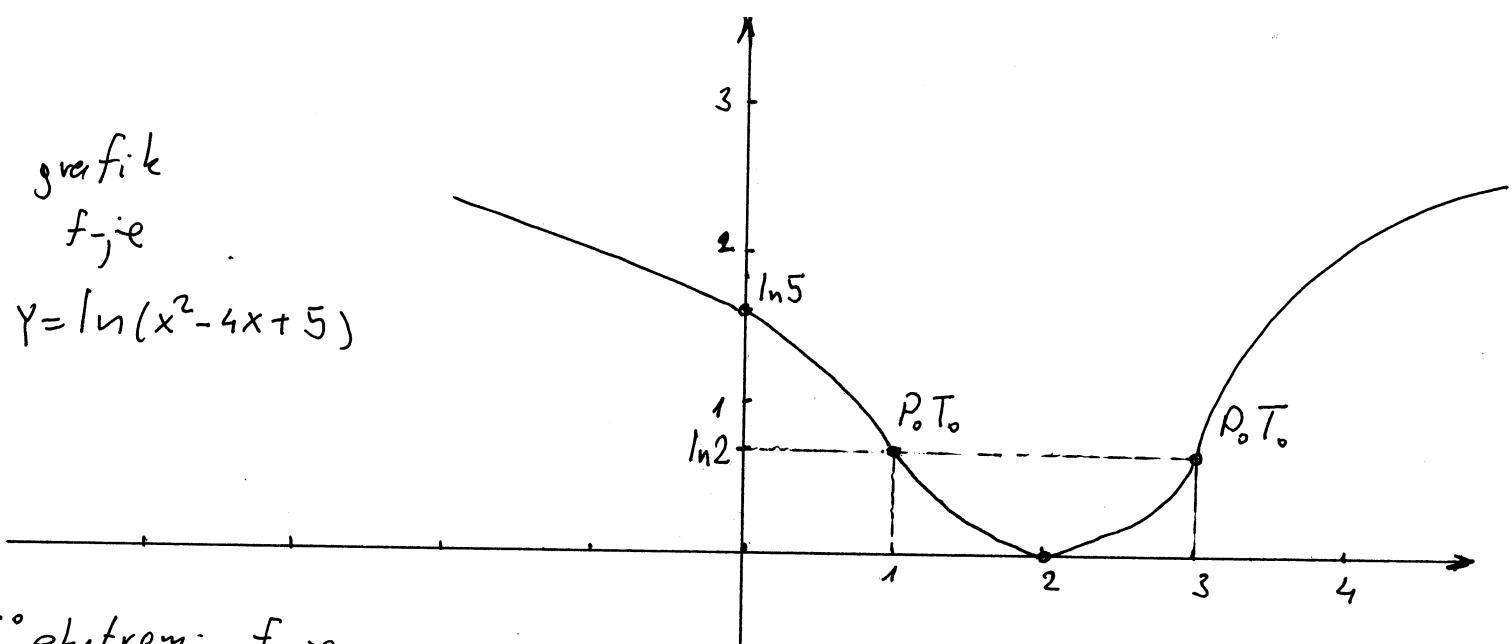
	$(-\infty, 2)$	$(2, +\infty)$
y'	-	+
y	↗	↗

rast
i padaži

grafik

f_{je}

$$y = \ln(x^2 - 4x + 5)$$



6° ekstremi f_{je}

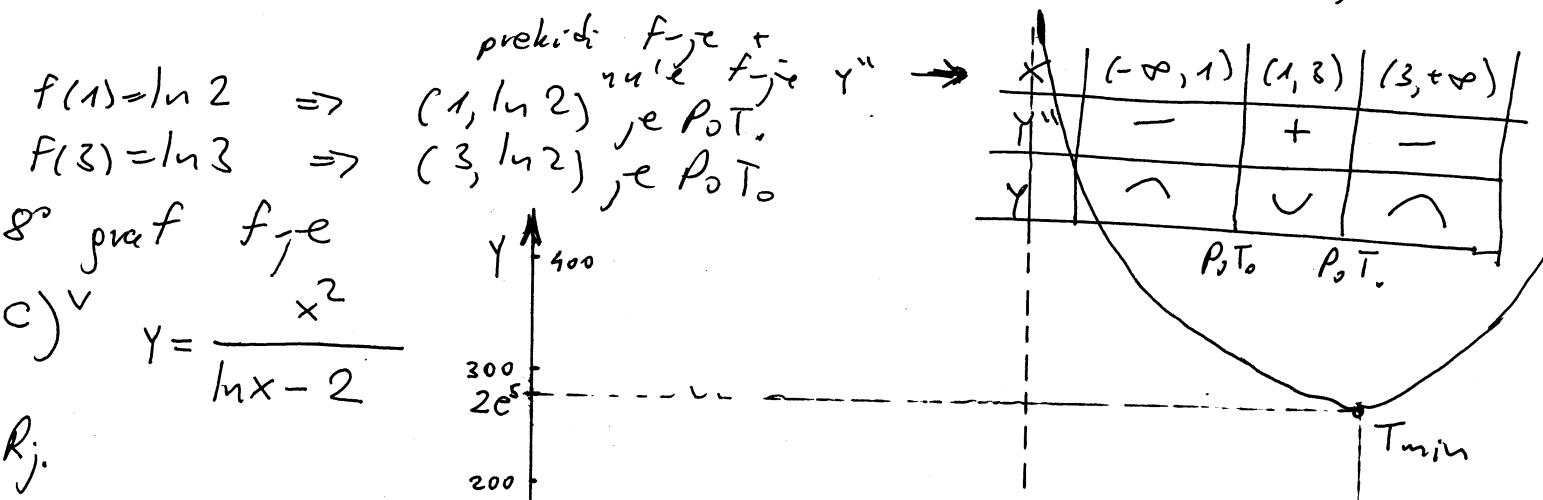
$y' = 0 \Rightarrow$ stacionarne tačke f_{je} u $x=2$

Na ogranici tečele u 5 $y'(2-0) < 0$; $y'(2+0) > 0 \Rightarrow f''(2) = 0$

7° preuzme tačke i intervali $(2, 0)$ je tačka minimum
konveksnosti i konkavnosti

$$y'' = \left(\frac{2x-4}{x^2-4x+5} \right)' = \frac{2 \cdot (x^2-4x+5) - (2x-4)(2x-4)}{(x^2-4x+5)^2} = \frac{2x^2-8x+10 - 4x^2 + 16x - 16}{(x^2-4x+5)^2}$$

$$y'' = \frac{-2x^2+8x-6}{(x^2-4x+5)^2}, \quad y'' = 0 \text{ a kada } -2x^2+8x-6 = 0 \\ t.j. -2(x-1)(x-3) = 0 \Rightarrow x_1=1, x_2=3$$



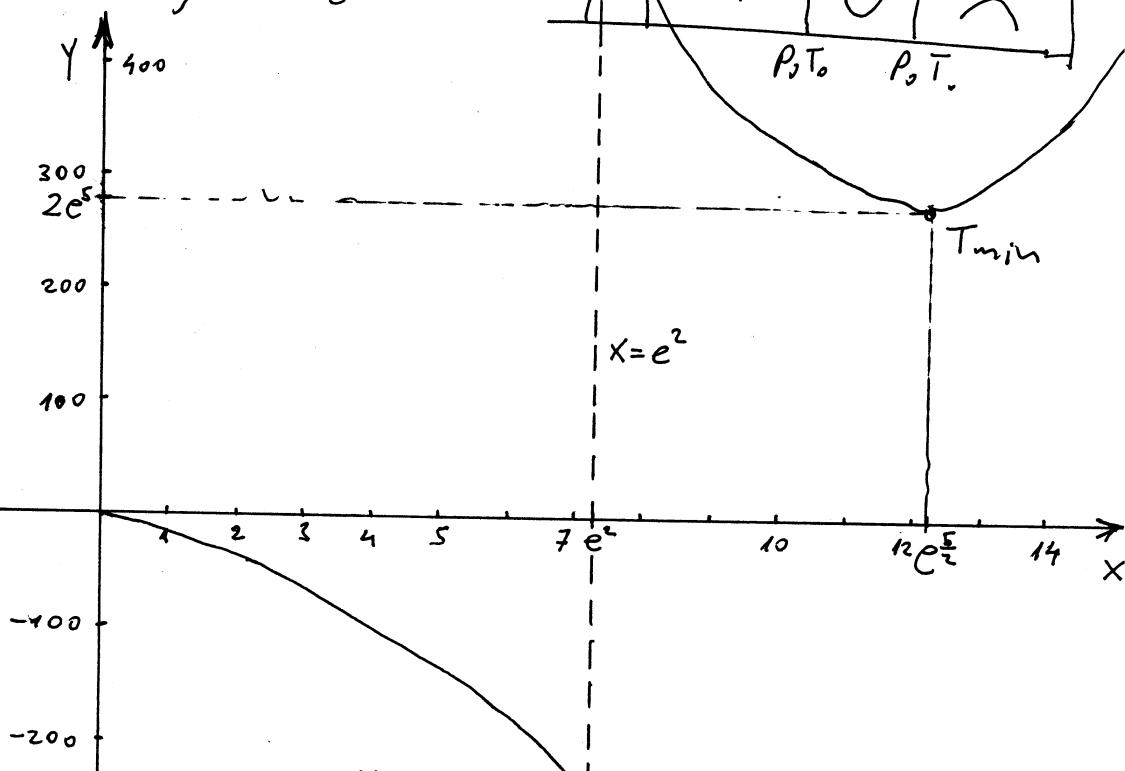
8° graf f_{je}

$$c) \vee \quad y = \frac{x^2}{\ln x - 2}$$

R.j.

$$y' = \frac{x(2/\ln x - 5)}{(\ln x - 2)^2}$$

$$y'' = \frac{2/\ln x - 11/\ln x + 16}{(\ln x - 2)^3}$$



Neodređeni integrali

Ako znamo izvod f-je npr. $y' = 3x^2$ postavlja se pitanje
koja je to f-ja y ?

Jedno rješenje $y = x^3$, ostala rješenja $y = x^3 - 1$
 $y = x^3 + 8$

Općenito svako rješenje je oblika $y = x^3 + C$, C -konstanta.

Primer 2: $y' = \frac{5}{2} 8^x \ln 8$

$$y = ? \quad y = \frac{5}{2} \cdot 8^x, \text{ općenito } y = \frac{5 \cdot 8^x}{2} + C$$

$$F'(x) = f(x) \Rightarrow \int f(x) dx = F(x) + C$$

$\int f(x) dx$ - neodređeni integral f-je $f(x)$

$F(x)$ - primitivna f-ja f-je $f(x)$

Integriranje je obrnuta operacija diferenciranja:

$$(\sin x)' = \cos x \Rightarrow \int \cos x dx = \sin x + C$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

Tablica integrala

$$1. \int 0 dx = C, \quad C \text{-konst.}$$

$$6. \int e^x dx = e^x + C$$

$$2. \int dx = x + C$$

$$7. \int \sin x dx = -\cos x + C$$

$$3. \int x^2 dx = \frac{x^{2+1}}{2+1} + C, \quad 2 \neq 1$$

$$8. \int \cos x dx = \sin x + C$$

$$4. \int \frac{dx}{x} = \ln|x| + C, \quad x \neq 0$$

$$9. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$5. \int a^x dx = \frac{a^x}{\ln a} + C, \quad 0 < a \neq 1$$

$$10. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$11. \int \sin x dx = -\cos x + C$$

$$15. \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, \quad |x| < 1$$

$$12. \int \cosh x dx = \sinh x + C$$

$$16. \int \frac{dx}{1+x^2} = \arctan x + C$$

$$13. \int \frac{dx}{\cosh^2 x} = -\tanh x + C$$

$$17. \int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln|x + \sqrt{x^2 \pm 1}| + C$$

$$14. \int \frac{dx}{\sinh^2 x} = -\coth x + C$$

$$18. \int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$I \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C, \quad a > 0$$

$$II \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C, \quad a \neq 0$$

$$III \int \frac{dx}{\sqrt{x^2 + a^2}} = \ln|x + \sqrt{x^2 + a^2}| + C, \quad a \neq 0$$

$$IV \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C, \quad a \neq 0$$

$$V \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C, \quad a \neq 0$$

$$VI \int \frac{dx}{\sin x} = \ln |\csc x| + C$$

$$VII \int \frac{dx}{\cos x} = \ln |\sec x| + C$$

Odredite sljedeće integrale:

$$\begin{aligned} \textcircled{1}_0 \int (2x^3 + 5x^2 - 7x - 6) dx &= \int 2x^3 dx + \int 5x^2 dx - \int 7x dx - \int 6 dx = \\ &= 2 \int x^3 dx + 5 \int x^2 dx - 7 \int x dx - 6 \int dx = 2 \cdot \frac{x^4}{4} + 5 \cdot \frac{x^3}{3} - 7 \cdot \frac{x^2}{2} - 6x + C \\ &= \frac{x^4}{2} + \frac{5x^3}{3} - \frac{7x^2}{2} - 6x + C \end{aligned}$$

$$\begin{aligned} \textcircled{2}_0 \int \frac{5x^7 + 2x^5 - x + 6}{x^3} dx &= \int \left(\frac{5x^7}{x^3} + \frac{2x^5}{x^3} - \frac{x}{x^3} + \frac{6}{x^3} \right) dx = \\ &= 5 \int x^4 dx + 2 \int x^2 dx - \int x^{-2} dx + 6 \int x^{-3} dx = 5 \cdot \frac{x^5}{5} + 2 \cdot \frac{x^3}{3} - \frac{x^{-1}}{-1} + 6 \cdot \frac{x^{-2}}{-2} + C \\ &= x^5 + \frac{2x^3}{3} + \frac{1}{x} - 3 \cdot \frac{1}{x^2} + C \end{aligned}$$

$$\begin{aligned} \textcircled{3}_0 \int \sqrt[3]{x \sqrt{x \sqrt{x}}} dx &= \int \sqrt[3]{x \sqrt[3]{\sqrt{x^2 \cdot x}}} dx = \int \sqrt[3]{x \sqrt[6]{x^3}} dx \\ &= \int \sqrt[6]{x^6 \cdot x^3} dx = \int \sqrt[12]{x^9} dx = \int x^{\frac{9}{12}} dx = \int x^{\frac{3}{4}} dx \\ &= \frac{x^{\frac{7}{4}}}{\frac{7}{4}} + C = \frac{4}{7} \sqrt[4]{x^7} + C \end{aligned}$$

$$\begin{aligned} \textcircled{4}_0 \int (x^2 + \sqrt{x})^2 dx &= \int (x^4 + 2x^2\sqrt{x} + x) dx = \int x^4 dx + 2 \int \underbrace{x^2 \cdot x^{\frac{1}{2}}}_{x^{\frac{5}{2}}} dx + \int x dx \\ &= \frac{x^5}{5} + 2 \cdot \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^2}{2} + C = \frac{x^5}{5} + \frac{4}{7} \sqrt{x^7} + \frac{x^2}{2} + C \end{aligned}$$

$$\begin{aligned} \textcircled{5}_0 \int \frac{x}{x+1} dx &= \int \frac{x+1-1}{x+1} dx = \int \left(1 - \frac{1}{x+1} \right) dx = \\ &= \int dx - \int \frac{dx}{x+1} = x - \ln|x+1| + C \end{aligned}$$

$$\begin{aligned}
 6. \int \frac{x^2}{x-1} dx &= \int \frac{x^2-1+1}{x-1} dx = \int \frac{x^2-1}{x-1} dx + \int \frac{1}{x-1} dx = \\
 &= \int \frac{(x-1)(x+1)}{(x-1)} dx + \int \frac{1}{x-1} dx = \int (x+1) dx + \int \frac{1}{x-1} dx = \frac{x^2}{2} + x + \ln|x-1| + C
 \end{aligned}$$

$$\begin{aligned}
 7. \int \frac{x^2}{x+2} dx &= \int \frac{x^2+4-4}{x+2} dx = \int \left(\frac{x^2-4}{x+2} + \frac{4}{x+2} \right) dx = \\
 &= \int \frac{(x-2)(x+2)}{x+2} dx + 4 \int \frac{dx}{x+2} = \int (x-2) dx + 4 \int \frac{dx}{x+2} = \frac{x^2}{2} - 2x + 4 \ln|x+2| + C
 \end{aligned}$$

// način bi bio da podjelimo x^2 sa $x+2$ pa izvadimo integral
od dobijenog rezultata $\sqrt{x^2} : (x+2) = x-2 + \frac{4}{x+2}$ $\int \frac{x^2}{x+2} dx = \int (x-2 + \frac{4}{x+2}) dx$

$$\begin{aligned}
 8. \int \frac{x^3}{x-3} dx &= \int \frac{x^3-27+27}{x-3} dx = \int \frac{x^3-27}{x-3} dx + \int \frac{27}{x-3} dx = \\
 &= \int \frac{(x-3)(x^2+3x+9)}{x-3} dx + \int \frac{27}{x-3} dx = \int (x^2+3x+9) dx + 27 \int \frac{dx}{x-3} \\
 &= \frac{x^3}{3} + \frac{3x^2}{2} + 9x + 27 \ln|x-3| + C
 \end{aligned}$$

$$\begin{aligned}
 9. \int \operatorname{tg}^2 x dx &= \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int dx \\
 &= \operatorname{tg} x - x + C
 \end{aligned}$$

$$10. \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx \quad R_j: -\frac{24}{17} \sqrt[12]{x^{12}} + \frac{4}{5} \sqrt[4]{x^5} + \frac{4}{3} \sqrt[4]{x^3} + C$$

$$11. \int \frac{e^{3x}+1}{e^x+1} dx \quad R_j: \frac{1}{2} e^{2x} - e^x + x + C$$

$$12. \int \frac{1}{\sin^2 2x} dx \quad R_j: -\frac{1}{2} \cdot \frac{\cos 2x}{\sin 2x} + C$$

Metoda zamjene

1 tip $\int f(ax+b) dx = \frac{1}{a} F(ax+b) + C$

$$\textcircled{1_0} \int \frac{dx}{2x+5} = \begin{cases} t=2x+5 \\ dt=2dx \\ dx=\frac{dt}{2} \end{cases} = \int \frac{\frac{dt}{2}}{t} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| + C = \frac{1}{2} \ln|2x+5| + C$$

$$\textcircled{2_0} \int \sin(4x+1) dx = \begin{cases} 4x+1=t \\ 4dx=dt \\ dx=\frac{dt}{4} \end{cases} = \int \sin t \cdot \frac{dt}{4} = \frac{1}{4} \int \sin t dt = -\frac{1}{4} \cos t + C = -\frac{1}{4} \cos(4x+1) + C$$

$$\textcircled{3_0} \int (3x-1)^9 dx = \begin{cases} 3x-1=t \\ 3dx=dt \\ dx=\frac{dt}{3} \end{cases} = \int t^9 \cdot \frac{dt}{3} = \frac{1}{3} \int t^9 dt = \frac{1}{3} \cdot \frac{t^{10}}{10} + C = \frac{(3x-1)^{10}}{30} + C$$

$$\textcircled{4_0} \int e^{1-3x} dx = \begin{cases} 1-3x=t \\ -3dx=dt \\ dx=-\frac{dt}{3} \end{cases} = \int e^t \cdot -\frac{dt}{3} = -\frac{1}{3} \int e^t dt = -\frac{1}{3} e^t + C = -\frac{1}{3} e^{1-3x} + C$$

$$\textcircled{5_0} \int \frac{dx}{\sqrt{1-(3x+2)^2}} = \begin{cases} 3x+2=t \\ 3dx=dt \\ dx=\frac{dt}{3} \end{cases} = \frac{1}{3} \int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{3} \arcsin t + C = \frac{1}{3} \arcsin(3x+2) + C$$

$$\textcircled{6_0} \int \cos(6x+4) dx \quad R_j: \quad \frac{1}{6} \sin(6x+4) + C$$

$$\textcircled{7_0} \int \frac{dx}{\cos^2(7x+8)} \quad R_j: \quad \frac{1}{7} \cdot \frac{\sin(7x+8)}{\cos(7x+8)}$$

$$\textcircled{8_0} \int \frac{dx}{1+(5x-2)^2} \quad R_j: \quad \frac{1}{5} \operatorname{arctg}(5x-2)$$

// tip $\int \frac{dx}{ax^2+b}$, $\int \frac{dx}{\sqrt{ax^2+b}}$, sejena $\sqrt{|a|} \cdot x = \sqrt{|b|} \cdot t$

$$\textcircled{1_0} \quad \int \frac{dx}{4x^2+9} = \int \frac{dx}{(2x)^2+3^2} = \begin{cases} 2x = 3t \\ 2dx = 3dt \\ dx = \frac{3}{2} dt \end{cases} = \frac{3}{2} \int \frac{dt}{(3t)^2+3^2} = \frac{3}{2} \int \frac{dt}{9t^2+9} = \\ = \frac{3}{2} \cdot \frac{1}{9} \int \frac{dt}{t^2+1} = \frac{1}{6} \arctan t + C = \frac{1}{6} \arctan \frac{2x}{3} + C$$

$$\textcircled{2_0} \quad \int \frac{dx}{\sqrt{2x^2+25}} = \int \frac{dx}{\sqrt{(\sqrt{2}x)^2+5^2}} = \begin{cases} \sqrt{2}x = 5t \\ \sqrt{2}dx = 5dt \\ dx = \frac{5}{\sqrt{2}} dt \\ t = \frac{\sqrt{2}}{5}x \end{cases} = \frac{5}{\sqrt{2}} \int \frac{dt}{\sqrt{25t^2+25}} = \\ = \frac{5}{\sqrt{2}} \cdot \frac{1}{5} \int \frac{dt}{\sqrt{t^2+1}} = \frac{1}{\sqrt{2}} \cdot \ln |t + \sqrt{t^2+1}| + C = \frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{2}}{5}x + \sqrt{\frac{2}{25}x^2+1} \right| + C$$

$$\textcircled{3_0} \quad \int \frac{dx}{5x^2-49} = \int \frac{dx}{(\sqrt{5}x)^2-7^2} = \begin{cases} \sqrt{5}x = 7t \\ \sqrt{5}dx = 7dt \\ dx = \frac{7}{\sqrt{5}} dt \\ t = \frac{\sqrt{5}x}{7} \end{cases} = \frac{7}{\sqrt{5}} \int \frac{dt}{49t^2-49} = \frac{7}{\sqrt{5}} \cdot \frac{1}{49} \int \frac{dt}{t^2-1} \\ = \frac{1}{7\sqrt{5}} \cdot \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C = \frac{1}{14\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}}{2}x-1}{\frac{\sqrt{5}}{2}x+1} \right| + C = \frac{1}{14\sqrt{5}} \ln \left| \frac{\sqrt{5}x-7}{\sqrt{5}x+7} \right|$$

$$\textcircled{4_0} \quad \int \frac{dx}{\sqrt{7-9x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}} = \begin{cases} 3x = \sqrt{7}t \\ 3dx = \sqrt{7}dt \\ dx = \frac{\sqrt{7}}{3} dt \\ t = \frac{3x}{\sqrt{7}} \end{cases} = \frac{\sqrt{7}}{3} \int \frac{dt}{\sqrt{7-7t^2}} = \frac{\sqrt{7}}{3} \cdot \frac{1}{\sqrt{7}} \int \frac{dt}{\sqrt{1-t^2}} \\ = \frac{1}{3} \arcsin t + C = \frac{1}{3} \arcsin \left(\frac{3x}{\sqrt{7}} \right) + C$$

$$\textcircled{5_0} \quad \int \frac{dx}{4x^2+11}, \quad R_j. \quad \frac{\sqrt{11}}{22} \arctan \frac{2\sqrt{11}x}{11} + C$$

$$\textcircled{6_0} \quad \int \frac{dx}{\sqrt{9x^2-16}}, \quad R_j. \quad \frac{1}{3} \ln |3x + \sqrt{9x^2-16}| + C$$

$$\textcircled{7_0} \quad \int \frac{dx}{\sqrt{2x^2+5}}, \quad R_j. \quad \frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{10}}{5}x + \sqrt{\frac{2}{5}x^2+1} \right| + C$$

$$\text{III tip } \int \frac{dx}{ax^2+bx+c}, \quad ax^2+bx+c = a(x-\lambda)^2 + \beta$$

$$\begin{aligned} 1. & \int \frac{dx}{x^2+6x+13}, \quad x^2+6x+13 = x^2+2 \cdot x \cdot 3 + 3^2 + 4 = (x+3)^2 + 4 \\ & I = \int \frac{dx}{(x+3)^2 + 2^2} = \left| \begin{array}{l} x+3=2t \\ dx=2dt \\ t=\frac{x+3}{2} \end{array} \right| = 2 \int \frac{dt}{4t^2+4} = 2 \cdot \frac{1}{4} \int \frac{dt}{t^2+1} = \frac{1}{2} \arct \varphi t + C \\ & = \frac{1}{2} \arct \varphi \frac{x+3}{2} + C \end{aligned}$$

$$\begin{aligned} 2. & \int \frac{dx}{\sqrt{2-x-x^2}}, \quad 2-x-x^2 = -x^2-x+2 = (-1)[x^2+x-2] = \\ & = (-1)(x^2+2 \cdot x \cdot \frac{1}{2} + (\frac{1}{2})^2 - (\frac{1}{2})^2 - 2) = \\ & = (-1)\left[\left(x+\frac{1}{2}\right)^2 - \frac{9}{4}\right] = \frac{9}{4} - \left(x+\frac{1}{2}\right)^2 \\ & I = \int \frac{dx}{\sqrt{\frac{9}{4} - \left(x+\frac{1}{2}\right)^2}} = \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}} = \left| \begin{array}{l} x+\frac{1}{2} = \frac{3}{2}t \\ dx = \frac{3}{2}dt \\ t = \frac{2x+1}{3} \end{array} \right| = \frac{3}{2} \int \frac{dt}{\sqrt{\frac{9}{4} - \frac{9}{4}t^2}} = \\ & = \frac{3}{2} \cdot \frac{1}{\frac{3}{2}} \int \frac{dt}{\sqrt{1-t^2}} = \frac{3}{2} \cdot \frac{2}{3} \cdot \arcsin t + C = \arcsin \frac{2x+1}{3} + C \end{aligned}$$

$$\begin{aligned} 3. & \int \frac{dx}{2x^2-7x+3}, \quad 2x^2-7x+3 = 2 \cdot \left(x^2 - \frac{7}{2}x + \frac{3}{2}\right) = 2 \cdot \left(x^2 - 2 \cdot x \cdot \frac{7}{4} + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 + \frac{3}{2}\right) = 2 \cdot \left[\left(x - \frac{7}{2}\right)^2 + \frac{-49+24}{16}\right] = 2 \left[\left(x - \frac{7}{2}\right)^2 - \frac{25}{16}\right] \end{aligned}$$

$$\begin{aligned} & I = \frac{1}{2} \int \frac{dx}{\left(x - \frac{7}{2}\right)^2 - \frac{25}{16}} = \frac{1}{2} \int \frac{dx}{\left(x - \frac{7}{2}\right)^2 - \left(\frac{5}{4}\right)^2} = \left| \begin{array}{l} x - \frac{7}{4} = \frac{5}{4}t \\ dx = \frac{5}{4}dt \\ t = \frac{4x-14}{5} \end{array} \right| = \frac{1}{2} \cdot \frac{5}{4} \int \frac{dt}{\frac{25}{16}t^2 - \frac{25}{16}} \\ & = \frac{5}{8} \cdot \frac{16}{5} \int \frac{dt}{t^2-1} = \frac{2}{5} \cdot \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C = \frac{1}{5} \ln \left| \frac{\frac{4x-12}{5}-1}{\frac{4x-2}{5}+1} \right| + C \\ & = \frac{1}{5} \ln \left| \frac{\frac{4x-12}{5}}{\frac{4x-2}{5}+1} \right| + C = \frac{1}{5} \ln \left| \frac{4x-12}{4x-2} \right| + C = \frac{1}{5} \ln \left| \frac{2x-6}{2x-1} \right| + C \end{aligned}$$

$$4. \int \frac{dx}{\sqrt{x^2+8x+25}}, \quad R: \quad \ln \left| \frac{x+4}{3} + \sqrt{\left(\frac{x+4}{3}\right)^2 + 1} \right| + C$$

$$\textcircled{5}_o \quad \int \frac{dx}{\sqrt{5x-x^2}} \quad R_j: \quad \arcsin \frac{2x-5}{5} + C$$

$$\textcircled{6}_o \quad \int \frac{dx}{3x^2+x-2} \quad R_j: \quad -\frac{1}{5} \ln \left| \frac{x+1}{3x-2} \right|$$

IV tip $\int \frac{f'(x)}{f(x)} dx, \quad \int \frac{f'(x)}{\sqrt{f(x)}} dx$

$$\textcircled{1}_o \quad \int ctg x dx = \int \frac{\cos x}{\sin x} dx = \begin{vmatrix} \sin x = t \\ \cos x dx = dt \end{vmatrix} = \int \frac{dt}{t} = \ln |t| + C = \ln |\sin x| + C$$

$$\textcircled{2}_o \quad \int \frac{3x^2+4x-4}{x^3+2x^2-4x+6} dx = \begin{vmatrix} x^3+2x^2-4x+6 = t \\ (3x^2+4x-4) dx = dt \end{vmatrix} = \int \frac{dt}{t} = \ln |t| + C = \ln |x^3+2x^2-4x+6| + C$$

$$\textcircled{3}_o \quad \int \frac{x-5}{\sqrt{x^2-10x+7}} dx = \begin{vmatrix} x^2-10x+7 = t \\ (2x-10) dx = dt \\ (x-5) dx = \frac{dt}{2} \end{vmatrix} = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{1}{2} \cdot 2 \sqrt{t} + C = \sqrt{x^2-10x+7} + C$$

$$\textcircled{4}_o \quad \int \frac{x-3}{x^2-6x+7} dx = \begin{vmatrix} x^2-6x+7 = t \\ (2x-6) dx = dt \\ (x-3) dx = \frac{dt}{2} \end{vmatrix} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |t| + C = \frac{1}{2} \ln |x^2-6x+7| + C$$

$$\textcircled{5}_o \quad \int \frac{x^3 dx}{\sqrt{x^4+1}} = \begin{vmatrix} x^4+1 = t \\ 4x^3 dx = dt \\ x^3 dx = \frac{dt}{4} \end{vmatrix} = \frac{1}{4} \int \frac{dt}{\sqrt{t}} = \frac{1}{4} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{1}{4} \cdot 2 \sqrt{t} + C = \frac{1}{2} \sqrt{x^4+1} + C$$

$$\textcircled{6}_o \quad \int \frac{3x^2}{\sqrt{x^3-2}} dx \quad R_j: \quad 2 \sqrt{x^3-2} + C$$

$$\textcircled{7}_o \quad \int tg x dx \quad R_j: \quad -\ln |\cos x| + C$$

$$\textcircled{8}_o \quad \int \frac{\sin x}{\sqrt{5 \cos x - 2}} dx \quad R_j: \quad -\frac{2}{5} \sqrt{5 \cos x - 2} + C$$

$$V \text{ tip } \int \frac{mx+n}{ax^2+bx+c} dx, \quad \int \frac{mx+n}{\sqrt{ax^2+bx+c}} dx$$

(1.) $I = \int \frac{x+4}{x^2-4x+5} dx$

$$(x^2-4x+5)' = 2x-4$$
 $x+4 = a \cdot (2x-4) + b, \quad a, b = ?$
 $x+4 = 2ax - 4a + b$
 $I = \int \frac{\frac{1}{2}(2x-4)+6}{x^2-4x+5} dx$
 $2a=1 \quad -4a+b=4$
 $a=\frac{1}{2} \quad -2+b=4$
 $b=6$
 $= \frac{1}{2} \underbrace{\int \frac{2x-4}{x^2-4x+5} dx}_{I_1} + 6 \underbrace{\int \frac{dx}{x^2-4x+5}}_{I_2} = \frac{1}{2} I_1 + 6 I_2 \quad x^2-2 \cdot x \cdot 2 + 2^2 - 2^2 + 5 =$
 $= (x-2)^2 + 1$

$I_1 = \left| \begin{array}{l} x^2-4x+5 = t \\ (2x-4)dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln |t| + C_1 = \ln |x^2-4x+5| + C_1$

$I_2 = \int \frac{dx}{(x-2)^2 + 1} = \left| \begin{array}{l} x-2=t \\ dx=dt \end{array} \right| = \int \frac{dt}{t^2+1} = \arctan t + C_2 =$
 $= \arctan(x-2) + C_2$

$I = \frac{1}{2} \ln |x^2-4x+5| + \arctan(x-2) + C$

(2.) $I = \int \frac{x}{\sqrt{x^2+2x-5}} dx$

 $(x^2+2x-5)' = 2x+2 \quad 2a=1 \Rightarrow a=\frac{1}{2}$
 $x=a(2x+2)+b \quad 2a+b=0$
 $x=2ax+2a+b \quad 2 \cdot \frac{1}{2} + b=0$
 $b=-1$

$I = \int \frac{\frac{1}{2}(2x+2)-1}{\sqrt{x^2+2x-5}} dx = \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x-5}} dx - \int \frac{dx}{\sqrt{x^2+2x-5}} = \frac{1}{2} I_1 - I_2$

$\int \frac{2x+2}{\sqrt{x^2+2x-5}} dx = \left| \begin{array}{l} x^2-2x-5 = t \\ (2x-2)dx=dt \end{array} \right| = \int \frac{dx}{\sqrt{t}} = \frac{1}{2} \sqrt{t} + C_1 = \frac{1}{2} \sqrt{x^2+2x-5} + C_1$

$x^2+2x-5 = x^2+2 \cdot x \cdot 1 + 1^2 - 1^2 - 5 =$
 $= (x+1)^2 - 6$

$I_2 = \int \frac{dx}{\sqrt{(x+1)^2-6}} = \left| \begin{array}{l} x+1 = \sqrt{6} t \\ dx = \sqrt{6} dt \\ t = \frac{x+1}{\sqrt{6}} \end{array} \right| =$

$$= \sqrt{6} \int \frac{dt}{\sqrt{6t^2 - 6}} = \frac{\sqrt{6}}{\sqrt{6}} \int \frac{dt}{\sqrt{t^2 - 1}} = \ln |t + \sqrt{t^2 - 1}| + C_2$$

$$= \ln \left| \frac{x+1}{\sqrt{6}} + \sqrt{\frac{(x+1)^2}{6} - 1} \right| + C_2$$

$$I = \sqrt{x^2 + 2x - 5} - \ln \left| \frac{x+1}{\sqrt{6}} + \sqrt{\frac{(x+1)^2}{6} - 1} \right| + C$$

(3) $I = \int \frac{6x-7}{x^2-4x+5} dx$

$$(x^2-4x+5)' = 2x-4$$

$$6x-7 = a(2x-4) + b$$

$$6x-7 = 2ax-4a+b$$

$$2a=6 \Rightarrow a=3$$

$$b-4a=-7 \Rightarrow b=5$$

$$x^2-4x+5 = x^2-2 \cdot x \cdot 2 + 2^2 + 5-2^2 = (x-2)^2 + 1$$

$$I = \int \frac{3(2x-4)+5}{x^2-4x+5} dx = 3 \int \frac{2x-4}{x^2-4x+5} dx + 5 \int \frac{dx}{x^2-4x+5} = 3I_1 + 5I_2$$

$$\int \frac{2x-4}{x^2-4x+5} dx = \left| \begin{array}{l} x^2-4x+5=t \\ (2x-4)dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln |t| + C_1 = \ln |x^2-4x+5| + C_1$$

$$I_2 = \int \frac{dx}{(x-2)^2 + 1} = \left| \begin{array}{l} x-2=t \\ dx=dt \end{array} \right| = \int \frac{dt}{t^2+1} = \arctan t + C_2 = \arctan(x-2) + C_2$$

$$I = 3 \ln |x^2-4x+5| + 5 \arctan(x-2) + C$$

(4) $\int \frac{3x+2}{\sqrt{x^2-8x-8}} dx$, Rj: $3\sqrt{x^2-8x-8} + 14 \ln \left| \frac{x-4}{5} + \sqrt{\left(\frac{x-4}{5}\right)^2 - 1} \right| + C$

(5) $\int \frac{3x+4}{\sqrt{-x^2+6x-8}} dx$, Rj: $-3\sqrt{-x^2+6x-8} + 13 \arcsin(x-3)$

(6) $\int \frac{2x+3}{\sqrt{4x^2+4x+3}} dx$

(7) $\int \frac{x-4}{x^2-5x+6} dx$

$$\text{VI tip} \quad \int f(g(x)) \cdot g'(x) dx = \begin{cases} g(x) = t \\ g'(x) dx = dt \end{cases} = \int f(t) dt$$

$$1. \int e^{\cos x} \cdot \sin x dx = \begin{cases} \cos x = t \\ -\sin x dx = dt \\ \sin x dx = -dt \end{cases} = \int e^t \cdot (-dt) = - \int e^t dt = -e^t + C = -e^{\cos x} + C$$

$$2. \int \frac{dx}{x \sqrt[5]{\ln x}} = \begin{cases} \ln x = t^5 \\ \frac{1}{x} dx = 5t^4 dt \\ t = \sqrt[5]{\ln x} \end{cases} = \int \frac{5t^4 dt}{t} = 5 \int t^3 dt = 5 \cdot \frac{t^4}{4} + C = \frac{5}{4} \sqrt[5]{\ln^4 x} + C$$

$$3. \int \frac{x^3 dx}{x^8 - 2} = \int \frac{x^3 dx}{(x^4)^2 - 2} = \begin{cases} x^4 = t \\ 4x^3 dx = dt \\ x^3 dx = \frac{1}{4} dt \end{cases} = \frac{1}{4} \int \frac{dt}{t^2 - 2} = \frac{1}{4} \cdot \frac{1}{2\sqrt{2}} \ln \left| \frac{t-\sqrt{2}}{t+\sqrt{2}} \right| + C = \frac{1}{8\sqrt{2}} \ln \left| \frac{x^4 - \sqrt{2}}{x^4 + \sqrt{2}} \right| + C$$

$$4. \int \frac{\sqrt[3]{\operatorname{tg} x}}{\cos^2 x} dx = \begin{cases} \operatorname{tg} x = t^3 \\ \frac{1}{\cos^2 x} dx = 3t^2 dt \end{cases} = \int \sqrt[3]{t^3} \cdot 3t^2 dt = 3 \int t^5 dt = 3 \cdot \frac{t^6}{6} + C = \frac{3}{4} \sqrt[3]{\operatorname{tg}^6 x} + C$$

$$5. \int x(1-x)^{10} dx = \begin{cases} 1-x = t \\ -dx = dt \\ dx = -dt \\ x = 1-t \end{cases} = \int (1-t) t^{10} \cdot (-dt) = - \int (t^{10} - t^{11}) dt = - \frac{t^{11}}{11} + \frac{t^{12}}{12} + C = \frac{(1-x)^{12}}{12} - \frac{(1-x)^{11}}{11} + C$$

$$6. \int \frac{x^2 - 1}{x^4 + 1} \frac{1/x^2}{1/x^2} dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \begin{cases} \text{treba da dobijemo zavojnu funkciju da} \\ \text{dobijemo} \left(1 - \frac{1}{x^2}\right) dx = dt \\ t = x + \frac{1}{x} \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt \\ \left(x + \frac{1}{x}\right)^2 = t^2 \Rightarrow x^2 + 2x \cdot \frac{1}{x} + \frac{1}{x^2} = t^2 \\ x^2 + \frac{1}{x^2} = t^2 - 2 \end{cases} = \int \frac{dt}{t^2 - 2} = \frac{1}{2\sqrt{2}} \ln \left| \frac{t-\sqrt{2}}{t+\sqrt{2}} \right| + C = \frac{1}{2\sqrt{2}} \ln \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| + C$$

$$\textcircled{7}_o \int \sqrt{\frac{\arcsin x}{1-x^2}} dx = \int \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}} dx = \begin{cases} \arcsin x = t^2 \\ \frac{dx}{\sqrt{1-x^2}} = 2t dt \\ t = \sqrt{\arcsin x} \end{cases} =$$

$$= \int \sqrt{t^2} 2t dt = 2 \int t^2 dt = 2 \cdot \frac{t^3}{3} + C = \frac{2}{3} \sqrt{\arcsin^3 x} + C$$

$$\textcircled{8}_o \int (x+4) \sqrt[5]{2x-1} dx = \begin{cases} 2x-1 = t^5 \\ 2dx = 5t^4 dt \\ dx = \frac{5}{2} t^4 dt \end{cases} = \left(\frac{t^5+1}{2} + 4 \right) \sqrt[5]{t^5} \cdot \frac{5}{2} t^4 dt$$

$$= \frac{5}{2} \int \frac{t^5+8}{2} t^5 dt = \frac{5}{4} \int (t^5+8) t^5 dt = \frac{5}{4} \int (t^{10} + 8t^5) dt =$$

$$= \frac{5}{4} \cdot \frac{t^{11}}{11} + \frac{5}{4} \cdot 8 \cdot \frac{t^6}{6} + C = \frac{5}{44} \sqrt[5]{(2x-1)^{11}} + \frac{15}{8} \sqrt[5]{(2x-1)^6} + C$$

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$$\int \frac{\ln x dx}{x \sqrt{1+\ln x}} = \begin{cases} \ln x = t \\ \frac{1}{x} dx = dt \end{cases} = \int \frac{t+1-1}{\sqrt{1+t}} dt = \int \frac{t+1}{\sqrt{1+t}} dt - \int \frac{dt}{\sqrt{1+t}}$$

$$= \int (1+t)^{\frac{1}{2}} dt - \int (1+t)^{-\frac{1}{2}} dt = \frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(1+t)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3} (1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3} (1+t)^{\frac{1}{2}} \cdot ((1+t)-3) + C = \frac{2}{3} \sqrt{1+\ln x} (\ln x - 2) + C$$

\textcircled{10}_o ^v ISPITNI

$$\int \frac{e^{3x} (10-2e^{3x})}{2e^{6x}-10e^{3x}+12} dx \quad R_j: -\frac{1}{6} \ln |e^{6x}-5e^{3x}+6| + \frac{5}{6} \ln \left| \frac{e^{3x}-3}{e^{3x}-2} \right| + C$$

$$\textcircled{11}_o \int \frac{1}{x^2} \sin \frac{1}{x} dx$$

$$\textcircled{12}_o \int x (2x+3)^7 dx$$

$$\textcircled{13}_o \int \frac{\arcsin x}{\sqrt{1-x^2}} dx$$

$$\textcircled{14}_o \int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}} dx$$

Metoda parcijalne integracije

$$\int u \, dv = uv - \int v \, du, \quad u=u(x), \quad v=v(x)$$

$$\textcircled{1.} \int x e^x \, dx = \left| \begin{array}{l} u=x \\ du=dx \end{array} \right. \quad \left. \begin{array}{l} dv=e^x \, dx \\ v=\int e^x \, dx = e^x \end{array} \right| = x e^x - \int e^x \, dx =$$

Da smo uzelj sujeve $u=e^x$, $dv=x \, dx$ dobili bi komplikovaniji rezultat.

$$\textcircled{2.} \int x^2 \sin 3x \, dx = \left| \begin{array}{l} u=x^2 \\ du=2x \, dx \end{array} \right. \quad \left. \begin{array}{l} dv=\sin 3x \, dx \\ v=\int \sin 3x \, dx = -\frac{1}{3} \cos 3x \end{array} \right| \stackrel{(*)}{=} \quad (*)$$

$$\int \sin 3x \, dx = \left| \begin{array}{l} 3x=t \\ 3 \, dx=dt \\ dt=3 \, dx \end{array} \right| = \frac{1}{3} \int \sin t \, dt = -\frac{1}{3} \cos t + C = -\frac{1}{3} \cos 3x + C$$

$$\stackrel{(*)}{=} -\frac{1}{3} x^2 \cos 3x + \frac{2}{3} \int x \cos 3x \, dx = -\frac{1}{3} x^2 \cos 3x + \frac{2}{3} I_1$$

$$I_1 = \int x \cos 3x \, dx = \left| \begin{array}{l} u=x \\ du=dx \end{array} \right. \quad \left. \begin{array}{l} dv=\cos 3x \, dx \\ v=\frac{1}{3} \sin 3x \end{array} \right| = \frac{1}{3} x \sin 3x - \frac{1}{3} \int \sin 3x \, dx$$

$$= \frac{1}{3} x \sin 3x - \frac{1}{3} \cdot \left(-\frac{1}{3} \right) \cos 3x + C_1 = \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C_1,$$

$$\int x^2 \sin 3x \, dx = -\frac{1}{3} x^2 \cos 3x + \frac{2}{3} x \sin 3x + \frac{2}{27} \cos 3x + C$$

$$\textcircled{3.} \int x^3 \ln x \, dx = \left| \begin{array}{l} u=\ln x \\ du=\frac{1}{x} \, dx \end{array} \right. \quad \left. \begin{array}{l} dv=x^3 \, dx \\ v=\frac{x^4}{4} \end{array} \right| = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C = \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C$$

$$\textcircled{4.} \int \arcsin x \, dx = \left| \begin{array}{l} u=\arcsin x \\ du=\frac{1}{\sqrt{1-x^2}} \, dx \end{array} \right. \quad \left. \begin{array}{l} dv=dx \\ v=x \end{array} \right| = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

$$= x \arcsin x - I_1$$

$$I_1 = \int \frac{x}{\sqrt{1-x^2}} dx = \begin{cases} 1-x^2 = t^2 \\ -2x dx = 2t dt \\ x dx = -t dt \\ t = \sqrt{1-x^2} \end{cases} = \int \frac{-t dt}{t} = - \int dt = -t + C_1 = -\sqrt{1-x^2} + C_1$$

$$\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + C$$

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$$\int \sin(\ln x) dx = \begin{cases} u = \sin(\ln x) \\ du = \cos(\ln x) \cdot \frac{1}{x} dx \end{cases} \quad \begin{cases} dv = dx \\ v = x \end{cases} = x \sin(\ln x) - \int \cos(\ln x) dx$$

$$\int \cos(\ln x) dx = \begin{cases} u = \cos(\ln x) \\ du = -\sin(\ln x) \cdot \frac{1}{x} dx \end{cases} \quad \begin{cases} dv = dx \\ v = x \end{cases} = x \cos(\ln x) + \int \sin(\ln x) dx$$

$$\underline{\underline{\int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx}}$$

$$2 \int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x) \quad /:2$$

$$\int \sin(\ln x) dx = \frac{x}{2} \sin(\ln x) - \frac{x}{2} \cos(\ln x) + C$$

$$(6) \int x \operatorname{arctg} x dx = \begin{cases} u = \operatorname{arctg} x \\ du = \frac{1}{1+x^2} dx \end{cases} \quad \begin{cases} dv = x dx \\ v = \frac{x^2}{2} \end{cases} = \frac{1}{2} x^2 \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$\int \frac{x^2}{x^2+1} dx = \int \frac{x^2+1-1}{x^2+1} dx = \int dx - \int \frac{1}{x^2+1} dx = x - \operatorname{arctg} x + C_1$$

$$\int x \operatorname{arctg} x dx = \frac{1}{2} x^2 \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arctg} x + C$$

$$(7) \int x^2 e^{-2x} dx \quad R_j: -\frac{x^2}{2} e^{-2x} - \frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C$$

$$(8) \int \frac{x^2}{(x^2+1)^2} dx \quad \text{uputaj: } \int x \cdot \frac{x}{(x^2+1)^2} dx = \begin{cases} u = x \\ dv = \frac{x dx}{(x^2+1)^2} \end{cases} = \dots$$

$$(9) \int x^3 \ln(2x+1) dx = \left| \begin{array}{l} u = \ln(2x+1) \\ du = \frac{1}{2x+1} \cdot 2 dx \end{array} \right. \left| \begin{array}{l} dv = x^3 dx \\ v = \frac{x^4}{4} \end{array} \right. =$$

$$= \frac{1}{4} x^4 \ln(2x+1) - \frac{1}{4} \cdot 2 \int \frac{x^4}{2x+1} dx = \frac{1}{4} x^4 \ln(2x+1) - \frac{1}{2} \Big|_1$$

$$\begin{aligned} x^4 : (2x+1) &= \frac{1}{2} x^3 - \frac{1}{4} x^2 + \frac{1}{8} x - \frac{1}{16} \\ &\underline{- \frac{1}{2} x^3} \end{aligned}$$

ostatok $\frac{1}{16}$

$$\underline{- \frac{1}{2} x^3 - \frac{1}{4} x^2}$$

$$x^4 = \left(\frac{1}{2} x^3 - \frac{1}{4} x^2 + \frac{1}{8} x - \frac{1}{16} \right) (2x+1) + \frac{1}{16}$$

$$\frac{1}{4} x^2$$

$$\underline{- \frac{1}{4} x^2 + \frac{1}{8} x}$$

$$-\frac{1}{8} x$$

$$\underline{- \frac{1}{8} x - \frac{1}{16}}$$

$$\text{ostatok } \frac{1}{16}$$

$$I_1 = \int \left(\frac{1}{2} x^3 - \frac{1}{4} x^2 + \frac{1}{8} x - \frac{1}{16} + \frac{\frac{1}{16}}{2x+1} \right) dx =$$

$$= \frac{1}{2} \cdot \frac{x^4}{4} - \frac{1}{4} \cdot \frac{x^3}{3} + \frac{1}{8} \cdot \frac{x^2}{2} - \frac{1}{16} x + \frac{1}{16} \cdot \frac{1}{2} \ln|2x+1| + C_1$$

$$\int x^3 \ln(2x+1) dx = \frac{1}{4} x^4 \ln(2x+1) - \frac{1}{16} x^4 + \frac{1}{24} x^3 - \frac{1}{32} x^2 + \frac{1}{32} x - \frac{1}{64} \ln|2x+1| + C$$

(10) 15 PITNI ZADATAK

$$I = \int \sqrt{x} \ln^2 x dx = \left| \begin{array}{l} u = \ln^2 x \\ du = 2 \ln x \cdot \frac{1}{x} dx \end{array} \right. \left| \begin{array}{l} dv = \sqrt{x} dx \\ v = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3} \sqrt{x^3} \end{array} \right. =$$

$$= \frac{2}{3} \sqrt{x^3} \ln^2 x - 2 \cdot \frac{2}{3} \int \frac{\sqrt{x^3}}{x} \ln x dx = \frac{2}{3} x \sqrt{x} \ln^2 x - \frac{4}{3} \int \sqrt{x} \ln x dx$$

$$\int \sqrt{x} \ln x dx = \left| \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \right. \left| \begin{array}{l} dv = \sqrt{x} dx \\ v = \frac{2}{3} \sqrt{x^3} \end{array} \right. = \frac{2}{3} x \sqrt{x} \ln x - \frac{2}{3} \int \frac{x \sqrt{x}}{x} dx =$$

$$= \frac{2}{3} x \sqrt{x} \ln x - \frac{2}{3} \cdot \frac{2}{3} \sqrt{x^3} + C = \frac{2}{3} x \sqrt{x} \ln x - \frac{4}{9} x \sqrt{x} + C$$

$$I = \frac{2}{3} x \sqrt{x} \ln^2 x - \frac{8}{9} x \sqrt{x} \ln x + \frac{16}{27} x \sqrt{x} + C$$

(11) ISPITNI ZADATAK

$$I = \int e^{3x} \cos 4x \, dx = \left| \begin{array}{l} u = e^{3x} \\ du = 3e^{3x} \, dx \end{array} \quad \begin{array}{l} dv = \cos 4x \, dx \\ v = \frac{1}{4} \sin 4x \end{array} \right| =$$

$$= \frac{1}{4} e^{3x} \sin 4x - \frac{3}{4} \int e^{3x} \sin 4x \, dx = \frac{1}{4} e^{3x} \sin 4x - \frac{3}{4} I_1$$

$$I_1 = \int e^{3x} \sin 4x \, dx = \left| \begin{array}{l} u = e^{3x} \\ du = 3e^{3x} \, dx \end{array} \quad \begin{array}{l} dv = \sin 4x \, dx \\ v = -\frac{1}{4} \cos 4x \end{array} \right| =$$

$$= -\frac{1}{4} e^{3x} \cos 4x + \frac{3}{4} \int e^{3x} \cos 4x \, dx = -\frac{1}{4} e^{3x} \cos 4x + \frac{3}{4} I_2$$

$$I_2 = \frac{1}{4} e^{3x} \sin 4x + \frac{3}{16} e^{3x} \cos 4x - \frac{9}{16} |$$

$$16 | = 4e^{3x} \sin 4x + 3e^{3x} \cos 4x - 9 |$$

$$I = \frac{4e^{3x} \sin 4x + 3e^{3x} \cos 4x}{25} + C$$

(12) ISPITNI ZADATAK

$$I = \int x^2 e^{3x} \, dx = \left| \begin{array}{l} u = x^2 \\ du = 2x \, dx \end{array} \quad \begin{array}{l} dv = e^{3x} \, dx \\ v = \frac{1}{3} e^{3x} \end{array} \right| = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} \, dx$$

$$\int x e^{3x} \, dx = \left| \begin{array}{l} u = x \\ du = dx \end{array} \quad \begin{array}{l} dv = e^{3x} \, dx \\ v = \frac{1}{3} e^{3x} \end{array} \right| = \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} \, dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C$$

$$I = \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C$$

(13) $I = \int \frac{\arcsin \frac{x}{2}}{\sqrt{2-x}} \, dx$

Uputa: $u = \arcsin \frac{x}{2}$ $\frac{du}{dx} = \frac{1}{\sqrt{1-\frac{x^2}{4}}} \cdot \frac{1}{2}$ $R_j: -2\sqrt{2-x} \cdot \arcsin \frac{x}{2} + 4\sqrt{2+x} + C$

(14) $I = \int e^x \sin x \, dx$ Uputa: $u = e^x$ $R_j: \frac{e^x(\sin x - \cos x)}{2} + C$

(15) $\int x^5 \ln x \, dx$

(17) $\int \frac{\ln(x^2+1)}{x^3} \, dx$

(16) $\int \frac{x^2 \, dx}{\cos^2 x}$

(18) $\int (\arcsin x)^2 \, dx$

Integralacija racionalnih funkcija

$P_n(x)$ - polinom stepena n

$Q_m(x)$ - polinom stepena m

npr. $p(x) = 5x^7 - 3x^2 + x + 8$, polinom 7-og stepena

Racionalna f-ja je količnik dva polinoma.

$s(x) = \frac{P_n(x)}{Q_m(x)}$. Za $n < m$ $s(x)$ je prava racionalna f-ja

Racionalnu f-ju razlažemo na proste razlomke.

Prosti razlomci su oblika:

$$\frac{A}{(ax+b)^n}, \quad \frac{Bx+C}{(ax^2+bx+c)^n}, \quad n \in \mathbb{N}$$

Izračunajte integrale:

1. $\int \frac{x}{(x-1)(x+1)^2} dx$

Lj. $\frac{x}{(x-1)(x+1)^2} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{(x+1)^2} \quad | \cdot (x-1)(x+1)^2$

$$x = a(x+1)^2 + b(x-1)(x+1) + c(x-1)$$

$$x = a(x^2+2x+1) + b(x^2-1) + c(x-1)$$

$$x = (a+b)x^2 + (2a+c)x + (a-b-c)$$

$$a+b = 0 \quad (1)$$

$$2a + c = 1 \quad (2)$$

$$a-b-c = 0 \quad (3)$$

$$\begin{aligned} (1) \quad a+b=0 \\ (2)+(3) \quad 3a-b=1 \\ \hline 4a=1 \end{aligned} \quad a=\frac{1}{4}$$

$$\frac{x}{(x-1)(x+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{2}}{(x+1)^2}$$

$$I = \frac{1}{4} \int \frac{dx}{x-1} - \frac{1}{4} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{dx}{(x+1)^2} =$$

$$= \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| + \frac{1}{2} \left[\frac{(x+1)^{-1}}{-1} \right] + C = \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2(x+1)} + C$$

(2.) $I = \int \frac{8x-31}{x^2-9x+14} dx$, $x^2-9x+14=0$, $x_{1,2} = \frac{9 \pm 5}{2}$

$D = 81-56$, $x_1 = 2$, $x_2 = 7$

Rj.: $\frac{8x-31}{x^2-9x+14} = \frac{8x-31}{(x-2)(x-7)} = \frac{a}{x-2} + \frac{b}{x-7}$ / $(x-2)(x-7)$

$8x-31 = a(x-7) + b(x-2)$

$8x-31 = (a+b)x + (-7a-2b)$

$\begin{array}{r} a+b=8 \\ -7a-2b=-31 \\ \hline 2a+2b=16 \\ -7a-2b=-31 \\ \hline -5a=-15 \end{array}$

$a=3$, $b=5$

$\frac{8x-31}{x^2-9x+14} = \frac{3}{x-2} + \frac{5}{x-7}$

$$I = 3 \int \frac{dx}{x-2} + 5 \int \frac{dx}{x-7} = 3 \ln|x-2| + 5 \ln|x-7| + C$$

(3.) $I = \int \frac{dx}{x^3-2x^2+x}$, $x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2$

Rj.: $\frac{1}{x^3-2x^2+x} = \frac{1}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$ / $x(x-1)^2$

$$1 = a(x-1)^2 + b x(x-1) + c x$$

$$1 = a(x^2-2x+1) + b(x^2-x) + cx$$

$$1 = (a+b)x^2 + (-2a-b+c)x + a$$

$$\begin{array}{rcl} a+b & = 0 \\ -2a-b+c & = 0 \\ \hline a & = 1 \\ b & = -1 & \Rightarrow c = 1 \end{array}$$

$$\frac{1}{x^3-2x^2+x} = \frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}$$

$$I = \int \frac{dx}{x} - \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} = \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$$

(4.) $I = \int \frac{x^3+x+1}{x^4-1} dx$, $x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$

Rj.: $\frac{x^3+x+1}{x^4-1} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}$ / $(x-1)(x+1)(x^2+1)$

$$x^3 + x + 1 = a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x-1)(x+1)$$

$$x^3 + x + 1 = a(x^3 + x + x^2 + 1) + b(x^3 + x - x^2 - 1) + (cx+d)(x^2 - 1)$$

$$x^3 + x + 1 = a(x^3 + x^2 + x + 1) + b(x^3 - x^2 + x - 1) + c(x^3 - x) + d(x^2 - 1)$$

$$x^3 + x + 1 = (a+b+c)x^3 + (a-b+d)x^2 + (a+b-c)x + (a-b-d)$$

$$\begin{array}{rcl} a+b+c & = 1 & (1) \\ a-b+d & = 0 & (2) \end{array}$$

$$\begin{array}{rcl} a+b-c & = 1 & (3) \\ a-b-d & = 1 & (4) \end{array}$$

$$(1)-(4): 2b+c+d=0$$

$$(2)-(4): 2d=-1 \Rightarrow d=-\frac{1}{2}$$

$$(3)-(4) \quad \underline{\underline{2b-c+d=0}}$$

$$2b=\frac{1}{2}$$

$$\begin{array}{rcl} 2b+c & = \frac{1}{2} \\ -2b-c & = \frac{1}{2} \end{array}$$

$$\underline{\underline{b=\frac{1}{4}}}$$

$$\frac{x^3+x+1}{x^4-1} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+1} + \frac{-\frac{1}{2}}{x^2+1} \quad 2c=0 \Rightarrow c=0 \quad a=1-\frac{1}{4}=0$$

$$I = \frac{3}{4} \int \frac{dx}{x-1} + \frac{1}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{x^2+1} = \frac{3}{4} \ln|x-1| + \frac{1}{4} \ln|x+1| - \frac{1}{2} \arctg x + C$$

$$\underline{\underline{a=\frac{3}{4}}}$$

(5) $\int \frac{x^3+3x-5}{(x^2+1)(x^2-x+1)} dx$, $x^2+1=0 \quad x^2-x+1=0$
 $D=-4<0 \quad D=1-4<0$
 $x^2+1; x^2-x+1$ se ne mogu razlavititi

Rj. $\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2-x+1} /((x^2+1)(x^2-x+1))$

$$x^3 + 3x - 5 = (ax+b)(x^2-x+1) + (cx+d)(x^2+1)$$

$$x^3 + 3x - 5 = a(x^3 - x^2 + x) + b(x^2 - x + 1) + c(x^3 + x) + d(x^2 + 1)$$

$$x^3 + 3x - 5 = (a+c)x^3 + (-a+b+d)x^2 + (a-b+c)x + (b+d)$$

$$\begin{array}{rcl} a+c & = 1 & (1) \\ -a+b+d & = 0 & (2) \\ a-b+c & = 3 & (3) \\ b+d & = -5 & (4) \end{array}$$

$$(1) \quad a+c = 1$$

$$(2)-(4) -a = 5 \Rightarrow a = -5$$

$$(3) \quad a-b+c = 3$$

$$d = -3$$

$$\underline{\underline{c=6}}$$

$$-5 - b + 6 = 3 \Rightarrow b = -2$$

$$\frac{x^3-3x-5}{(x^2+1)(x^2-x+1)} = \frac{-5x-2}{x^2+1} + \frac{6x-3}{x^2-x+1}$$

$$\overline{(x^2+1)} = 2x$$

$$-5x-2 = 2 \cdot 2x + 3$$

$$-5x-2 = 2 \cdot \frac{-5}{2}x - 2$$

$$I = \int \frac{-5x-2}{x^2+1} dx + 3 \int \frac{2x-1}{x^2-x+1} dx = I_1 + 3I_2$$

$$I_1 = \int \frac{2 \cdot \frac{-5}{2} x - 2}{x^2 + 1} dx = -\frac{5}{2} \int \frac{2x}{x^2 + 1} dx - 2 \int \frac{dx}{x^2 + 1} = -\frac{5}{2} \ln|x^2 + 1| - 2 \arctan(x^2 + 1) + C_1$$

$$I_2 = \int \frac{2x-1}{x^2-x+1} dx = \left| \begin{array}{l} x^2-x+1=t \\ (2x-1)dx=dt \end{array} \right| = \int \frac{dt}{t} = \ln|x^2-x+1| + C_2$$

$$I = -\frac{5}{2} \ln|x^2+1| - 2 \arctan(x^2+1) + 3 \ln|x^2-x+1| + C$$

6.) $I = \int \frac{3}{x(x+1)^3} dx$

$$\text{Rj: } \frac{3}{x(x+1)^3} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2} + \frac{d}{(x+1)^3} \quad | \cdot x(x+1)^3$$

$$3 = a(x+1)^3 + b x(x+1)^2 + c x(x+1) + d x$$

Ako uvjetimo $x=0$ u gornju jednakost dobijemo $3=a$

$$\text{tj. } a=3 : \quad 3 = 3(x+1)^3 + b x(x+1)^2 + c x(x+1) + d x$$

$$\text{za } x=-1 \text{ imamo } 3 = d \cdot (-1) \Rightarrow d = -3$$

$$3 = 3(x+1)^3 + b x(x+1)^2 + c x(x+1) - 3 x$$

$$\text{za } x=1 : \quad 3 = 3 \cdot 2^3 + b \cdot 1 \cdot 2^2 + c \cdot 1 \cdot 2 - 3 \cdot 1 \Rightarrow 4b + 2c = -18$$

$$\text{za } x=-2 : \quad 3 = 3 \cdot (-1)^3 + b(-2)(-1)^2 + c(-2)(-1) - 3 \cdot (-2) \Rightarrow -2b + 2c = 0 \quad \left. \right\}$$

$$\Rightarrow b=c=-3$$

$$\frac{3}{x(x+1)^3} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-3}{(x+1)^2} + \frac{-3}{(x+1)^3} \quad \left. \begin{aligned} \int \frac{dx}{(x+1)^3} &= \int (x+1)^{-3} dx = \frac{(x+1)^{-2}}{-2} + C_1 \\ &= -\frac{1}{2(x+1)^2} + C_1 \end{aligned} \right]$$

$$I = 3 \int \frac{dx}{x} - 3 \int \frac{dx}{x+1} - 3 \int \frac{dx}{(x+1)^2} - 3 \int \frac{dx}{(x+1)^3} = 3 \ln|x| - 3 \ln|x+1| + \frac{3}{x+1} + \frac{3}{2(x+1)^2} + C$$

7.) $\int \frac{2x-3}{(x^2-3x+2)^3} dx \quad \text{Rj: } -\frac{1}{2(x^2-3x+2)^2} + C$

8.) $\int \frac{x^3+x+1}{x(x^2+1)} dx \quad \text{Rj: } x + \ln \left| \frac{x}{\sqrt{x^2+1}} \right| + C$

9.) $\int \frac{x^4}{x^4-1} dx \quad \text{Rj: } x + \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \arctan x + C$

10.) $\int \frac{dx}{(x^2-4x+3)(x^2+4x+5)} \quad \text{Rj: } \frac{1}{52} \ln|x-3| - \frac{1}{20} \ln|x-1| + \frac{1}{65} \ln(x^2+4x+5) + \frac{7}{130} \arctan(x) + C$

$$11_0 \quad I = \int \frac{-x^5 - 2x^2 + 2}{(x-1)(x^2 + 2x + 1)} dx$$

$$R_j: (x-1)(x^2 + 2x + 1) = x^3 + 2x^2 + x - x^2 - 2x - 1 = x^3 + x^2 - x - 1$$

$$\begin{aligned} & (-x^5 - 2x^2 + 2) : (x^3 + x^2 - x - 1) = -x^2 + x - 2 - \frac{x}{x^3 + x^2 - x - 1} \\ & - \frac{-x^5 - x^4 + x^3 + x^2}{=} \\ & \quad \frac{x^4 - x^3 - 3x^2 + 2}{=} \\ & \quad - \frac{x^4 + x^3 - x^2 - x}{=} \\ & \quad \frac{-2x^3 - 2x^2 + x + 2}{=} \\ & \quad - \frac{-2x^3 - 2x^2 + 2x + 2}{=} \end{aligned}$$

$$I = \int \left(-x^2 + x - 2 - \frac{x}{(x-1)(x^2 + 2x + 1)} \right) dx$$

$$= - \int x^2 dx + \int x dx - 2 \int dx - \int \frac{x}{(x-1)(x+1)^2} dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} - 2x - I_1, \text{ integral } I_1 \text{ smo odredili u zadatku 1}$$

$$I = -\frac{x^3}{3} + \frac{x^2}{2} - 2x - \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{2(x+1)} + C$$

$$12_0 \quad I = \int \frac{x^5 - 2x^3 + x - 1}{x^3 - 2x^2 + x} dx$$

$$\begin{aligned} & (x^5 - 2x^3 + x - 1) : (x^3 - 2x^2 + x) = x^2 + 2x + 1 - \frac{1}{x^3 - 2x^2 + x} \\ & - \frac{x^5 - 2x^4 + x^3}{=} \\ & \quad \frac{2x^4 - 3x^3 + x - 1}{=} \\ & \quad - \frac{2x^4 - 4x^3 + 2x^2}{=} \\ & \quad \frac{x^3 - 2x^2 + x - 1}{=} \\ & \quad - \frac{x^3 - 2x^2 + x}{=} \end{aligned}$$

$$\begin{aligned} & I = \int \left(x^2 + 2x + 1 - \frac{1}{x^3 - 2x^2 + x} \right) dx \\ & = \int x^2 dx + 2 \int x dx + \int dx - \int \frac{dx}{x(x-1)^2} \end{aligned}$$

$$= \frac{x^3}{3} + 2 \cdot \frac{x^2}{2} + x - I_2, \text{ integral } I_2 \text{ smo odredili u zadatku 3.}$$

$$I = \frac{x^3}{3} + x^2 + x - \ln|x| + \ln|x-1| + \frac{1}{x-1} + C$$

$$+ \frac{11}{2\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C$$

$$13_0 \quad I = \int \frac{x^5 + 2x^3 - 4}{(x-2)(x^2 + x + 1)} dx$$

$$R_j: I = \frac{x^3}{8} + \frac{x^2}{2} + 4x + \frac{49}{7} \ln|x-2| + \frac{5}{14} \ln|x^2 + x + 1| +$$

$$14. \int \frac{x^5 - 60x^3 + 73x^2 + 171}{x^2 - 9x + 14} dx$$

$$\begin{aligned}
 R_j: & (x^5 - 60x^3 + 73x^2 + 171) : (x^2 - 9x + 14) = x^3 + 9x^2 + 7x + 10 + \frac{-8x + 31}{x^2 - 9x + 14} \\
 & - \frac{x^5 - 9x^4 + 14x^3}{x^2 - 9x + 14} \\
 & = \frac{9x^4 - 74x^3 + 73x^2 + 171}{x^2 - 9x + 14} \\
 & - \frac{9x^4 - 81x^3 + 126x^2}{x^2 - 9x + 14} \\
 & = \frac{7x^3 - 53x^2 + 171}{x^2 - 9x + 14} \\
 & - \frac{7x^3 - 63x^2 + 98x}{x^2 - 9x + 14} \\
 & = \frac{10x^2 - 98x + 171}{x^2 - 9x + 14} \\
 & - \frac{10x^2 - 90x + 140}{x^2 - 9x + 14} \\
 & = \frac{-8x + 31}{x^2 - 9x + 14}
 \end{aligned}$$

$$\begin{aligned}
 I &= \frac{x^4}{4} + 9 \cdot \frac{x^3}{3} + 7 \cdot \frac{x^2}{2} + 10x - \Big|_3, \text{ integral } I_3 \text{ smo odredili} \\
 & \text{u zadatku broj 2.} \\
 I &= \frac{1}{4}x^4 + 3x^3 + \frac{7}{2}x^2 + 10x - 3\ln|x-2| + 5\ln|x-7| + C
 \end{aligned}$$

$$15. \int \frac{x^7 - 2x^6 + x^5 + x^4 + 2x^2}{x^4 - 1} dx$$

$$\begin{aligned}
 R_j: & (x^7 - 2x^6 + x^5 + x^4 + 2x^2) : (x^4 - 1) = x^3 - 2x^2 + x + 1 + \frac{x^3 + x + 1}{x^4 - 1} \\
 & - \frac{x^7 - x^3}{x^4 - 1} \\
 & = \frac{-2x^6 + x^5 + x^4 + x^3 + 2x^2}{x^4 - 1} \\
 & - \frac{-2x^6 + 2x^2}{x^4 - 1} \\
 & = \frac{x^5 + x^4 + x^3}{x^4 - 1} \\
 & - \frac{x^5 - x}{x^4 - 1} \\
 & - \frac{x^4 - 1}{x^3 + x + 1}
 \end{aligned}$$

$$\begin{aligned}
 I &= \int (x^3 - 2x^2 + x + 1 + \frac{x^3 + x + 1}{x^4 - 1}) dx = \\
 & = \int x^3 dx - 2 \int x^2 dx + \int x dx + \int dx + \\
 & + \int \frac{x^3 + x + 1}{x^4 - 1} dx
 \end{aligned}$$

$$I = \frac{x^4}{4} - 2 \cdot \frac{x^3}{3} + \frac{x^2}{2} + x + \Big|_4, \text{ integral } I_4 \text{ smo odredili u zadatku 4}$$

$$I = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + x + \frac{3}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| - \frac{1}{2}\arctan x + C$$

$$\begin{aligned}
 16. \int \frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2} dx & R_j: I = \frac{x^2}{2} - 2x + \frac{2}{x} + 2\ln|x^2 + 2x + 2| - \\
 & - 2\arctan(x+1) + C
 \end{aligned}$$

$$17. \int \frac{-2x^7 - x^6 - x^3 + 6x^2 - x}{(x^2+1)(x^2-x+1)} dx$$

$$\begin{aligned}
 R_j: & (x^2+1) \cdot (x^2-x+1) = x^4 - x^3 + x^2 + x^2 - x + 1 = x^4 - x^3 + 2x^2 - x + 1 \\
 & (-2x^7 - x^6 - x^3 + 6x^2 - x) : (x^4 - x^3 + 2x^2 - x + 1) = -2x^3 - 3x^2 + x + 5 + \frac{x^3 + 3x - 5}{x^4 - x^3 + 2x^2 - x + 1} \\
 & = -2x^7 + 2x^6 - 4x^5 + 2x^4 - 2x^3 \\
 & \quad - \frac{-3x^6 + 4x^5 - 2x^4 + x^3 + 6x^2 - x}{-3x^6 + 3x^5 - 6x^4 + 3x^3 - 3x^2} \\
 & \quad = \frac{x^5 + 4x^4 - 2x^3 + 3x^2 - x}{x^5 - x^4 + 2x^3 - x^2 + x} \\
 & \quad = \frac{5x^4 - 4x^3 + 10x^2 - 2x}{5x^4 - 5x^3 + 10x^2 - 5x + 5} \\
 & \quad \quad \quad x^3 + 3x - 5
 \end{aligned}$$

$$\begin{aligned}
 I &= -2 \cdot \frac{x^4}{4} - 3 \cdot \frac{x^3}{3} + \frac{x^2}{2} + 5x + \frac{1}{5}, \text{ integral } I_5 \text{ smo odredili u} \\
 &\quad \text{zadataku broj 5} \\
 I &= -\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2 + 5x - \frac{5}{2}\ln|x^2+1| - 2 \arctg(x^2+1) + 3\ln|x^2-x+1| + C
 \end{aligned}$$

$$18. \int \frac{x^5 + x^4 - 8}{x^3 - 4x} dx$$

$$19. \int \frac{x^5 + 2}{x^3 - 1} dx$$

Integracija iracionalnih f-ja

Iracionalne f-je su one f-je koje su izražene preko korijena polinoma.

I metoda - Metoda Ostrogradskog

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}} dx = Q_{n-1}(x) \cdot \sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\begin{aligned}
 10. \quad I &= \int \frac{3x^3}{\sqrt{x^2+4x+5}} dx \quad R_j: \\
 &= (ax^2+bx+c)\sqrt{x^2+4x+5} + \lambda \int \frac{dx}{\sqrt{x^2+4x+5}} \quad / \frac{d}{dx} \\
 & \frac{3x^3}{\sqrt{x^2+4x+5}} = (2ax+5)\sqrt{x^2+4x+5} + (ax^2+bx+c) \cdot \frac{2x+4}{2\sqrt{x^2+4x+5}} + \lambda \cdot \frac{1}{\sqrt{x^2+4x+5}} \quad / \cdot \sqrt{x^2+4x+5}
 \end{aligned}$$

$$3x^3 = (2ax+b)(x^2+4x+5) + (ax^2+bx+c)(x+2) + \lambda$$

$$3x^3 = \underline{2ax^3} + \underline{8ax^2} + \underline{10ax} + b(x^2+4x+5) + \underline{ax^2} + \underline{bx^2} + \underline{cx} + 2ax^2 + 2bx + 2c + \lambda$$

$$3x^3 = (2a+a)x^3 + (8a+5+b+b+2a)x^2 + (10a+4b+c+2b)x + 5b+2c+\lambda$$

$$3x^3 = 3ax^3 + (10a+2b)x^2 + (10a+6b+c)x + 5b+2c+\lambda$$

$$\text{uz } x^3: \quad 3a = 3 \Rightarrow a=1$$

$$\text{uz } x^2: \quad 10a+2b=0 \Rightarrow b=-5$$

$$\text{uz } x: \quad 10a+6b+c=0 \Rightarrow c=20$$

$$\text{uz } x^0: \quad \underline{5b+2c+\lambda=0} \quad \lambda = -15$$

$$I = (x^2 - 5x + 20) \sqrt{x^2 + 4x + 5} - 15 \int \frac{dx}{\sqrt{x^2 + 4x + 5}}$$

$$x^2 + 4x + 5 = x^2 + 2x \cdot 2 + 2^2 - 2^2 + 5 = (x+2)^2 + 1$$

$$I = (x^2 - 5x + 20) \sqrt{x^2 + 4x + 5} - 15 \int \ln |x+2 + \sqrt{(x+2)^2 + 1}| + C$$

$$(2) I = \int \frac{3x+1}{\sqrt{2x^2-x+1}} dx \stackrel{R:}{=} a \sqrt{2x^2-x+1} + \lambda \int \frac{dx}{\sqrt{2x^2-x+1}} / \frac{1}{dx}$$

$$\frac{3x+1}{\sqrt{2x^2-x+1}} = a \cdot \frac{4x-1}{2\sqrt{2x^2-x+1}} + \lambda \cdot \frac{1}{\sqrt{2x^2-x+1}} / \cdot 2\sqrt{2x^2-x+1}$$

$$6x+2 = a(4x-1) + 2\lambda \Rightarrow 4a = 6 \quad a = \frac{6}{4} = \frac{3}{2}$$

$$\frac{2\lambda-a=2}{\lambda=\frac{7}{4}}$$

$$I = \frac{3}{2} \sqrt{2x^2-x+1} + \frac{7}{4} \int \frac{dx}{\sqrt{2x^2-x+1}} = \frac{3}{2} \sqrt{2x^2-x+1} + \frac{7}{4} I_1$$

$$2x^2 - x + 1 = 2\left(x^2 - \frac{1}{2}x + \frac{1}{2}\right) = 2\left(x^2 - 2 \cdot x \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{1}{2}\right) = 2\left[\left(x - \frac{1}{4}\right)^2 + \frac{7}{16}\right]$$

$$I_1 = \int \frac{dx}{\sqrt{2\left[\left(x-\frac{1}{4}\right)^2 + \frac{7}{16}\right]}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x-\frac{1}{4}\right)^2 + \frac{7}{16}}} = \begin{cases} x - \frac{1}{4} = \frac{\sqrt{7}}{4}t & | \cdot 4 \\ dx = \frac{\sqrt{7}}{4}dt & \\ 4x-1 = \sqrt{7}t & \end{cases} = \frac{1}{\sqrt{2}} \int \frac{\frac{\sqrt{7}}{4}dt}{\sqrt{\frac{7}{16}t^2 - \frac{7}{16}}} = \frac{1}{\sqrt{2}} \cdot \frac{\frac{\sqrt{7}}{4}dt}{\sqrt{\frac{7}{16}(t^2-1)}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{7}}{4} \cdot \frac{1}{\sqrt{7}} \int \frac{dt}{\sqrt{t^2-1}} = \frac{1}{\sqrt{2}} \ln |t + \sqrt{t^2+1}| + C = \frac{1}{\sqrt{2}} \ln \left| \frac{4x-1}{\sqrt{7}} + \sqrt{\left(\frac{4x-1}{\sqrt{7}}\right)^2 + 1} \right| + C$$

$$I = \frac{3}{2} \sqrt{2x^2 - x + 1} + \frac{7}{4\sqrt{2}} \ln \left| \frac{4x-1}{\sqrt{7}} + \sqrt{\left(\frac{4x-1}{\sqrt{7}} \right)^2 + 1} \right| + C$$

Metodom Ostrogradskog rješavamo; integrante obliku

$$\int \sqrt{ax^2 + bx + c} dx = \int \frac{ax^2 + bx + c}{\sqrt{ax^2 + bx + c}} dx$$

$$(3) \int \sqrt{x^2 + 1} dx \stackrel{f_j.}{=} (ax + b)\sqrt{x^2 + 1} + \lambda \int \frac{dx}{\sqrt{x^2 + 1}} \quad \left| \frac{d}{dx} \right.$$

$$\sqrt{x^2 + 1} = a\sqrt{x^2 + 1} + (ax + b) \frac{2x}{2\sqrt{x^2 + 1}} + \lambda \cdot \frac{1}{\sqrt{x^2 + 1}} \quad \left| \sqrt{x^2 + 1} \right.$$

$$x^2 + 1 = \underline{a(x^2 + 1)} + \underline{(ax^2 + bx)} + \lambda \equiv$$

$$x^2: \quad a + a = 1 \quad x: \quad b = 0 \quad x^2: \quad a + \lambda = 1$$

$$2a = 1 \quad \lambda = 1 - \frac{1}{2}$$

$$a = \frac{1}{2} \quad \lambda = \frac{1}{2}$$

$$\int \sqrt{x^2 + 1} dx = \frac{1}{2}x\sqrt{x^2 + 1} + \frac{1}{2}\ln|x + \sqrt{x^2 + 1}| + C$$

$$(4) \int \frac{2x^2 - 3x}{\sqrt{x^2 - 2x + 5}} dx$$

$$(5) \int \sqrt{x^2 - 2x - 1} dx$$

$$(6) \int \frac{x^5}{\sqrt{1-x^2}} dx \quad f_j. - \frac{8+4x^2+3x^4}{15} \sqrt{1-x^2}$$

$$(7) \int x^4 \sqrt{1-x^2} dx \quad \text{uputa: } \int \frac{x^4(1-x^2)}{\sqrt{1-x^2}} dx$$

II metoda $\int R(x, \sqrt[n]{ax+b}) dx$, suposta $ax+b=t^n$

$$(1) \int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}} = \begin{cases} 2x-1 = t^4 \\ 2dx = 4t^3 dt \\ dx = \frac{1}{2}t^3 dt \\ t = \sqrt[4]{2x-1} \end{cases} \quad \left| :2 \right. = 2 \int \frac{t^3 dt}{\sqrt{t^4} - \sqrt[4]{t^4}} = 2 \int \frac{t^3 dt}{t^2 - t} =$$

$$= 2 \int \frac{t^2}{t-1} dt = 2 \int \frac{t^2 - 1 + 1}{t-1} dt = 2 \int \frac{t^2 - 1}{t-1} dt + \int \frac{dt}{t-1} =$$

$$= 2 \int \frac{(t-1)(t+1)}{t-1} dt + 2 \int \frac{dt}{t-1} = 2 \int (t-1) dt + 2 \int \frac{1}{t-1} dt = 2 \cdot \frac{t^2}{2} - 2t + 2 \ln|t-1| + C$$

$$= \sqrt[4]{(2x-1)^2} + 2\sqrt[4]{(2x-1)} + 2|\ln|\sqrt[4]{2x-1}-1|| + C = \sqrt{2x-1} + 2\sqrt[4]{2x-1} + 2|\ln|\sqrt[4]{2x-1}-1|| + C$$

(2)

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}} dx = \left| \begin{array}{l} x=t^4 \\ dx=4t^3 dt \\ t=\sqrt[4]{x} \end{array} \right| = \int \frac{\sqrt[3]{1+t}}{t^2} \cdot 4t^3 dt =$$

$$= 4 \int t \sqrt[3]{1+t} dt = \left| \begin{array}{l} 1+t=u^3 \\ dt=3u^2 du \\ u=\sqrt[3]{1+t} \\ t=u^3-1 \end{array} \right| = 4 \int (u^2-1) \sqrt[3]{u^3} \cdot 3u^2 du =$$

$$= 12 \int (u^6-u^3) du = 12 \left(\frac{u^7}{7} - \frac{u^4}{4} \right) + C = \frac{12}{7} u^7 - 3u^4 + C =$$

$$= \frac{12}{7} \sqrt[3]{(1+t)^7} - 3 \sqrt[3]{(1+t)^4} + C = \frac{12}{7} \sqrt[3]{(1+\sqrt[4]{x})^7} - 3 \sqrt[3]{(1+\sqrt[4]{x})^4} + C$$

(3)

$$\int \sqrt{\frac{x+1}{x-1}} dx$$

ako stavim vježbu $\frac{x+1}{x-1} = t^2$ dobijemo

$$x+1 = t^2(x-1)$$

$$x+1 = t^2x - t^2$$

$$x - t^2x = -t^2 - 1$$

$$(1-t^2)x = -t^2 - 1 \quad | : (1-t^2)$$

$$x = \frac{t^2+1}{t^2-1}$$

$$dx = d\left(\frac{t^2+1}{t^2-1}\right)$$

$$dx = \frac{2t(t^2-1)-(t^2+1)\cdot 2t}{(t^2-1)^2} dt$$

$$dx = \frac{2t^3-2t-2t^3-2t}{(t^2-1)^2} dt$$

$$I = \int t \cdot \frac{-4t}{(t^2-1)^2} dt = -4 \int t \cdot \frac{t}{(t^2-1)^2} dt = -4 I_1$$

$$dx = \frac{-4t}{(t^2-1)^2} dt$$

$$I_1 = \int t \cdot \frac{t}{(t^2-1)^2} dt = \left| \begin{array}{l} u=t \\ du=dt \end{array} \right. \quad dv = \frac{t}{(t^2-1)^2} dt$$

$$v = \int \frac{t}{(t^2-1)^2} dt = \left| \begin{array}{l} t^2-1=z \\ 2t dt = dz \\ t dt = \frac{dz}{2} \end{array} \right| = \int \frac{\frac{1}{2} dz}{z^2} =$$

$$= \frac{1}{2} \int z^{-2} dz = \frac{1}{2} \cdot \frac{z^{-1}}{-1} = -\frac{1}{2} \cdot \frac{1}{z} = -\frac{1}{2(t^2-1)}$$

$$= -\frac{1}{2} \cdot \frac{t}{t^2-1} + \frac{1}{2} \int \frac{dt}{t^2-1} = -\frac{1}{2} \cdot \frac{t}{t^2-1} + \frac{1}{2} \cdot \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C,$$

$$I = -4 \cdot \left(-\frac{1}{2} \cdot \frac{t}{t^2-1} + \frac{1}{4} \ln \left| \frac{t-1}{t+1} \right| \right) + C =$$

$$= 2 \cdot \frac{\sqrt{\frac{x+1}{x-1}}}{\frac{x+1}{x-1} - 1} - 1 \ln \left| \frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1} \right| + C$$

$$\textcircled{4}_o \quad \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} \quad R_j: \quad 6\sqrt[6]{x} - 3\sqrt[3]{x} + 2\sqrt{x} - 6 \ln(1 + \sqrt[6]{x}) + C$$

$$\textcircled{5}_o \quad \int \frac{\sqrt{x+1} + 2}{(x+1)^2 - \sqrt{x+1}} dx \quad R_j: \ln \left| \frac{(\sqrt{x+1} - 1)^2}{x+2 + \sqrt{x+1}} \right| - \frac{2}{\sqrt{3}} \arctan \frac{2\sqrt{x+1} + 1}{\sqrt{3}} + C$$

$$\textcircled{6}_o \quad \int \frac{dx}{(2-x)\sqrt{1-x}} \quad R_j: -2 \arctan \sqrt{1-x} + C$$

III metoda - integrali koji se mogu riješiti racionaliziraju

$$\textcircled{1}_o \quad I = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx$$

$$R_j: \quad I = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} \cdot \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} dx = \int \frac{x+1 - 2 \cdot \sqrt{x+1} \cdot \sqrt{x-1} + x-1}{x+1 - (x-1)} dx = \\ -\frac{1}{2} \int (2x - 2\sqrt{x^2-1}) dx = \int x dx - \int \sqrt{x^2-1} dx = \frac{x^2}{2} - I_1$$

$$I_1 = \int \sqrt{x^2-1} dx = (ax+b)\sqrt{x^2-1} + \lambda \int \frac{dx}{\sqrt{x^2-1}} \quad / \frac{d}{dx}$$

$$\sqrt{x^2-1} = a \cdot \sqrt{x^2-1} + (ax+b) \frac{2x}{2\sqrt{x^2-1}} + \lambda \cdot \frac{1}{\sqrt{x^2-1}} \quad / \sqrt{x^2-1}$$

$$x^2-1 = a(x^2-1) + (ax^2+bx) + \lambda$$

$$x^2: \quad a+a=1 \Rightarrow a=\frac{1}{2}, \quad x: \quad b=0, \quad x^2: \quad -a+\lambda=-1 \quad \lambda = -\frac{1}{2}$$

$$I_1 = \int \sqrt{x^2 - 1} dx = \frac{1}{2} \sqrt{x^2 - 1} - \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C$$

$$I = \frac{1}{2} x^2 - \frac{1}{2} \sqrt{x^2 - 1} + \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C$$

$$\textcircled{2}_0 \int \frac{x}{\sqrt{x+2} + \sqrt{x+1}} dx$$

$$\textcircled{2}_j \int \frac{x}{\sqrt{x+2} + \sqrt{x+1}} \cdot \frac{\sqrt{x+2} - \sqrt{x+1}}{\sqrt{x+2} - \sqrt{x+1}} dx = \int \frac{x\sqrt{x+2} - x\sqrt{x+1}}{x+2 - (x+1)} dx = \\ = \int (x\sqrt{x+2} - x\sqrt{x+1}) dx = \int x\sqrt{x+2} dx - \int x\sqrt{x+1} dx = I_1 - I_2$$

$$I_1 = \int x\sqrt{x+2} dx = \left| \begin{array}{l} x+2 = t^2 \\ \downarrow x = t^2 - 2 \\ x = t^2 - 2 \\ t = \sqrt{x+2} \end{array} \right| = \int (t^2 - 2) \cdot t \cdot 2t dt = 2 \int (t^4 - 2t^2) dt = 2 \cdot \frac{t^5}{5} - 4 \cdot \frac{t^3}{3} + C_1 \\ = \frac{2}{5} \sqrt{(x+2)^5} - \frac{4}{3} \sqrt{(x+2)^3} + C_1$$

$$I_2 = \int x\sqrt{x+1} dx = \left| \begin{array}{l} x+1 = t^2 \\ x = t^2 - 1 \\ dx = 2t dt \\ t = \sqrt{x+1} \end{array} \right| = \int (t^2 - 1) \cdot t \cdot 2t dt = 2 \int (t^4 - t^2) dt = \\ = 2 \cdot \frac{t^5}{5} - 2 \cdot \frac{t^3}{3} + C_2 = \frac{2}{5} \sqrt{(x+1)^5} - \frac{2}{3} \sqrt{(x+1)^3} + C_2$$

$$I = \frac{2}{5} \sqrt{(x+2)^5} - \frac{4}{3} \sqrt{(x+2)^3} - \frac{2}{5} \sqrt{(x+1)^5} + \frac{2}{3} \sqrt{(x+1)^3} + e$$

$$\textcircled{3}_0 \int \frac{dx}{x - \sqrt{x^2 - 1}} \quad \textcircled{3}_j \frac{1}{2} x^2 + \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + e$$

$$\textcircled{4}_0 \int \frac{dx}{\sqrt{x^2 + 1} - x}$$

$$\textcircled{5}_0 \int \frac{\sqrt{x^2 + 2x + 2}}{x} dx$$

$$\text{IV metoda} \int \frac{Mx + N}{(x-\alpha)^n \sqrt{ax^2 + bx + c}} dx, n \in \mathbb{N}, M, N, a, b, c \in \mathbb{R}$$

uvodimo mijenju $x - \alpha = \frac{1}{t}$

$$1. \int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \left| \begin{array}{l} x+1 = \frac{1}{t} \\ dx = -\frac{1}{t^2} dt \\ t = \frac{1}{x+1} \end{array} \right| = \int \frac{-\frac{1}{t^2} dt}{t \sqrt{\left(\frac{1}{t}-1\right)^2 + \frac{1}{t} - 1 + 1}} = - \int \frac{dt}{t \sqrt{\frac{1}{t^2} - \frac{2}{t} + 1 + \frac{1}{t}}} =$$

$$= - \int \frac{dt}{\sqrt{t^2 - t + 1}} = \left| \begin{array}{l} t^2 - t + 1 = \\ = t^2 - 2 \cdot t \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 \\ = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4} \end{array} \right| = - \int \frac{dt}{\sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}}} = \left| \begin{array}{l} t - \frac{1}{2} = \frac{\sqrt{3}}{2} z \\ dt = \frac{\sqrt{3}}{2} dz \end{array} \right|$$

$$= - \frac{\sqrt{3}}{2} \int \frac{dz}{\sqrt{\frac{3}{4}z^2 + \frac{3}{4}}} = - \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}} \int \frac{dz}{\sqrt{z^2 + 1}} = - \ln |z + \sqrt{z^2 + 1}| + C =$$

$$= - \ln \left| \frac{\frac{2t-1}{\sqrt{3}} + \sqrt{\left(\frac{2t-1}{\sqrt{3}}\right)^2 + 1}}{\sqrt{3}} \right| + C = - \ln \left| \frac{\frac{2}{x+1} - 1}{\sqrt{3}} + \sqrt{\frac{\left(\frac{2}{x+1} - 1\right)^2}{3} + 1} \right| + C$$

$$2. \int \frac{dx}{(x-1)^3 \sqrt{x^2 + 3x + 1}} = \left| \begin{array}{l} x-1 = \frac{1}{t} \\ dx = -\frac{1}{t^2} dt \\ x = \frac{1}{t} + 1 \\ t = \frac{1}{x-1} \end{array} \right| = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t^3} \sqrt{\left(\frac{1}{t}+1\right)^2 + 3\left(\frac{1}{t}+1\right) + 1}} = - \int \frac{t dt}{\sqrt{\frac{1}{t^2} + \frac{5}{t} + 5}}$$

$$= - \int \frac{t dt}{\sqrt{\frac{1+5t+5t^2}{t^2}}} = - \int \frac{t^2}{\sqrt{5t^2+5t+1}} dt = (at+b)\sqrt{5t^2+5t+1} + \lambda \int \frac{dt}{\sqrt{5t^2+5t+1}} / \frac{d}{dt}$$

$$\frac{-t^2}{\sqrt{5t^2+5t+1}} = a\sqrt{5t^2+5t+1} + (at+b) \frac{10t+5}{2\sqrt{5t^2+5t+1}} + \lambda \cdot \frac{1}{\sqrt{5t^2+5t+1}} / \cdot 2\sqrt{5t^2+5t+1}$$

$$-2t^2 = 2a \cdot (5t^2 + 5t + 1) + a(10t + 5) + b(10t + 5) + 2\lambda$$

$$t^2: 10a + 10a = -2 \quad t: 10a + 5a + 10b = 0 \quad \text{za } b = \frac{3}{2} \quad -\frac{2}{10} + \frac{15}{20} = -2\lambda$$

$$a = -\frac{1}{10}$$

$$15a + 10b = 0$$

$$b = \frac{3}{20} \quad -2\lambda = \frac{11}{20}$$

$$I = \left(-\frac{1}{10}t + \frac{3}{20}\right) \sqrt{5t^2+5t+1} - \frac{11}{40} \int \frac{dt}{\sqrt{5t^2+5t+1}}$$

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$$\lambda = -\frac{11}{40}$$

$$3. \int \frac{dx}{x^2 \sqrt{x^2-x+1}}$$

$$4. \int \frac{(3x+2) dx}{(x+1) \sqrt{x^2+3x+3}}$$

$$5. \int \frac{dx}{x^3 \sqrt{x^2+1}}$$

V metoda - integral binomnog diferencijala

$$\int x^m (a+bx^n)^p dx \quad (a, b \in \mathbb{R}; m, n, p \in \mathbb{R})$$

Integracija je moguća ako

$$1^{\circ} p \in \mathbb{Z}, \text{ uvodimo supenu } x=t^r, s = NZS(m_1, n_2), m = \frac{m_1}{m_2}, n = \frac{n_1}{n_2}$$

$$2^{\circ} \frac{m+1}{n} \in \mathbb{Z}, \text{ uvodimo supenu } a+bx^n = t^{p_2}, p = \frac{p_1}{p_2}$$

$$3^{\circ} \frac{m+1}{n} + p \in \mathbb{Z}, \text{ uvodimo supenu } ax^{-n} + b = t^{p_2}, p_2 \text{ nazivnik od } p$$

$$\textcircled{1}_0 \int \frac{dx}{x^2 (\sqrt{1+x^2})^3} = \int x^{-2} (1+x^2)^{-\frac{3}{2}} dx = \begin{cases} m=-2, n=2, p=-\frac{3}{2} \\ p \notin \mathbb{Z}, \text{ nije } 1^{\circ} \\ \frac{m+1}{n} = \frac{-2+1}{2} = -\frac{1}{2} \notin \mathbb{Z}, \text{ nije } 2^{\circ} \\ \frac{m+1}{n} + p = -\frac{1}{2} - \frac{3}{2} = -2 \in \mathbb{Z}, 3^{\circ} \end{cases}$$

$$\text{Supena: } x^{-2} + 1 = t^2$$

$$x^{-2} = t^2 - 1$$

$$x^2 = (t^2 - 1)^{-1}$$

$$x = (t^2 - 1)^{-\frac{1}{2}}$$

$$\begin{aligned} dx &= -\frac{1}{2}(t^2 - 1)^{-\frac{3}{2}} \cdot 2t dt \\ &\quad \left| \begin{array}{l} dx = -t(t^2 - 1)^{-\frac{3}{2}} dt \\ \int (t^2 - 1) \left(1 + \frac{1}{t^2 - 1} \right)^{-\frac{3}{2}} \cdot (-t)(t^2 - 1)^{-\frac{3}{2}} dt \\ = \int (t^2 - 1) \cdot \frac{t^{-3}}{(t^2 - 1)^{-\frac{3}{2}}} (-t) \cdot (t^2 - 1)^{-\frac{3}{2}} dt \end{array} \right. \\ &= \int (-1 + t^{-2}) dt = -t - \frac{1}{t} + C = \frac{-x^{-2} - 2}{\sqrt{x^{-2} + 1}} + C \end{aligned}$$

VI metoda - Eulerove supene $\int R(x, \sqrt{ax^2 + bx + c}) dx$

$$1^{\circ} \text{ za } a > 0 \text{ uzimamo supenu } \sqrt{ax^2 + bx + c} = \pm \sqrt{a} x + t \text{ racionalan f-jači}$$

$$2^{\circ} \text{ za } c > 0 \text{ uzimamo supenu } \sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$$

$$3^{\circ} \text{ za } b^2 - 4ac > 0 \text{ uzimamo supenu } \sqrt{a(x-x_1)(x-x_2)} = t(x-x_1)$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\textcircled{1}_0 1 = \int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}}$$

Rj. 3°

$$\text{Supena } \sqrt{(-x-1)(x+2)} = t(x+1)^2$$

Integralacija trigonometričkih funkcija

1 tip $\int \sin^m x \cdot \cos^n x \, dx$ ($m, n \in \mathbb{N}_0$)

ako je m neparan uvodimo smjeru $\cos x = t$

ako je n neparan uvodimo smjeru $\sin x = t$

ako su m i n parni koristimo formule

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\begin{aligned} 1_0 \int \sin^3 x \cdot \cos^{12} x \, dx &= \int \sin x \cdot \sin^2 x \cdot \cos^{12} x \, dx = \int \underline{\sin x} (\underline{1 - \cos^2 x}) \underline{\cos^{12} x} \, dx \\ &= \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ \sin x \, dx = -dt \end{array} \right| = \int (1 - t^2) \cdot t^{12} \cdot (-dt) = - \int (t^{12} - t^{14}) \, dt = - \int t^{12} \, dt + \\ &+ \int t^{14} \, dt = - \frac{t^{13}}{13} + \frac{t^{15}}{15} + C = - \frac{1}{13} \cos^{13} x + \frac{1}{15} \cos^{15} x + C \end{aligned}$$

$$\begin{aligned} 2_0 \int \sin^4 x \cdot \cos^5 x \, dx &= \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx \\ &= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right| = \int t^4 \cdot (1 - t^2)^2 \, dt = \\ &= \int t^4 (1 - 2t^2 + t^4) \, dt = \int (t^8 - 2t^6 + t^4) \, dt = \frac{t^9}{9} - 2 \cdot \frac{t^7}{7} + \frac{t^5}{5} + C \\ &= \frac{1}{9} \sin^9 x - \frac{2}{7} \sin^7 x + \frac{1}{5} \sin^5 x + C \end{aligned}$$

$$3_0 \int \sin^2 x \cdot \cos^{10} x \, dx$$

$$5_0 \int \sin^5 x \, dx$$

$$R: -\frac{1}{5} \cos^5 x + \frac{2}{3} \cos^3 x - \cos x + C$$

$$4_0 \int \sin^2 x \cdot \cos^3 x \, dx$$

$$6_0 \int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx = \int \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx =$$

$$= \int \frac{1+2\cos 2x + \cos^2 2x}{4} dx = \frac{1}{4} \int dx + \frac{1}{4} \int \cos 2x dx + \frac{1}{4} \int \cos^2 2x dx =$$

$$= \frac{1}{4}x + \frac{1}{2} \cdot \frac{1}{2} \sin 2x + \frac{1}{4} \int \frac{1+\cos 4x}{2} dx = \frac{1}{4}x + \frac{1}{4} \sin 2x + \frac{1}{8} \left(\int dx + \int \cos 4x dx \right)$$

$$\boxed{\int \cos 2x dx - \left| \begin{array}{l} 2x=t \\ 2dx=dt \\ dx=\frac{dt}{2} \end{array} \right| = \int \cos t \cdot \frac{dt}{2} = \frac{1}{2} \sin t + C = \frac{1}{2} \sin 2x + C}$$

$$= \frac{1}{4}x + \frac{1}{4} \sin 2x + \frac{1}{8}x + \frac{1}{8} \cdot \frac{1}{4} \sin 4x + C = \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

Napomena: Zadatak možemo riješiti i parcijalnom integracijom $\int \cos^4 x dx = \int \cos^3 x \cdot \cos x dx = \left| \begin{array}{l} u = \cos^3 x \\ dv = \cos x dx \end{array} \right.$

$$\textcircled{7}_0 \quad \int \sin^2 x \cdot \cos^2 x dx = \int \left(\frac{1-\cos 2x}{2} \right) \left(\frac{1+\cos 2x}{2} \right) dx = \frac{1}{4} \int (1-\cos^2 2x) dx$$

$$= \frac{1}{4} \int dx - \frac{1}{4} \int \cos^2 2x dx = \frac{1}{4}x - \frac{1}{4} \int \frac{1+\cos 4x}{2} dx = \frac{1}{4}x - \frac{1}{8} \int (1+\cos 4x) dx$$

$$= \frac{1}{4}x - \frac{1}{8} \int dx - \frac{1}{8} \int \cos 4x dx = \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8} \cdot \frac{1}{4} \sin 4x + C =$$

$$= \frac{1}{8}x - \frac{1}{32} \sin 4x + C$$

$$\textcircled{8}_0 \quad \int \sin^6 x dx$$

$$\textcircled{11}_0 \quad \int \sin^4 2x dx$$

$$\textcircled{9}_0 \quad \int \sin^2 x \cdot \cos^4 x dx$$

$$R_j: \quad \frac{3}{8}x - \frac{1}{8} \sin 4x + \frac{1}{64} \sin 8x + C$$

$$\textcircled{10}_0 \quad \int \cos^6 x dx$$

$$R_j: \quad \frac{5}{16}x - \frac{1}{48} \sin^3 2x + \frac{1}{4} \sin 2x + \frac{3}{64} \sin 4x + C$$

II tip $\int \sin \alpha x \cdot \sin \beta x dx, \int \sin \alpha x \cdot \cos \beta x dx, \int \cos \alpha x \cdot \cos \beta x dx$

Koristimo formule:

$$\sin \alpha x \cdot \sin \beta x = \frac{1}{2} [\cos(\alpha - \beta)x - \cos(\alpha + \beta)x]$$

$$\sin \alpha x \cdot \cos \beta x = \frac{1}{2} [\sin(\alpha + \beta)x + \sin(\alpha - \beta)x]$$

$$\cos \alpha x \cdot \cos \beta x = \frac{1}{2} [\cos(\alpha + \beta)x + \cos(\alpha - \beta)x]$$

$$\begin{aligned} 1. \quad \int \sin 4x \cdot \sin 2x dx &= \frac{1}{2} \int (\cos 2x - \cos 6x) dx = \frac{1}{2} \int \cos 2x dx - \\ &- \frac{1}{2} \int \cos 6x dx = \frac{1}{2} \cdot \frac{1}{2} \sin 2x - \frac{1}{2} \cdot \frac{1}{6} \sin 6x + C = \frac{1}{4} \sin 2x - \frac{1}{12} \sin 6x + C \end{aligned}$$

$$\begin{aligned} 2. \quad \int \sin 3x \cdot \cos 5x dx &= \frac{1}{2} \int (\sin 8x + \sin(-2x)) dx = \frac{1}{2} \int \sin 8x dx - \\ &- \frac{1}{2} \int \sin 2x dx = \frac{1}{2} \cdot \left(-\frac{1}{8}\right) \cos 8x - \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cos 2x + C = -\frac{1}{16} \cos 8x + \frac{1}{4} \cos 2x + C \end{aligned}$$

$$\begin{aligned} 3. \quad \int \cos x \cdot \cos 3x \cdot \cos 5x dx &= \int \frac{1}{2} (\cos 4x + \cos(-2x)) \cos 5x dx = \\ &= \frac{1}{2} \int (\cos 4x + \cos 2x) \cos 5x dx = \frac{1}{2} \int \cos 4x \cos 5x dx + \frac{1}{2} \int \cos 2x \cos 5x dx = \\ &= \frac{1}{2} \cdot \frac{1}{2} \int (\cos 8x + \cos x) dx + \frac{1}{2} \cdot \frac{1}{2} \int (\cos 7x + \cos 3x) dx = \\ &= \frac{1}{4} \int \cos 8x dx + \frac{1}{4} \int \cos x dx + \frac{1}{4} \int \cos 7x dx + \frac{1}{4} \int \cos 3x dx = \\ &= \frac{1}{4} \cdot \frac{1}{9} \sin 9x + \frac{1}{4} \sin x + \frac{1}{4} \cdot \frac{1}{7} \sin 7x + \frac{1}{4} \cdot \frac{1}{3} \sin 3x + C \\ &= \frac{1}{36} \sin 9x + \frac{1}{4} \sin x + \frac{1}{28} \sin 7x + \frac{1}{12} \sin 3x + C \end{aligned}$$

$$4. \quad \int \sin x \cdot \sin 2x \cdot \sin 4x dx \quad \text{Rj.} \quad -\frac{1}{20} \cos 5x - \frac{1}{12} \cos 3x + \frac{1}{28} \cos 7x + \frac{1}{4} \cos x + C$$

$$5. \int \sin 2x \cdot \cos 3x \cdot \sin 5x \, dx$$

$$6. \int \sin^2 \frac{x}{2} \cos 3x \, dx = \int \frac{1 - \cos 2 \cdot \frac{x}{2}}{2} \cos 3x \, dx = \frac{1}{2} \int (1 - \cos x) \cdot \cos 3x \, dx$$

$$= \frac{1}{2} \int (\cos 3x - \cos x \cdot \cos 3x) \, dx = \frac{1}{2} \int \cos 3x \, dx - \frac{1}{2} \int \cos x \cos 3x \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{3} \sin 3x - \frac{1}{2} \cdot \frac{1}{2} \int (\cos 4x + \cos 2x) \, dx = \frac{1}{6} \sin 3x - \frac{1}{4} \int \cos 4x \, dx$$

$$- \frac{1}{4} \int \cos 2x \, dx = \frac{1}{6} \sin 3x - \frac{1}{16} \sin 4x - \frac{1}{8} \sin 2x + C$$

$$7. \int \sin^2 2x \cdot \cos^2 3x \, dx$$

$$8. \int \sin^3 x \cdot \cos^2 2x \, dx$$

III tip $\int R(\sin x, \cos x) \, dx$, R - racionalna f-ja

koristimo smjenu $\operatorname{tg} \frac{x}{2} = t \Rightarrow$

$$\Rightarrow dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\operatorname{tg} \frac{x}{2} = t$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} : \cos^2 \frac{x}{2} = \frac{2 \operatorname{tg} \frac{x}{2}}{\operatorname{tg}^2 \frac{x}{2} + 1} = \frac{2t}{t^2 + 1}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} : \cos^2 \frac{x}{2} = \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{\operatorname{tg}^2 \frac{x}{2} + 1} = \frac{1 - t^2}{1 + t^2}$$

$$\operatorname{tg} \frac{x}{2} = t \Rightarrow \frac{x}{2} = \arctg t \Rightarrow x = 2 \arctg t \Rightarrow dx = \frac{2dt}{1+t^2}$$

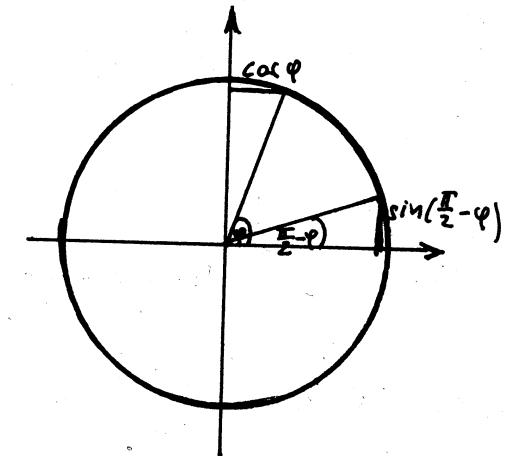
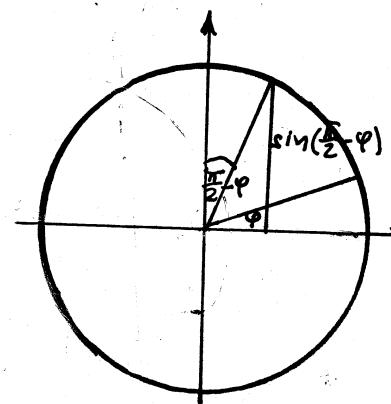
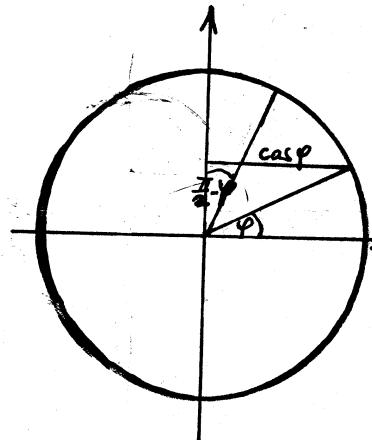
$$1_0 \int \frac{dx}{\sin x} = \left| \begin{array}{l} \operatorname{tg} \frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \end{array} \right| = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln |t| + C = \ln |\operatorname{tg} \frac{x}{2}| + C$$

$$2_0 \int \frac{dx}{\cos x} = \left| \begin{array}{l} \sin x = \cos(\frac{\pi}{2}-x) \\ \cos x = \sin(\frac{\pi}{2}-x) \\ \text{OBJASNIVNO} \end{array} \right| = \int \frac{dx}{\sin(\frac{\pi}{2}-x)} = \left| \begin{array}{l} \frac{\pi}{2}-x = t \\ -dx = dt \\ dx = -dt \end{array} \right| =$$

$$= - \int \frac{dt}{\sin t} \stackrel{1. zad.}{=} - \ln |\operatorname{tg} \frac{t}{2}| + C = - \ln |\operatorname{tg}(\frac{\pi}{4} - \frac{x}{2})| + C =$$

$$= \ln |\operatorname{tg}(\frac{\pi}{4} - \frac{x}{2})|^{-1} = \ln |\operatorname{ctg}(\frac{\pi}{4} - \frac{x}{2})| + C = \left| \begin{array}{l} \operatorname{ctg} x = \frac{\cos x}{\sin x} = \\ = \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)} = \operatorname{tg}(\frac{\pi}{2}-x) \end{array} \right|$$

$$= \ln |\operatorname{tg} \frac{\pi}{2} - (\frac{\pi}{4} - \frac{x}{2})| + C = \ln |\operatorname{tg}(\frac{x}{2} + \frac{\pi}{4})| + C$$



$$3_0 \int \frac{dx}{5-4\sin x+3\cos x} = \left| \begin{array}{l} \operatorname{tg} \frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{array} \right| = \int \frac{\frac{2dt}{1+t^2}}{5-4 \cdot \frac{2t}{1+t^2} + 3 \cdot \frac{1-t^2}{1+t^2}} =$$

$$= 2 \int \frac{\frac{dt}{1+t^2}}{\frac{5+5t^2-8t+3-3t^2}{1+t^2}} = 2 \int \frac{dt}{2t^2-8t+8} = \int \frac{dt}{t^2-4t+4}$$

$$= \int \frac{dt}{(t-2)^2} = \left| \begin{array}{l} t-2 = z \\ dt = dz \end{array} \right| = \int \frac{dz}{z^2} = -\frac{1}{z} + C = -\frac{1}{t-2} + C = -\frac{1}{\operatorname{tg} \frac{x}{2} - 2} + C$$

$$\begin{aligned}
 4. & \int \frac{\cos x + 2 \sin x}{4 \cos x + 3 \sin x} dx = \left| \begin{array}{l} \operatorname{tg} \frac{x}{2} = t \\ dx = \frac{2 dt}{1+t^2} \end{array} \right. \quad \begin{array}{l} \cos x = \frac{1-t^2}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \end{array} \quad = \\
 & = \int \frac{\frac{1-t^2}{1+t^2} + 2 \frac{2t}{1+t^2}}{4 \cdot \frac{1-t^2}{1+t^2} + 3 \frac{2t}{1+t^2}} \cdot \frac{2 dt}{1+t^2} = \int \frac{\frac{1-t^2+4t}{1+t^2}}{\frac{4-4t^2+6t}{1+t^2}} \cdot \frac{2 dt}{1+t^2} = \\
 & = \int \frac{-t^2+4t+1}{(-2t^2+3t+2)(1+t^2)} dt = \dots
 \end{aligned}$$

$$5. \int \frac{dx}{8-4 \sin x + 7 \cos x} \quad 6. \int \frac{\cos x + \sin x}{\cos x - 2 \sin x} dx$$

IV tip

$$\int R(\sin^2 x, \sin x \cos x, \cos^2 x) dx \quad R - \text{racionalna} \\
 \text{ili} \quad \int R(\operatorname{tg} x) dx \quad f_j^{-1}$$

uvodimo smjenu $\operatorname{tg} x = t \Rightarrow$

$$\Rightarrow dx = \frac{dt}{1+t^2}, \quad \sin^2 x = \frac{t^2}{1+t^2}, \quad \cos^2 x = \frac{1}{1+t^2},$$

$$\sin x \cos x = \frac{t}{1+t^2}$$

$$\operatorname{tg} x = t \Rightarrow x = \arctg t \Rightarrow dx = \frac{dt}{1+t^2}$$

$$\sin^2 x = \frac{\sin^2 x : \cos^2 x}{\sin^2 x + \cos^2 x : \cos^2 x} = \frac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x + 1} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{\cos^2 x : \cos^2 x}{\sin^2 x + \cos^2 x : \cos^2 x} = \frac{1}{\operatorname{tg}^2 x + 1} = \frac{1}{1+t^2}$$

$$\sin x \cos x = \frac{\sin x \cos x : \cos^2 x}{\sin^2 x + \cos^2 x : \cos^2 x} = \frac{\operatorname{tg} x}{\operatorname{tg}^2 x + 1} = \frac{t}{1+t^2}$$

$$\textcircled{1} \int \frac{dx}{\cos^4 x} = \left| \begin{array}{l} \operatorname{tg} x = t \\ dx = \frac{dt}{1+t^2} \end{array} \right| \quad \begin{array}{l} \cos^2 x = \frac{1}{1+t^2} \\ \sin^2 x = \frac{t^2}{1+t^2} \end{array} = \int \frac{\frac{dt}{1+t^2}}{\left(\frac{1}{1+t^2}\right)^2} = \int \frac{(1+t^2)^2}{4t^2} dt =$$

$$= \int (1+t^2) dt = \int dt + \int t^2 dt = t + \frac{t^3}{3} + C = \operatorname{tg} x + \frac{1}{3} \operatorname{tg}^3 x + C$$

$$\textcircled{2} \int \frac{dx}{\sin^2 x - 4 \sin x \cos x + 5 \cos^2 x} = \left| \begin{array}{l} \operatorname{tg} x = t \\ \sin^2 x = \frac{t^2}{1+t^2} \\ \cos^2 x = \frac{1}{1+t^2} \end{array} \right| \quad \begin{array}{l} dx = \frac{dt}{1+t^2} \\ \sin x \cos x = \frac{t}{1+t^2} \end{array} = \frac{t}{1+t^2}$$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2} - 4 \cdot \frac{t}{1+t^2} + 5 \cdot \frac{1}{1+t^2}} = \int \frac{dt}{t^2 - 4t + 5} = \int \frac{dt}{(t-2)^2 + 1} = \arct \operatorname{tg}(t-2) + C$$

$$= \arct \operatorname{tg}(\operatorname{tg} x - 2) + C$$

$$\textcircled{3} \int \operatorname{tg}^3 x dx = \left| \begin{array}{l} \operatorname{tg} x = t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int t^3 \cdot \frac{dt}{1+t^2} = \int \frac{t^3 + t - t}{1+t^2} dt = \int \frac{t+t^3}{1+t^2} dt -$$

$$- \int \frac{t}{1+t^2} dt = \int \frac{t(1+t^2)}{1+t^2} dt - \frac{1}{2} \int \frac{2t dt}{1+t^2} = \left| \begin{array}{l} t^2 = s \\ 2t dt = ds \end{array} \right| =$$

$$= \int t dt - \frac{1}{2} \int \frac{ds}{1+s} = \frac{t^2}{2} - \frac{1}{2} \ln|1+s| + C = \frac{1}{2} t^2 - \frac{1}{2} \ln|t^2+1| + C =$$

$$= \frac{1}{2} \operatorname{tg}^2 x - \frac{1}{2} \ln|\operatorname{tg}^2 x + 1| + C = \frac{1}{2} \operatorname{tg}^2 x - \frac{1}{2} \ln \left| \frac{\sin^2 x}{\cos^2 x} + 1 \right| + C = \frac{1}{2} \operatorname{tg}^2 x - \frac{1}{2} \ln \left| \frac{1}{\cos^2 x} \right| + C$$

$$= \frac{1}{2} \operatorname{tg}^2 x + \ln \left| \frac{1}{\cos^2 x} \right|^{-\frac{1}{2}} + C = \frac{1}{2} \operatorname{tg}^2 x + \ln |\cos x|^{\frac{1}{2}} + C = \frac{1}{2} \operatorname{tg}^2 x + \ln |\cos x| + C$$

$$\textcircled{4} \int \frac{\cos x + 2 \sin x}{4 \cos x + 3 \sin x} dx = \left| \begin{array}{l} \cos x \\ \sin x \end{array} \right| = \int \frac{1+2 \operatorname{tg} x}{4+3 \operatorname{tg} x} dx = \left| \begin{array}{l} \operatorname{tg} x = t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{1+2t}{4+3t} \cdot \frac{dt}{1+t^2}$$

$$\frac{1+2t}{(4+3t)(1+t^2)} = \frac{a}{4+3t} + \frac{bt+c}{1+t^2}$$

$$\textcircled{5} \int \frac{dx}{\sin^4 x}$$

$$\textcircled{6} \int \frac{dx}{3 \cos^2 x + 4 \sin^2 x}$$

$$\textcircled{7} \int \frac{\operatorname{tg} x}{\operatorname{tg}^2 x - 2 \operatorname{tg} x - 3} dx \rightarrow \text{Rj. } -\frac{1}{10} x + \frac{3}{40} \ln |\operatorname{tg} x - 3| + \frac{1}{8} \ln |\operatorname{tg} x + 1| + \frac{1}{5} \ln |\cos x| + C$$

Određeni integral

Određeni integral smo računati pomoću
Newton-Leibnizove formule

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a), \text{ gde je } F'(x) = f(x)$$

$$(1) \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\cos^2 x} = \operatorname{tg} x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \operatorname{tg} \frac{\pi}{3} - \operatorname{tg} \frac{\pi}{6} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

$$(2) \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x dx = -\cos x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -(\cos \frac{\pi}{3} - \cos \frac{\pi}{4}) = -\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = -\frac{1-\sqrt{2}}{2}$$

$$(3) \int_0^{\sqrt{2}} \frac{dx}{z+x^2} = \frac{1}{\sqrt{z}} \operatorname{arctg} \frac{x}{\sqrt{z}} \Big|_0^{\sqrt{2}} = \frac{1}{\sqrt{z}} \left(\operatorname{arctg} \frac{\sqrt{2}}{\sqrt{z}} - \operatorname{arctg} \frac{0}{\sqrt{z}} \right) = \frac{1}{\sqrt{z}} \cdot \frac{\pi}{4} = \frac{\sqrt{z}\pi}{28}$$

$$(4) \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \arcsin \frac{\sqrt{3}}{2} - \arcsin \frac{1}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$(5) \int_a^b x^m dx = \frac{x^{m+1}}{m+1} \Big|_a^b = \frac{1}{m+1} (b^{m+1} - a^{m+1}) = \frac{1}{5} (e-1)^5$$

$$(6) \int_0^1 (e^x - 1)^4 e^x dx \stackrel{(*)}{=} \frac{1}{5} (e^x - 1)^5 \Big|_0^1 = \frac{1}{5} [(e^1 - 1)^5 - (e^0 - 1)^5] =$$

$$\int_0^1 (e^x - 1)^4 e^x dx = \left| \begin{array}{l} e^x - 1 = t \\ e^x dx = dt \end{array} \right| = \int_0^1 t^4 dt = \frac{(e^x - 1)^5}{5} \dots (*)$$

$$\textcircled{7} \int_2^3 \sqrt[3]{x-1} dx \stackrel{(A)}{=} \frac{3}{4} \sqrt[3]{(x-1)^4} \Big|_2^3 = \frac{3}{4} \left[\sqrt[3]{(9-1)^4} - \sqrt[3]{(2-1)^4} \right] = \frac{3}{4} \left[\sqrt[3]{8^3} - \sqrt[3]{1} \right] \stackrel{(1)}{=}$$

$$\begin{aligned} \int \sqrt[3]{x-1} dx &= \left| \begin{array}{l} x-1=t^3 \\ dx=3t^2 dt \\ t=\sqrt[3]{x-1} \end{array} \right| = \int \sqrt[3]{t^3} \cdot 3t^2 dt = 3 \int t^3 dt = 3 \cdot \frac{t^4}{4} + C = \\ &= \frac{3}{4} \sqrt[3]{(x-1)^4} + C \quad \dots (A) \end{aligned}$$

$$\stackrel{(1)}{=} \frac{3}{4} [8\sqrt[3]{8}-1] = \frac{3}{4} (8 \cdot 2 - 1) = \frac{3}{4} \cdot 15 = \frac{45}{4}$$

$$\textcircled{8} \int_0^{\ln 5} \frac{\sqrt{e^x-1}}{1+3e^{-x}} dx \stackrel{(*)*)}{=} 2\sqrt{e^x-1} \Big|_0^{\ln 5} - 4 \arctg \frac{\sqrt{e^x-1}}{2} \Big|_0^{\ln 5} \stackrel{(2)}{=}$$

$$\begin{aligned} \int \frac{\sqrt{e^x-1}}{1+3e^{-x}} dx &= \int \frac{e^x \sqrt{e^x-1}}{e^x+3} dx = \left| \begin{array}{l} e^x-1=t^2 \\ e^x dx=2t dt \\ t=\sqrt{e^x-1} \\ e^x=t^2+1 \end{array} \right| = \int \frac{\sqrt{t^2} \cdot 2t dt}{t^2+1+3} \\ &= 2 \int \frac{t^2+4-4}{t^2+4} dt = 2 \int dt - 2 \int \frac{4}{t^2+4} = 2 \cdot t - 8 \cdot \frac{1}{2} \arctg \frac{t}{2} + C \\ &= 2t - 4 \arctg \frac{t}{2} + C = 2\sqrt{e^x-1} - 4 \arctg \frac{\sqrt{e^x-1}}{2} + C \quad \dots (*)*) \end{aligned}$$

$$\stackrel{(2)}{=} 2(\sqrt{5-1} - \sqrt{1-1}) - 4 \left(\arctg \frac{\sqrt{4}}{2} - \arctg \frac{0}{2} \right) = 4 - \pi$$

$$\textcircled{9} \int_0^{\frac{1}{2}} \sqrt{1-x^2} dx \stackrel{(\Delta A)}{=} \frac{1}{2} \arcsin x \Big|_0^{\frac{1}{2}} + \frac{1}{4} \sin(2 \arcsin x) \Big|_0^{\frac{1}{2}} = \frac{1}{2} \cdot \frac{\pi}{6} + \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{2\pi + 3\sqrt{3}}{24}$$

$$\int \sqrt{1-x^2} dx = \left| \begin{array}{l} x=\sin t \\ dx=\cos t dt \\ t=\arcsin x \end{array} \right| = \int \sqrt{1-\sin^2 t} \cdot \cos t dt = \left| \begin{array}{l} 1-\sin^2 t+\cos^2 t \\ \cos^2 t \end{array} \right|$$

$$= \int \cos^2 t \cdot \cos t dt = \int \cos^3 t dt = \frac{1}{2} \int (1+\cos 2t) dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C \quad \dots (\Delta A) \\ = \frac{1}{2} \arcsin x + \frac{1}{4} \sin(2 \arcsin x) + C$$

$$10. \int_{0}^4 \frac{dx}{1 + \sqrt{2x+1}}$$

uputa: uvođimo varijablu $2x+1 = t^2$

$$\text{rj. } 2 - \ln 2$$

$$11. \int_0^1 \frac{dx}{\sqrt{2-x^2+x}}$$

uputa: $-x^2 + x + 2 = \dots = \frac{9}{4} - (x - \frac{1}{2})^2$

$$x-1 = \frac{3}{2}t \quad \dots \quad \text{rj. } 2 \arcsin \frac{1}{3}$$

$$12. \int_3^{29} \frac{\sqrt[3]{(x-2)^2}}{\sqrt[3]{(x-2)^2} + 3} dx$$

uputa: uvođimo varijablu $x-2 = t^3$

$$\text{rj. } 8 + \frac{3\sqrt{3}\pi}{2}$$

Osnovne određenog integrala

$$a) \int_a^a f(x) dx = 0$$

$$b) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$c) \int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx$$

$$d) \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$e) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx + c$$

$$13. \int_{-3}^2 |x-1| dx = \int_{-3}^2 |x-1| = \begin{cases} x-1, & x \geq 1 \\ -(x-1), & x < 1 \end{cases} = \int_{-3}^1 -(x-1) dx + \int_1^2 (x-1) dx$$

$$= - \int_{-3}^1 x dx + \int_{-3}^1 1 dx + \int_1^2 x dx - \int_1^2 1 dx = -\frac{1}{2}x^2 \Big|_{-3}^1 + x \Big|_{-3}^1 + \frac{1}{2}x^2 \Big|_1^2 - x \Big|_1^2 = -\frac{1}{2}(1-9) +$$

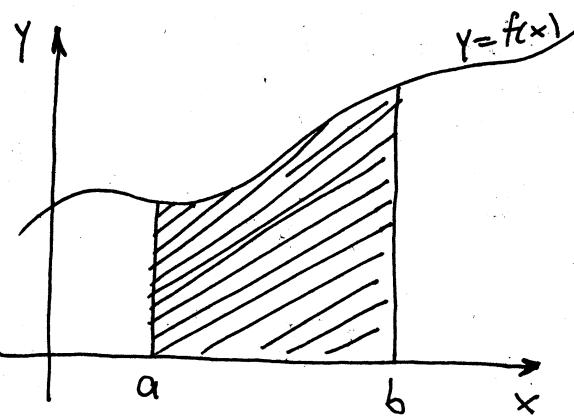
$$+ (1-(-3)) + \frac{1}{2}(4-1) - (2-1) = 4 + 4 + \frac{3}{2} - 1 = \frac{17}{2}$$

$$14. \int_{-2}^6 |x^2 - 4x - 5| dx$$

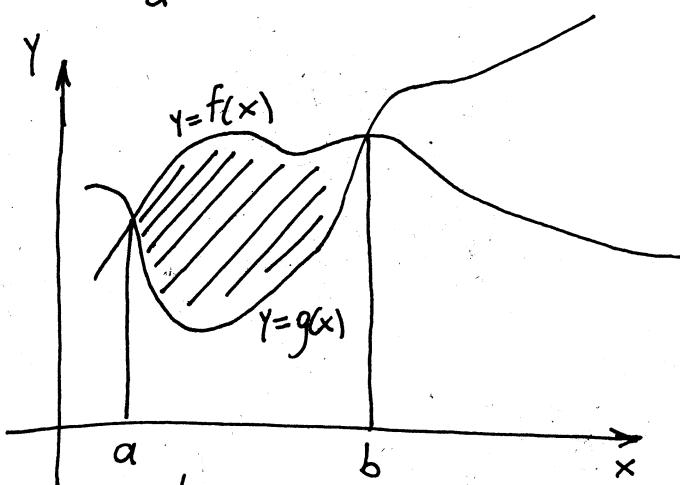
$$15. \int_{-1}^4 \sqrt{x^2(x-3)^2} dx = \int_{-1}^4 |x| \cdot |x-3| dx = \dots$$

Primjena određenog integrala

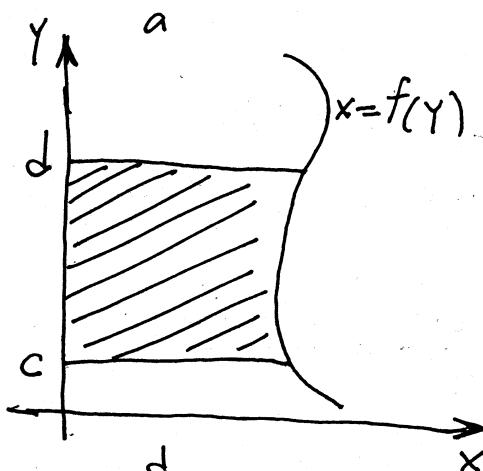
Izračunavanje površine ravne figure



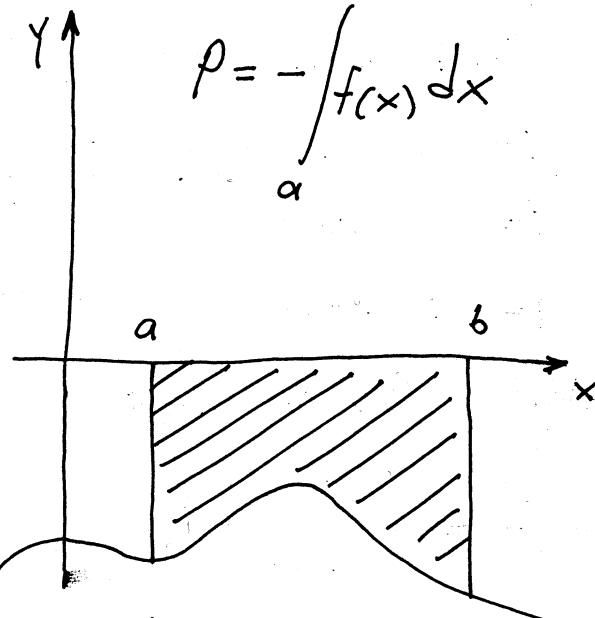
$$P = \int_a^b f(x) dx$$



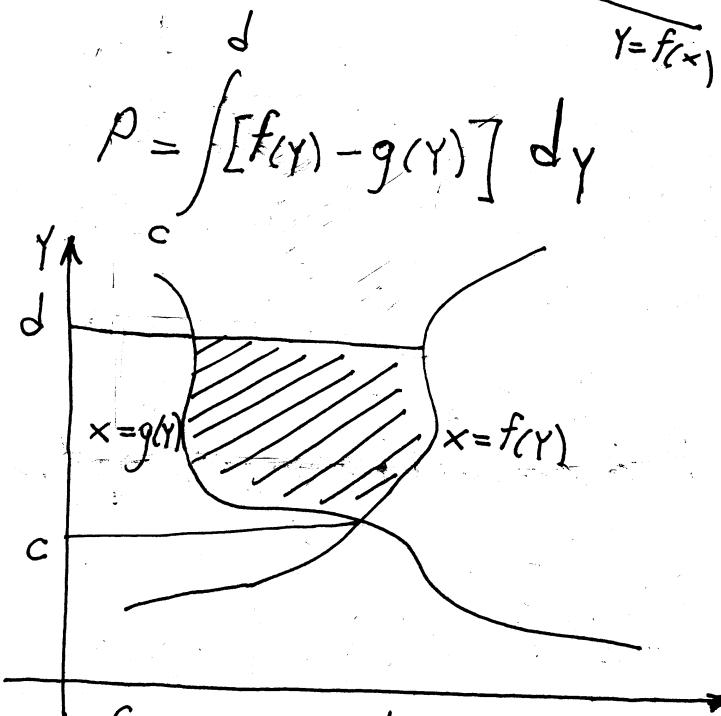
$$P = \int_a^b [f(x) - g(x)] dx$$



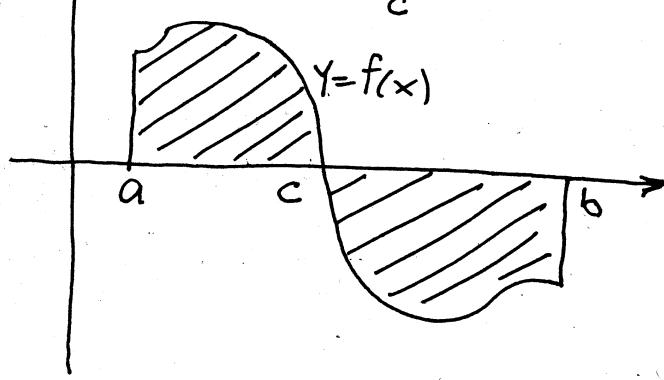
$$P = \int_c^d f(y) dy$$



$$P = - \int_a^b f(x) dx$$



$$P = \int_a^c f(x) dx + \left| \int_c^b f(x) dx \right|$$



1. Izračunati površinu ravne figure koja je ograničena linijama $y = 4 - (x-2)^2$ i $y = 0$.

Rj.

$$y = 4 - (x-2)^2$$

$$y = 4 - (x^2 - 4x + 4)$$

$$y = -x^2 + 4x$$

$$y = -x(x-4)$$

Nale A(0,0) ; B(4,0)

$$-\frac{b}{2a} = -\frac{4}{2 \cdot (-1)} = 2$$

$$\frac{4ac-b^2}{4a} = \frac{0-16}{-4} = 4$$

Tjeme parabole $y = 4 - (x-2)^2$, je u tački (2,4).

$$P = \int_0^4 (-x^2 + 4x) dx = \int_0^4 (-x^2) dx + \int_0^4 4x dx = -\frac{x^3}{3} \Big|_0^4 + 4 \cdot \frac{x^2}{2} \Big|_0^4 = -\frac{1}{3}(4^3 - 0^3)$$

$$+ 2(4^2 - 0^2) = -\frac{1}{3} \cdot 64 + 32 = \frac{96}{3} - \frac{64}{3} = \frac{32}{3}$$

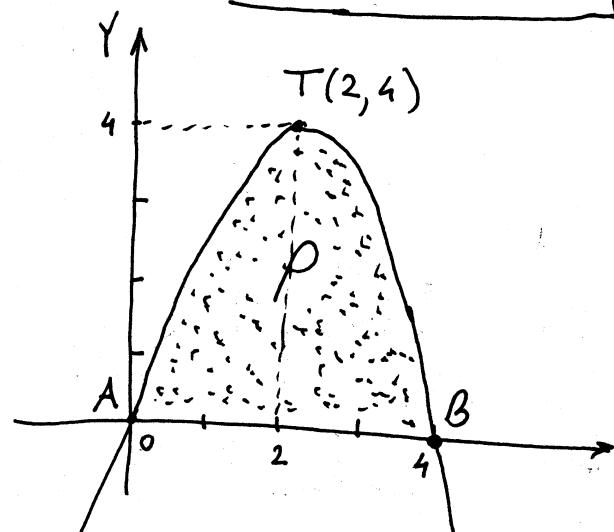
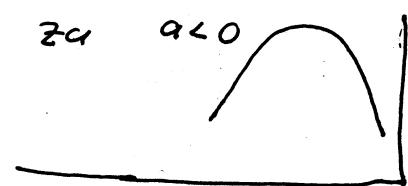
Kriva $y = ax^2 + bx + c$ ima grafik u obliku parabole.

Tjeme parabole $T\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

za $a > 0$



za $a < 0$



2. Izračunati površinu ravne figure koja je ograničena krivom $y = x^2 - 4x + 3$ i pravama $y = 0$, $x = 0$; $x = 2$.

Rj. $y = x^2 - 4x + 3$

$$D = 16 - 12 = 4$$

$$x_{1,2} = \frac{4 \pm 2}{2}$$

Nule krive

$$A(1,0) \text{ i } B(3,0)$$

$$-\frac{b}{2a} = -\frac{-4}{2} = 2$$

$$\frac{4ac-b^2}{4a} = \frac{12-16}{4} = -1$$

Tjeme krive $y = x^2 - 4x + 3$ je u tački $T(2, -1)$.

$C(0,3)$ je tačka presjeka krive sa y -osom

$$P = P_1 + P_2$$

$$P_1 = \int_{0}^1 (x^2 - 4x + 3) dx = \frac{x^3}{3} \Big|_0^1 - 4 \cdot \frac{x^2}{2} \Big|_0^1 + 3x \Big|_0^1 =$$

$$= \frac{1}{3}(1-0) - 2(1-0) + 3(1-0) =$$

$$= \frac{1}{3} + 1 = \frac{4}{3}$$

$$P_2 = - \int_{1}^2 (x^2 - 4x + 3) dx = - \left(\frac{x^3}{3} \Big|_1^2 - 4 \cdot \frac{x^2}{2} \Big|_1^2 + 3x \Big|_1^2 \right) = - \left(\frac{1}{3}(8-1) - 2(4-1) + 3 \cdot 1 \right)$$

$$= - \left(\frac{7}{3} - 6 + 3 \right) = - \left(-\frac{2}{3} \right) = \frac{2}{3}$$

$P = \frac{4}{3} + \frac{2}{3} = 2$ tražena površina ravne figure.

3. Izračunati površinu ravne figure kojeg čine parabola $y = x^2 - 2x + 2$ i prava $x + 2y - 9 = 0$.

Rj:

$$\text{prava } x + 2y - 9 = 0$$

$$2y = -x + 9$$

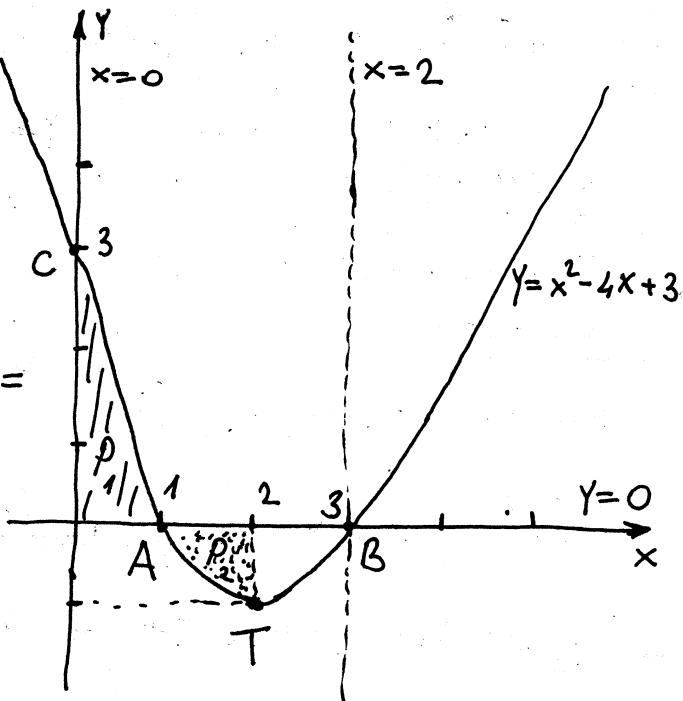
$$y = -\frac{1}{2}x + \frac{9}{2}$$

prava prolazi kroz tačke $A(0, \frac{9}{2})$ i $B(9, 0)$.

$$-\frac{b}{2a} = -\frac{-2}{2} = 1$$

$$\frac{4ac - b^2}{4a} = \frac{8-4}{4} = 1$$

$T(1,1)$ je tjemelj parabole



Trebamo naci još tačke presjeka prave i parabole.

$$Y = x^2 - 2x + 2$$

$$x + 2Y - 9 = 0$$

$$Y = x^2 - 2x + 2$$

$$x = -2Y + 9$$

$$Y = (-2Y + 9)^2 - 2(-2Y + 9) + 2$$

$$Y_1 = \frac{13}{4} \Rightarrow x = -2 \cdot \frac{13}{4} + 9 = -\frac{13}{2} + \frac{18}{2} = \frac{5}{2}$$

$$Y_2 = 5 \Rightarrow x = -2 \cdot 5 + 9 = -1$$

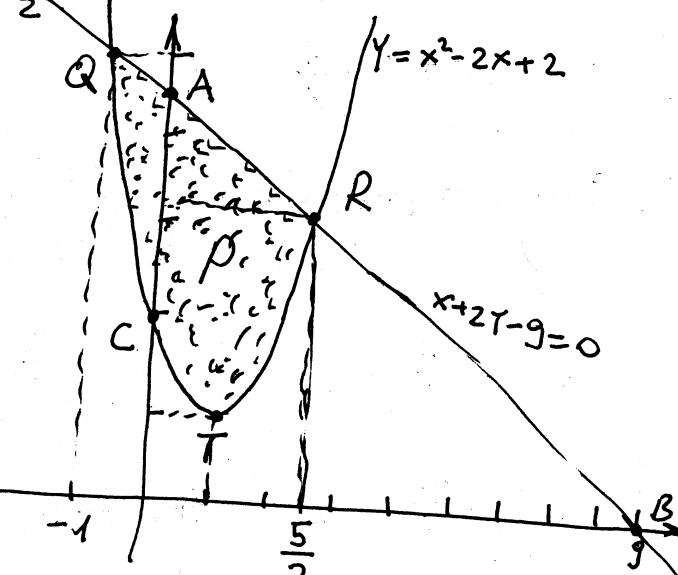
$$Y = 4Y^2 - 36Y + 81 + 4Y - 18 + 2$$

$$4Y^2 - 32Y + 65 = 0$$

$$\Delta = 33^2 - 16 \cdot 65 = 49$$

$$Y_{1,2} = \frac{33 \pm 7}{8} \quad Y_1 = \frac{26}{8} = \frac{13}{4}$$

$$Y_2 = 5$$



Tačke presjeka prave i parabole

su $R(\frac{5}{2}, \frac{13}{4})$; $Q(-1, 5)$

$$P = \int_{-1}^{\frac{5}{2}} \left[\left(-\frac{1}{2}x + \frac{9}{2} \right) - (x^2 - 2x + 2) \right] dx$$

$$\begin{aligned} \int_{-1}^{\frac{5}{2}} \left(-\frac{1}{2}x + \frac{9}{2} \right) dx &= -\frac{1}{2} \cdot \frac{x^2}{2} \Big|_{-1}^{\frac{5}{2}} + \frac{9}{2}x \Big|_{-1}^{\frac{5}{2}} = -\frac{1}{4} \left(\frac{25}{4} - 1 \right) + \frac{9}{2} \left(\frac{5}{2} - (-1) \right) \\ &= -\frac{1}{4} \cdot \frac{21}{4} + \frac{9}{2} \cdot \frac{7}{2} = \frac{231}{16} \\ \int_{-1}^{\frac{5}{2}} (x^2 - 2x + 2) dx &= \frac{x^3}{3} \Big|_{-1}^{\frac{5}{2}} - 2 \cdot \frac{x^2}{2} \Big|_{-1}^{\frac{5}{2}} + 2x \Big|_{-1}^{\frac{5}{2}} = \frac{1}{3} \left(\frac{125}{8} - (-1) \right) - \left(\frac{25}{4} - 1 \right) + \\ &+ 2 \left(\frac{5}{2} - (-1) \right) = \frac{1}{3} \cdot \frac{133}{8} - \frac{21}{4} + 2 \cdot \frac{7}{2} = \frac{133}{24} + \frac{7}{4} = \frac{175}{24} \end{aligned}$$

$$P = \frac{231}{16} - \frac{175}{24} = \frac{231}{48} - \frac{175}{48} = \frac{686}{96} = \frac{343}{48}$$

4.) Izracunati površinu ravne figure koja je ogranicena krivom $y^2 = 2x + 1$ i pravom $y = 2x - 1$.
R.j. prava $y = 2x - 1$ prolazi kroz tačke $A(0, -1)$; $B(\frac{1}{2}, 0)$.

$$y^2 = 2x + 1$$

$$2x = y^2 - 1$$

$$x = \frac{1}{2}y^2 - \frac{1}{2}$$

$$x=0 \Rightarrow y^2 = 1$$

$$A(0, -1) ; B(0, 1)$$

su tačke presjeka
f-je sa Y-osiom

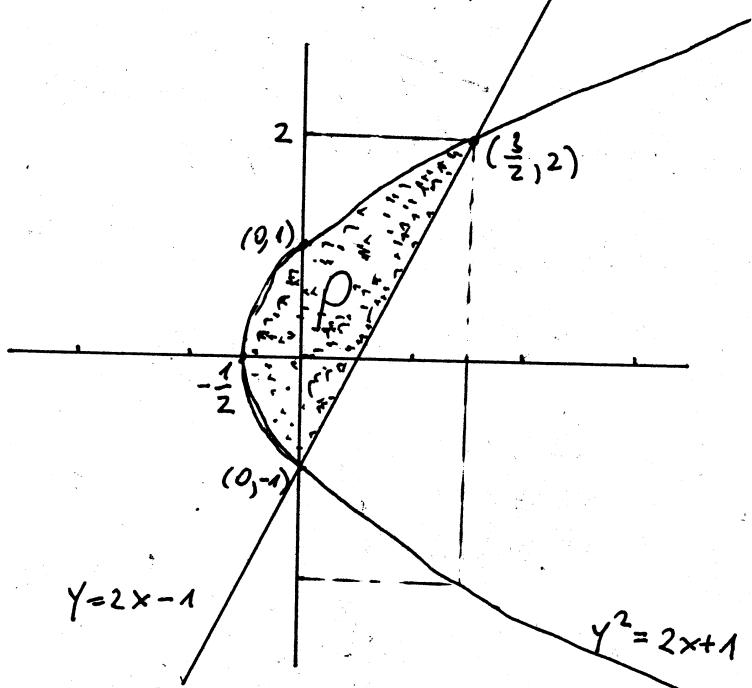
$$C(-\frac{1}{2}, 0) je nula f-je$$

$$D=1 \quad -\frac{D}{4a} = -\frac{1}{4 \cdot \frac{1}{2}}$$

$$-\frac{b}{2a} = -\frac{0}{2 \cdot \frac{1}{2}} = 0$$

$$T(-\frac{1}{2}, 0)$$

je tjemne parabole



$$\rho = \int_{-1}^2 \left[\frac{1}{2}y + \frac{1}{2} - \left(\frac{1}{2}y^2 - \frac{1}{2} \right) \right] dy = \frac{1}{2} \int_{-1}^2 (y+1-y^2+1) dy = \frac{1}{2} \int_{-1}^2 (-y^2+y+2) dy =$$

$$= \frac{1}{2} \cdot \left[\left(-\frac{y^3}{3} \right) \Big|_{-1}^2 + \frac{y^2}{2} \Big|_{-1}^2 + 2y \Big|_{-1}^2 \right] = \frac{1}{2} \left[-\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2(2+1) \right] =$$

$$= \frac{1}{2} \left(-3 + \frac{3}{2} + 6 \right) = \frac{1}{2} \cdot \frac{9}{2} = \frac{9}{4}$$

tražena površina

Kriva oblika $x=ay^2+bx+c$
ima grafik u obliku parabole



$$a < 0 \qquad a > 0$$

Tjeme krive se traži

$$po formuli T\left(-\frac{D}{4a}, -\frac{b}{2a}\right)$$

Tražimo još tačke presjeka
krive i prave

$$\begin{aligned} y &= 2x - 1 \\ y^2 &= 2x + 1 \end{aligned}$$

$$(2x-1)^2 = 2x+1$$

$$4x^2 - 4x + 1 - 2x - 1 = 0$$

$$4x^2 - 6x = 0$$

$$2x(2x-3) = 0$$

$$x=0 \quad v \quad x=\frac{3}{2}$$

D(0, -1) ; E(\frac{3}{2}, 2) su
tačke presjeka krive i prave

$$y = 2x - 1 \Rightarrow x = \frac{1}{2}y + \frac{1}{2}$$

$$y^2 = 2x + 1 \Rightarrow x = \frac{1}{2}y^2 - \frac{1}{2}$$

$$\int_{-1}^2 (-y^2 + y + 2) dy =$$

$$= \frac{1}{2} \left[-\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2(2+1) \right] =$$

$$= \frac{1}{2} \left(-3 + \frac{3}{2} + 6 \right) = \frac{1}{2} \cdot \frac{9}{2} = \frac{9}{4}$$

5. Izračunati površinu figure koju u koordinatnom sistemu čine linije $y = 3 + 2x - x^2$; $x + y = 3$. Rj. $\frac{9}{2}$
6. Na paraboli $y = 4 - x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine dva te parabola, povučena normala i y-osa. Rj. normala $y = \frac{1}{4}x - \frac{1}{2}$, $P = \frac{35}{6}$
7. Izračunati površinu figure ograničene linijama $y = \frac{1}{3}x^2$ i $y = \frac{1}{6}x^2 - 2x$. Rj. $P = 48$.

8. Izračunati površinu ravne figure ograničene krivim $x^2 + y^2 = 5$, $y = \frac{\sqrt{6}}{2}x$ (ako je $y \geq 0$), $y \geq \frac{\sqrt{6}}{2}x$; $x \geq 0$.

Rj. $(x - p)^2 + (y - q)^2 = r^2$ je jednačina kruga poluprečnika r sa centrom u tački (p, q)

$x^2 + y^2 = 5$ ima centar u tački $(0, 0)$ i poluprečnik $\sqrt{5}$

Nadimo presječne tačke kruga $x^2 + y^2 = 5$ i prave $y = \frac{\sqrt{6}}{2}x$.

$$x^2 + y^2 = 5$$

$$y = \frac{\sqrt{6}}{2}x$$

$$x^2 + \frac{6}{4}x^2 = 5$$

$$\frac{5}{2}x^2 = 5$$

$$x^2 = 2$$

$$x_1 = -\sqrt{2}, x_2 = \sqrt{2}$$

$$x^2 + y^2 = 5$$

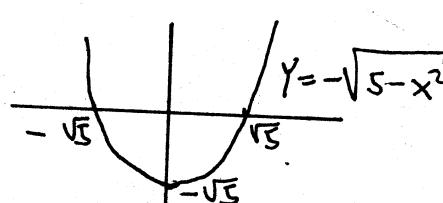
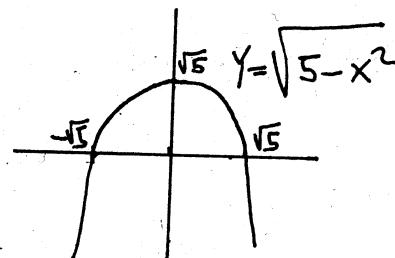
$$y^2 = 5 - x^2$$

$$y = \pm \sqrt{5 - x^2}$$

$$A(-\sqrt{2}, -\sqrt{3})$$

$$; B(\sqrt{2}, \sqrt{3})$$

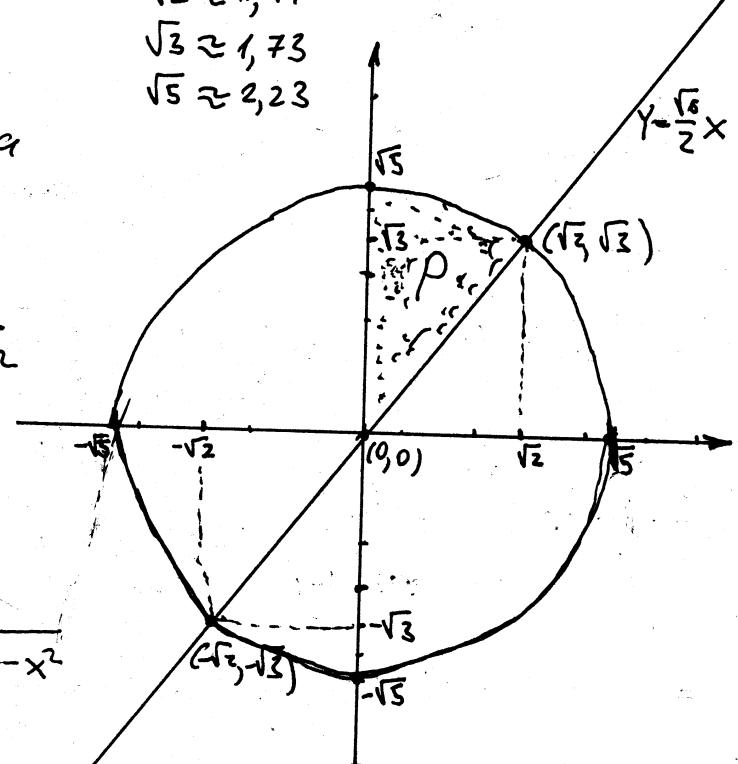
su tačke presjeka
kruga i
prave



$$\sqrt{2} \approx 1,41$$

$$\sqrt{3} \approx 1,73$$

$$\sqrt{5} \approx 2,23$$



$$\rho = \int_0^{\sqrt{2}} \left(\sqrt{5-x^2} - \frac{\sqrt{6}}{2}x \right) dx = \int_0^{\sqrt{2}} \sqrt{5-x^2} dx - \frac{\sqrt{6}}{2} \int_0^{\sqrt{2}} x dx = I_1 - \frac{\sqrt{6}}{2} \cdot \frac{x^2}{2} \Big|_0^{\sqrt{2}} = I_1 - \frac{\sqrt{6}}{4} \cdot 2$$

$$\int \sqrt{5-x^2} dx = \int \frac{5-x^2}{\sqrt{5-x^2}} dx = (ax+b) \sqrt{5-x^2} + \lambda \int \frac{dx}{\sqrt{5-x^2}} \quad / \frac{d}{dx}$$

$$\frac{5-x^2}{\sqrt{5-x^2}} = a \sqrt{5-x^2} + (ax+b) \frac{-2x}{2\sqrt{5-x^2}} + \lambda \cdot \frac{1}{\sqrt{5-x^2}} \quad / \cdot \sqrt{5-x^2}$$

$$5-x^2 = a(5-x^2) + (-ax^2 - bx) + \lambda$$

$$\begin{aligned} x^2: \quad -a-a &= -1 & x^1: \quad -b &= 0 & x^0: \quad 5a+\lambda &= 5 \\ -2a &= -1 & b &= 0 & \lambda &= 5 - \frac{5}{2} \\ a &= \frac{1}{2} & & & \lambda &= \frac{5}{2} \end{aligned}$$

$$\int \sqrt{5-x^2} dx = \frac{1}{2} \times \sqrt{5-x^2} + \frac{5}{2} \arcsin \frac{x}{\sqrt{5}} + C$$

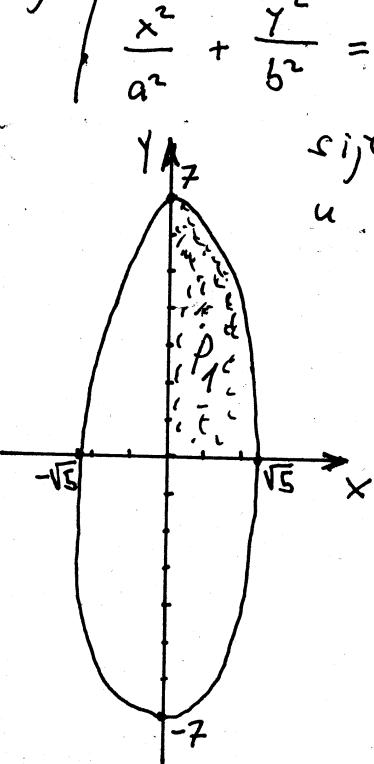
$$\int_0^{\sqrt{2}} \sqrt{5-x^2} dx = \frac{1}{2} \times \sqrt{5-x^2} \Big|_0^{\sqrt{2}} + \frac{5}{2} \arcsin \frac{x}{\sqrt{5}} \Big|_0^{\sqrt{2}} = \frac{\sqrt{6}}{2} + \frac{5}{2} \arcsin \frac{\sqrt{2}}{\sqrt{5}}$$

$$\rho = I_1 - \frac{\sqrt{6}}{2} = \frac{5}{2} \arcsin \frac{\sqrt{10}}{5} \quad \begin{matrix} \text{tražena} \\ \text{površina} \end{matrix}$$

9. Izračunati površinu elipse $\frac{x^2}{5} + \frac{y^2}{49} = 1$.

R.j.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ je jednačina elipse koja x-osi sijeće u tačkama $(a, 0)$ i $(-a, 0)$, a Y-osi u tačkama $(0, b)$ i $(0, -b)$, sa centrom u $(0, 0)$



$$\rho_{\text{elipse}} = 4 \cdot \rho_1 \quad \frac{x^2}{5} + \frac{y^2}{49} = 1 \quad 1 \cdot 5 \cdot 49$$

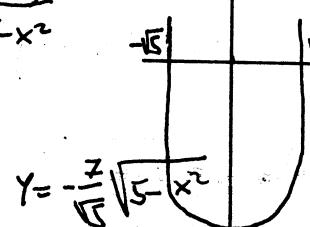
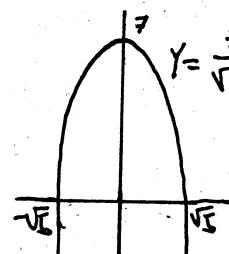
$$5y^2 = 245 - 49x^2$$

$$5y^2 = 49(5-x^2) \quad | :5$$

$$y^2 = \frac{49}{5}(5-x^2)$$

$$49x^2 + 5y^2 = 245$$

$$y = \pm \frac{z}{\sqrt{5}} \sqrt{5-x^2}$$



$$P_1 = \int_0^{\sqrt{5}} \frac{1}{\sqrt{5}} \sqrt{5-x^2} dx = \frac{1}{\sqrt{5}} \int_0^{\sqrt{5}} \sqrt{5-x^2} dx \stackrel{(*)}{=} \frac{1}{\sqrt{5}} \cdot \frac{5\pi}{4} = \frac{35\pi}{4\sqrt{5}}$$

$$\int \sqrt{5-x^2} dx \xrightarrow[\text{zadatka}]{\text{iz prethodnog}} \frac{1}{2} x \sqrt{5-x^2} + \frac{5}{2} \arcsin \frac{x}{\sqrt{5}} + C$$

$$\int_0^{\sqrt{5}} \sqrt{5-x^2} dx = \left[\frac{1}{2} x \sqrt{5-x^2} + \frac{5}{2} \arcsin \frac{x}{\sqrt{5}} \right]_0^{\sqrt{5}} = \frac{5}{2} \arcsin 1 = \frac{5}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{4}$$

.. (*)

$$P = 4P_1 = \frac{35\pi}{\sqrt{5}} = 7\sqrt{5}\pi \text{ tražena površina elipse}$$

10. Na parabolu $y=x^2+x+1$ je povučena tangenta u tački sa apcisorom $x=1$. Izračunati površinu koju obrazuje data parabola sa povučenom tangentom i osom simetrije parabole

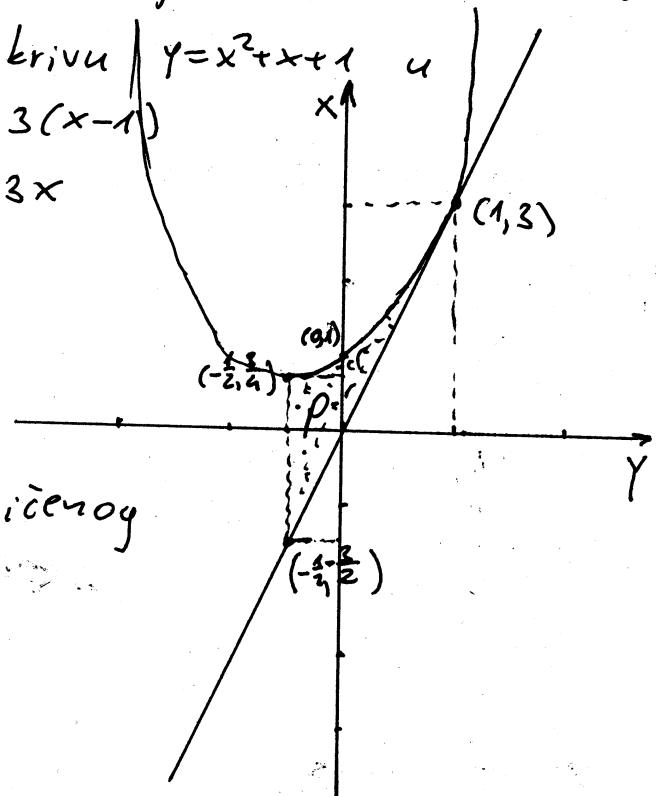
Uputa: $t: Y-y_0=Y'(x_0)(x-x_0)$ jednačina tangente u tački (x_0, y_0)

$$y=x^2+x+1 \quad \text{jednačina tangente na krivu}$$

$$Y'=2x+1 \quad \text{tački } (1,3) \quad Y-3=3(x-1)$$

$$Y'(1)=3$$

$$Y(1)=3 \quad P = \int_{-\frac{1}{2}}^{\frac{1}{2}} (x^2+x+1-3x) dx = \dots = \frac{9}{8}$$



11. Izračunati površinu lika ograničenog krivim $x^2+y^2-4x=0$, $y^2=2x$ ako je $y \geq 0$.

$$Uputa: x^2+y^2-4x=0$$

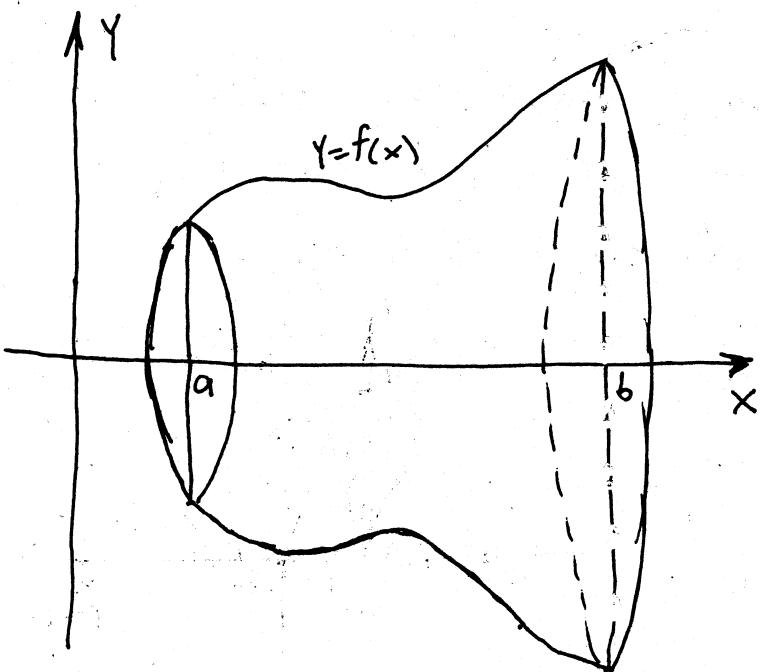
$$x^2-4x+4-4+y^2=0$$

$$(x-2)^2+y^2=4$$

ovo je jednačina kruga sa centrom u tački $(2,0)$ poluprečnika 2

$$P = \int_0^2 (\sqrt{4x-x^2} - \sqrt{2x}) dx = \dots = \pi - \frac{8}{3}$$

II Zapremina rotacionog tijela

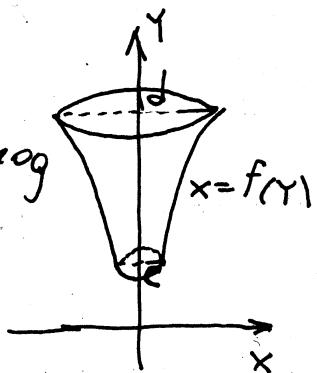


zapremina tijela
dobijenog rotacijom
tijela krive $y = f(x)$
oko x -ose

$$V_x = \pi \cdot \int_a^b [f(x)]^2 dx$$

$$V_y = \pi \int_c^d [f(y)]^2 dy$$

- zapremina tijela dobijenog
rotacijom tijela krive
 $x = f(y)$ oko y -ose



Ako je kriva dana u parametarskom obliku:

$$x = \alpha(t)$$

$$y = \beta(t)$$

$$t_1 \leq t \leq t_2$$

$$V_x = \pi \int_{t_1}^{t_2} [\beta(t)]^2 |\alpha'(t)| dt$$

$$V_y = \pi \int_{t_1}^{t_2} [\alpha(t)]^2 |\beta'(t)| dt$$

1. Izračunati zapreminu tijela koje nastaje rotacijom krive $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oko y -ose.

Rješenje

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2 \cdot \left(1 - \frac{y^2}{b^2}\right)$$

$$x^2 = a^2 \cdot \frac{b^2 - y^2}{b^2}$$

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$$

A diagram of an ellipse centered at the origin of a 2D Cartesian coordinate system. The ellipse has vertices on the x-axis at $(-a, 0)$ and $(a, 0)$, and vertices on the y-axis at $(0, b)$ and $(0, -b)$.

$$V_y = \pi \int_{-b}^b [f(y)]^2 dy = \pi \int_{-b}^b \frac{a^2}{b^2} (b^2 - y^2) dy = 2\pi \frac{a^2}{b^2} \int_0^b (b^2 - y^2) dy =$$

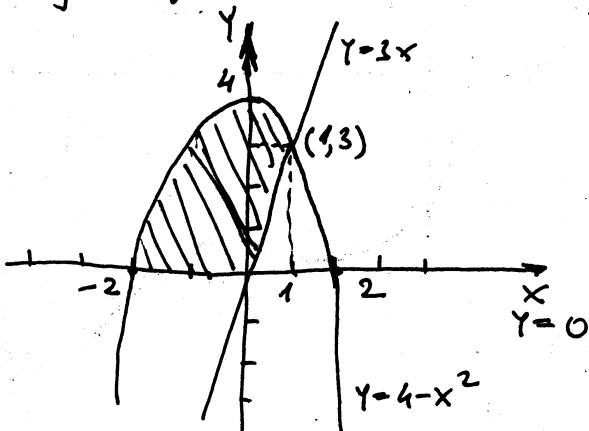
Parna f-ja (simetrična u odnosu na $y=0$)

$$= 2\pi \frac{a^2}{b^2} \left(b^2 \gamma \Big|_0^b - \frac{\gamma^3}{3} \Big|_0^b \right) = 2\pi \frac{a^2}{b^2} \left(b^3 - \frac{1}{3} b^3 \right) = 2\pi \frac{a^2}{b^2} \cdot \frac{2}{3} b^3 = \frac{4\pi a^2 b}{3}$$

- 20) Figura u ravni ograničena parabolom $y=4-x^2$ i pravama $y \geq 3x$, $y \geq 0$ rotira oko x -ose.
Izračunati zapreminu dobijenog tijela.

Rj:

$y = 4 - x^2$	$x_1 = -4 \Rightarrow y_1 = -12$
$y = 3x$	$x_2 = 1 \Rightarrow y_2 = 3$
$3x = 4 - x^2$	$A(-4, -12)$ i
$x^2 + 3x - 4 = 0$	$B(1, 3)$ su tačke
$D = 9 + 16 = 25$	projekta prave
$x_{1,2} = \frac{-3 \pm 5}{2}$	i parbole



$$V_x = V_1 - V_2, \quad V_1 = \pi \int_{-2}^1 (4-x^2)^2 dx, \quad V_2 = \pi \int_0^1 (3x)^2 dx$$

$$V_1 = \pi \int_{-2}^1 (16 - 8x^2 + x^4) dx = \pi \left(16x \Big|_{-2}^1 - 8 \frac{x^3}{3} \Big|_{-2}^1 + \frac{x^5}{5} \Big|_{-2}^1 \right) = \pi \left(16 \cdot 3 - \frac{8}{3} \cdot 9 + \frac{1}{5} \cdot 33 \right) \\ = \pi \left(48 - 24 + \frac{53}{5} \right) = \frac{158}{5} \pi$$

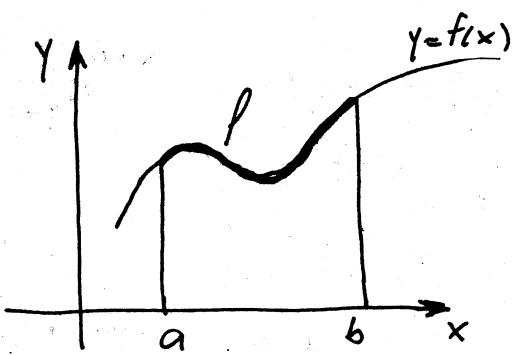
$$V_2 = \pi \cdot 9 \int_0^1 x^2 dx = 9\pi \frac{x^3}{3} \Big|_0^1 = 3\pi(1-0) = 3\pi$$

$$V = V_1 - V_2 = \frac{158}{5}\pi - 3\pi = \frac{158\pi - 15\pi}{5} = \frac{138}{5}\pi$$

- 3.) Izračunati zapreminu tijela nastalog obrtanjem oko x -ose figure omeđene krivom $y = \arcsin x$ i pravama $x=1$ i $y=0$. Uputa: parcijalna integracija 2x

- 4.) Izračunati zapreminu tijela koje nastaje rotacijom ravne figure ograničene parabolom $y=6-x-x^2$ i prave $y=0$ oko x -ose.

III Dužina luka krive



$$l = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

dužina luka krive $y = f(x)$

$$l = \int_c^d \sqrt{1 + [f'(y)]^2} dy$$

dužina luka krive $x = f(y)$

Ako je kriva data

u parametarskom obliku: t_2

$$x = x(t)$$

$$y = y(t)$$

$$t_1 \leq t \leq t_2$$

$$\Rightarrow l = \int_{t_1}^{t_2} \sqrt{(\dot{x})^2 + (\dot{y})^2} dt$$

$$\text{gdje je } \dot{x} = \frac{dx}{dt}$$

$$\text{i } \dot{y} = \frac{dy}{dt} \quad (\text{izvod po } t)$$

1. Izračunati dužinu luka krive $y = \frac{x^2}{2} - \frac{\ln x}{4}$ ašo, že $1 \leq x \leq 3$.

Rj. $l = \int_a^b \sqrt{1 + [f'(x)]^2} dx, \quad y = f(x) = \frac{x^2}{2} - \frac{\ln x}{4} = \frac{1}{2}x^2 - \frac{1}{4}\ln x$

$$y' = \frac{1}{2} \cdot 2x - \frac{1}{4} \cdot \frac{1}{x} = x - \frac{1}{4x}$$

$$\begin{aligned} l &= \int_1^3 \sqrt{1 + \left(x - \frac{1}{4x}\right)^2} dx = \int_1^3 \sqrt{1 + x^2 - 2x \cdot \frac{1}{4x} + \frac{1}{16x^2}} dx = \int_1^3 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}} dx \\ &= \int_1^3 \sqrt{\left(x + \frac{1}{4x}\right)^2} dx = \int_1^3 \left(x + \frac{1}{4x}\right) dx = \frac{x^2}{2} \Big|_1^3 + \frac{1}{4} \ln x \Big|_1^3 = \frac{1}{2}(9-1) + \frac{1}{4}(\ln 3 - \ln 1) \\ &= 4 + \frac{1}{4} \ln 3 = 4 + \ln \sqrt[4]{3} \end{aligned}$$

$$\boxed{\frac{1}{4} \ln 3 = \ln 3^{\frac{1}{4}}}$$

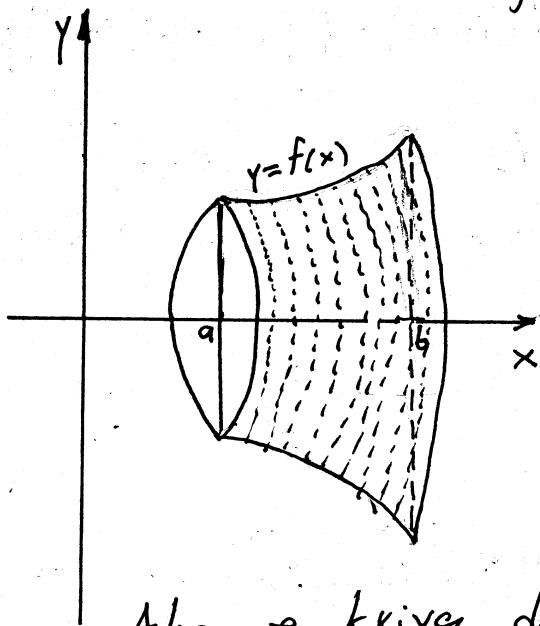
3. Izračunati dužinu luka krive

a) $y = \sqrt{2x-x^2} - 1$, ako je $\frac{1}{4} \leq x \leq 1$

b) $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$, ako je $1 \leq y \leq e$

IV Komplanacija obrtne površi

Komplanacija lat. postupak za izračunavanje površina dijelova zakrivljenih ploha



površina omotača tijela dobijenog rotacijom dijela krive $y = f(x)$ oko x -ose

$$P = 2\pi \int_a^b |f(x)| \cdot \sqrt{1 + (f'(x))^2} dx$$

Ako je kriva data u parametarskom obliku

$$x = \alpha(t)$$

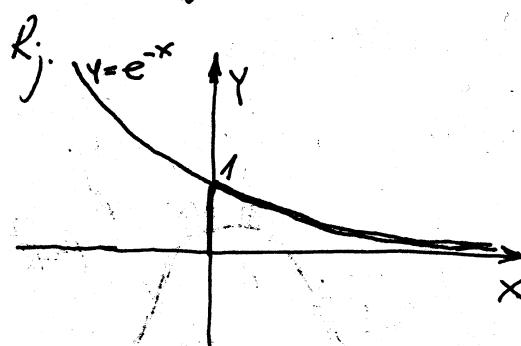
$$y = \beta(t)$$

$$t_1 \leq t \leq t_2$$

$$\Rightarrow P = 2\pi \int_{t_1}^{t_2} |\beta(t)| \cdot \sqrt{(\dot{x})^2 + (\dot{y})^2} dt$$

gdje je $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$

1. Izračunati površinu omotača tijela koje nastaje rotacijom krive $y = e^{-x}$ oko x -ose za $x \geq 0$.



$$y = e^{-x}, \quad y' = e^{-x} \cdot (-1) = -e^{-x}$$

$$P = 2\pi \int_0^{+\infty} e^{-x} \cdot \sqrt{1 + e^{-2x}} dx$$

$$= 2\pi \lim_{R \rightarrow +\infty} \int_0^R e^{-x} \sqrt{1 + e^{-2x}} dx$$

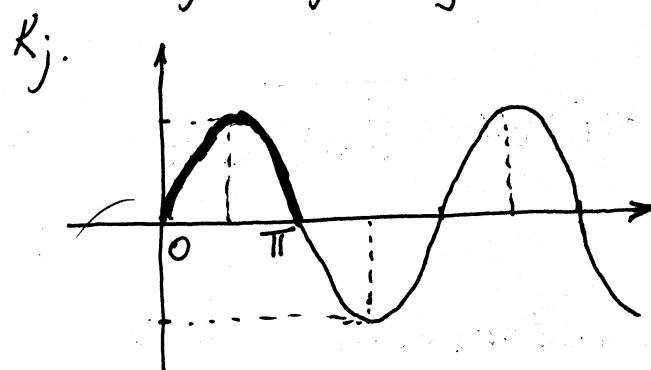
$$\int_0^R e^{-x} \sqrt{1+e^{-2x}} dx = \begin{cases} u = e^{-x} & x=0 \Rightarrow u=1 \\ du = -e^{-x} dx & x=R \Rightarrow u=e^{-R} \\ -du = e^{-x} dx & \end{cases} = \int_1^{e^{-R}} \sqrt{1+u^2} \cdot (-du)$$

$$= - \int_1^{e^{-R}} \sqrt{1+u^2} du = \int_{e^{-R}}^1 \sqrt{1+u^2} du \quad \begin{array}{l} \text{radeno} \\ \text{rauji} \\ \text{(metoda} \\ \text{Ostrograd.)} \end{array}$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2} e^{-R} \sqrt{1+e^{-2R}} + \frac{1}{2} \ln(1+\sqrt{2}) - \frac{1}{2} \ln(e^{-R} + \sqrt{1+e^{-2R}}), \quad \begin{array}{l} e^{-2R} \rightarrow 0, R \rightarrow \infty \\ e^{-R} \rightarrow 0, R \rightarrow \infty \end{array}$$

$$P = 2\pi \lim_{R \rightarrow +\infty} \int_0^R e^{-x} \sqrt{1+e^{-2x}} dx = 2\pi \left(\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(1+\sqrt{2}) \right) = \pi (\sqrt{2} + \ln(1+\sqrt{2}))$$

(2) Izračunati površinu omotača tijela koje nastaje rotacijom jednog svoda sinusoida $y = \sin x$ oko x -ose.



$$y = \sin x$$

$$y' = \cos x$$

$$P = 2\pi \int_0^{\pi} \sin x \cdot \sqrt{1+\cos^2 x} dx =$$

$$= \begin{cases} \cos x = t & x=0 \Rightarrow t=1 \\ -\sin x dx = dt & x=\pi \Rightarrow t=-1 \\ \sin x dx = -dt & \end{cases} = 2\pi \int_{-1}^{-1} \sqrt{1+t^2} (-dt) = -2\pi \int_{-1}^{-1} \sqrt{1+t^2} dt$$

$$= 2\pi \int_{-1}^1 \sqrt{1+t^2} dt = 4\pi \int_0^1 \sqrt{1+t^2} dt \quad \begin{array}{l} \text{radeno} \\ \text{rauji} \\ \text{(metoda} \\ \text{Ostrograd.)} \end{array}$$

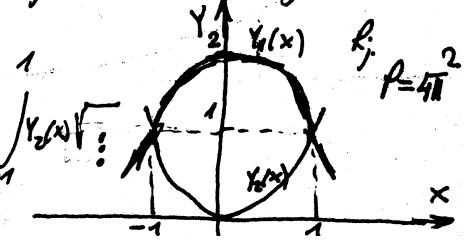
$$= 4\pi \cdot \frac{\sqrt{2}}{2} + 4\pi \cdot \frac{1}{2} \left(\ln(1+\sqrt{2}) - \ln 1 \right) = 2\sqrt{2}\pi + 2\pi \ln(1+\sqrt{2})$$

(3) Izračunati površinu tijela koje nastaje rotacijom kružnice $x^2 + (y-1)^2 = 1$ oko x -ose.

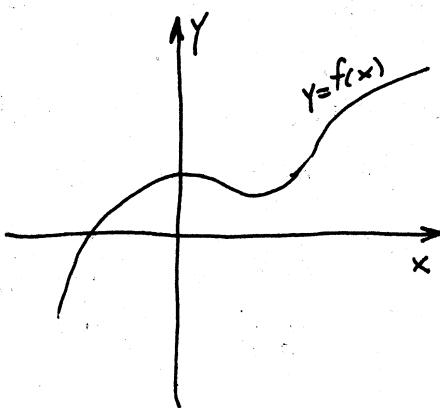
Uputa: $y_1(x) = 1 + \sqrt{1-x^2}$
 $y_2(x) = 1 - \sqrt{1-x^2}$

$$P = P_1 + P_2 = 2\pi \int_{-1}^1 y_1(x) \sqrt{1+(y'_1)^2} dx + \int_{-1}^1 y_2(x) \sqrt{1+(y'_2)^2} dx$$

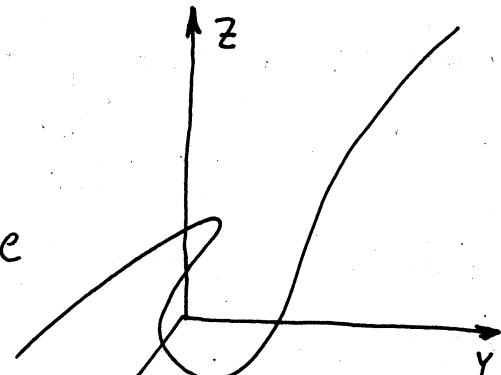
Tijelo koje nastaje rotacijom kružnice $x^2 + (y-1)^2 = 1$ oko x -ose je TORUS (ŠLAUF).



F-je daje i više promjenjivih



$y = f(x)$
 $F(x, y) = 0$
 f-ja jedne promjenjive
 $f: \mathbb{R} \rightarrow \mathbb{R}$



$z = f(x, y)$
 $F(x, y, z) = 0$
 f-ja dvije promjenjive
 $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

npr. a) $f(x, y) = x^2 + y^2 - x^y$

$f(2, 1) = 2^2 + 1^2 - 2^1 = 3$ \rightarrow f-ja dvije promjenjive (skalarna f-ja)

b) $f(x, y) = (\sin x, \cos y)$ \rightarrow f-ja dvije promjenjive (vektorska f-ja)
 $f(0, 0) = (0, 1)$ u ovom slučaju je $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

c) $f(x, y) = \sqrt{\cos \pi(x^2 + y^2)}$ \rightarrow f-ja dvije promjenjive

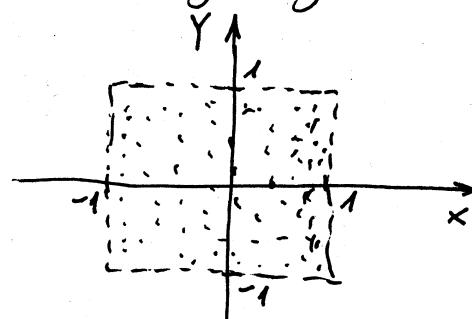
$f: \mathbb{R}^n \rightarrow \mathbb{R}$ f-je više promjenjivih

$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ puta}}$ ne \mathbb{N} , $f(x_1, x_2, \dots, x_n) = A$

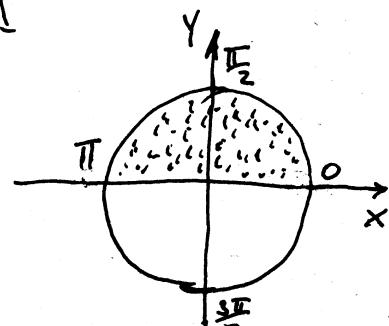
1. Naći domen f-je i predstaviti ga grafički:

a) $z = \sqrt{1-x^2} + \sqrt{1-y^2}$

Rj: $1-x^2 \geq 0$ $1-y^2 \geq 0$
 $x^2 \leq 1$ $y^2 \leq 1$
 $x \in [-1, 1]$ $y \in [-1, 1]$



b) $z = \sqrt{\sin \pi(x^2 + y^2)}$ Rj: $2k \leq x^2 + y^2 \leq 2k+1$, $k \in \mathbb{Z}$



Parcijalni izvodi f-ja daje i više promjenjivih

$z = z(x, y)$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}$ parcijalni izvodi

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$ totalni diferencijjal prvog reda
f-je daje promjenjive

$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2$ totalni diferencijjal drugog reda

$u = u(x, y, z)$ - f-ja tri promjenjive

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$

(2) Odrediti parcijalne izvode i totalne diferencijale

f-ja:

a) $z = \frac{x}{y}$ Rj: $\frac{\partial z}{\partial x} = \frac{1}{y}$, $\frac{\partial z}{\partial y} = -\frac{x}{y^2}$ $\left[\left(\frac{1}{y}\right)' = (y^{-1})' = (-1)y^{-2} = -\frac{1}{y^2} \right]$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = \frac{1}{y} dx - \frac{x}{y^2} dy = \frac{y dx - x dy}{y^2}$$

b) $z = x^2 \cdot \sqrt{y}$

Rj: $\frac{\partial z}{\partial x} = 2x\sqrt{y}$, $\frac{\partial z}{\partial y} = x^2 \cdot \frac{1}{2} y^{-\frac{1}{2}} = \frac{x^2}{2\sqrt{y}}$, $dz = 2x\sqrt{y} dx + \frac{x^2}{2\sqrt{y}} dy$

c) $z = \arctan \frac{y}{x}$

Rj: $\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(\frac{y}{x}\right)_x' = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(\frac{y}{x}\right)_y' = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$dz = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = \frac{-y dx + x dy}{x^2 + y^2}$$

d)

$$z = (2x^2 y^2 - x + 1)^3$$

Rj: $dz = \frac{(2x-1)dx + 2ydy}{2\sqrt{x^2 + y^2 - x + 1}}$

e) $z = \arcsin \frac{x}{y}$

Ekstremne vrijednosti f-ja daju promjenjivih

$$z = z(x, y)$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial y} &= 0 \end{aligned} \quad \left. \begin{array}{l} \text{sistem} \end{array} \right\}$$

rješenjem sistema dobijamo stacionarne tačke koje mogu ali i ne moraju biti ekstrem
 npr. $M(x_1, y_1)$ jedna stacionarna tačka

$$A = \frac{\partial^2 z}{\partial x^2}(x_1, y_1)$$

$$D = AC - B^2$$

$$B = \frac{\partial^2 z}{\partial x \partial y}(x_1, y_1)$$

$D > 0$ f-ja ima ekstrem u tački $M(x_1, y_1)$

$$C = \frac{\partial^2 z}{\partial y^2}(x_1, y_1)$$

a) $A > 0 \quad (C > 0) \quad z_{\min}$

b) $A < 0 \quad (C < 0) \quad z_{\max}$

$D < 0$ f-ja nema ekstrem

$D = 0$ treba ispitati ponašanje f-je u njenoj okolini

$$\Delta z = z(x_1 + \epsilon, y_1 + \eta) - z(x_1, y_1) \quad \text{- privištaj f-je}$$

$\Delta z \geq 0$ za sve (x, y) iz neke okoline tačke M_1

\Rightarrow u tački M_1 f-ja ima minimum

$\Delta z \leq 0$ za sve (x, y) iz neke okoline tačke M_1

\Rightarrow u tački M_1 f-ja ima maximum

$\Delta z(x_1, y_1)$ promjenjivog znaka \Rightarrow u tački M_1 nema ekstrema

1.

Odrediti ekstreme f-je $z(x, y) = x^2 - 6xy + y^3 + 3x + 6y$.

$$R_j: \frac{\partial z}{\partial x} = 2x - 6y + 3$$

$$2x - 6y + 3 = 0$$

$$D = 36 - 20 = 16$$

$$-6x + 3y^2 + 6 = 0 \quad | :3$$

$$y_{1,2} = \frac{6 \pm 4}{2}$$

$$\frac{\partial z}{\partial y} = -6x + 3y^2 + 6$$

$$2x - 6y + 3 = 0$$

$$y_1 = 1 \Rightarrow x_1 = \frac{3}{2}$$

$$-2x + y^2 + 2 = 0$$

$$y_2 = 5 \Rightarrow x_2 = \frac{27}{2}$$

+

$$y^2 - 6y + 5 = 0$$

$M(\frac{3}{2}, 1)$ i $N(\frac{27}{2}, 5)$ su stacionarne tačke

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$M(\frac{3}{2}, 1)$$

$$D = AC - B^2 = 12 - 36 < 0$$

$$\frac{\partial^2 z}{\partial y^2} = 6y$$

$$A = 2$$

f-ja u tački M

$$\frac{\partial^2 z}{\partial x \partial y} = -6$$

$$B = -6$$

$$C = 6$$

nema ekstrem

$$N\left(\frac{27}{2}, 5\right)$$

$A=2 \quad D=AC-B^2=60-36 > 0$ f-ja u tački N ima ekstrem

$$B=-6$$

$$C=30$$

$A > 0$ f-ja z u tački $N\left(\frac{27}{2}, 5\right)$ ima minimum

$$Z_{\min} = \left(\frac{27}{2}\right)^2 - 6 \cdot \frac{27}{2} \cdot 5 + 5^3 + 3 \cdot \frac{27}{2} + 6 \cdot 5 = \frac{27^2}{4} - 15 \cdot 27 + 5^3 + \\ + \frac{81}{2} + 30 = \frac{729 - 1620 + 500 + 162 + 120}{4} = \frac{-109}{4}$$

(2) Naći sve ekstreme f-je $Z(x, y) = 8xy + \frac{1}{x} + \frac{1}{y}$.

$$\begin{array}{l} R_j: \frac{\partial Z}{\partial x} = 8y - \frac{1}{x^2} = 0 \quad 8y - \frac{1}{x^2} = 0 \quad 8x - \frac{1}{\frac{1}{x^2}} = 0 \\ \frac{\partial Z}{\partial y} = 8x - \frac{1}{y^2} = 0 \quad \underline{8x - \frac{1}{y^2} = 0} \quad 8x - \frac{1}{64x^4} = 0 \\ \underline{y = \frac{1}{8x^2}} \quad 8x(1 - 8x^3) = 0 \\ \underline{8x - \frac{1}{y^2} = 0} \quad 8x(1 - 2x)(1 + 2x + 4x^2) = 0 \\ x_1 = 0 \Rightarrow y \text{ nije definisana} \\ x_2 = \frac{1}{2} \Rightarrow y_2 = \frac{1}{8 \cdot \frac{1}{4}} = \frac{1}{2} \end{array}$$

$M\left(\frac{1}{2}, \frac{1}{2}\right)$ je stacionarna tačka.

$$\begin{array}{l} \frac{\partial^2 Z}{\partial x^2} = (-x^{-2})' = \frac{2}{x^3} \quad M\left(\frac{1}{2}, \frac{1}{2}\right) \quad D = AC - B^2 = 16^2 - 8^2 > 0 \\ \frac{\partial^2 Z}{\partial x \partial y} = 8 \quad \frac{\partial^2 Z}{\partial y^2} = \frac{2}{y^3} \quad A = 16 \quad f-ja u tački M ima \\ B = 8 \quad \text{ekstrem} \\ C = 16 \quad A > 0 \quad f-ja ima minimum \\ Z_{\min}\left(\frac{1}{2}, \frac{1}{2}\right) = 8 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{2}} = 2 + 2 + 2 = 6 \quad Z_{\min} = 6 \end{array}$$

(3) Naći ekstremne vrijednosti f-je $Z = x^3 + xy^2 - 6xy$.

$$\begin{array}{l} R_j: \frac{\partial Z}{\partial x} = 3x^2 + y^2 - 6y = 0 \\ \frac{\partial Z}{\partial y} = 2xy - 6x = 0 \\ \underline{3x^2 + y^2 - 6y = 0} \\ \underline{2xy - 6x = 0} \\ 2x(y - 3) = 0 \Rightarrow x = 0 \text{ ili } y = 3 \end{array}$$

$$x=0: 3 \cdot 0 + y^2 - 6y = 0$$

$$y(y-6) = 0 \Rightarrow y_1 = 0, y_2 = 6$$

$$y=3: 3x^2 + 9 - 18 = 0$$

$$3x^2 = 9 \Rightarrow x_{1,2} = \pm \sqrt{3}$$

Stacionarne tačke su: $M_1(0, 0)$, $M_2(0, 6)$, $M_3(-\sqrt{3}, 3)$; $M_4(\sqrt{3}, 3)$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\text{za } M_1(0,0), \quad A=0, \quad D=AC-B^2=-36 < 0$$

$$B=-6 \\ C=0$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2y - 6$$

$$\text{za } M_2(0,6), \quad A=0, \quad D=AC-B^2=-36 < 0$$

$$B=6 \\ C=0$$

$$\frac{\partial^2 z}{\partial y^2} = 2x$$

f-ja u tački M_1
nema ekstrem

$$\text{za } M_3(-\sqrt{3}, 3), \quad \left. \begin{array}{l} A=-6\sqrt{3} \\ B=0 \\ C=-2\sqrt{3} \end{array} \right\} \Rightarrow D=AC-B^2,$$

$$D=12 \cdot 3 - 0 = 36 > 0$$

f-ja u tački M_3 ima ekstrem

$A < 0 \Rightarrow$ f-ja ima maksimum

$$z_{\max}(-\sqrt{3}, 3) = -3\sqrt{3} - 9\sqrt{3} + 18\sqrt{3} = 6\sqrt{3}$$

$$\text{za } M_4(\sqrt{3}, 3), \quad \left. \begin{array}{l} A=6\sqrt{3} \\ B=0 \\ C=2\sqrt{3} \end{array} \right\} \quad \begin{array}{l} D=AC-B^2=12 \cdot 3 > 0 \\ A>0 \end{array} \quad \begin{array}{l} \text{f-ja u tački } M_4 \\ \text{ima ekstrem} \end{array}$$

$$z_{\min}(\sqrt{3}, 3) = 3\sqrt{3} + 9\sqrt{3} - 18\sqrt{3} = -6\sqrt{3}$$

4. Nadi ekstreme f-je $z = \frac{2}{3}x^3 + 3y^2 + 6xy - 2x + 6y$.

$$\begin{aligned} R_j: \quad \frac{\partial z}{\partial x} &= 2x^2 + 6y - 2 & 2x^2 + 6y - 2 &= 0 & D = 9 + 16 = 25 \\ \frac{\partial z}{\partial y} &= 6y + 6x + 6 & -6x + 6y + 6 &= 0 \\ && 2x^2 - 6x - 8 &= 0 & x_{1,2} = \frac{3 \pm 5}{2} \\ && x^2 - 3x - 4 &= 0 & x_1 = -1, \quad x_2 = 4 \\ && & & y_1 = 0, \quad y_2 = -6 - 2 \\ && & & y_2 = -5 \end{aligned}$$

$M(-1, 0)$ i $N(4, -5)$

su stacionarne tačke

$$\frac{\partial^2 z}{\partial x^2} = 4x$$

$$\text{za tačku } M(-1,0), \quad A=-4, \quad D=AC-B^2=-24-36 \\ B=6 \\ C=6 \quad D < 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6$$

$$\text{za tačku } N(4,-5)$$

$$\frac{\partial^2 z}{\partial y^2} = 6$$

$$A=16, \quad B=6, \quad C=6, \quad D=AC-B^2=96-36 > 0$$

f-ja u tački N ima ekstrem, $A>0$ f-ja ima minimum

$$z_{\min}(4, -5) = \frac{2}{3} \cdot 4^3 + 3 \cdot 25 - 6 \cdot 20 - 8 - 30 = \frac{128}{3} - 83 = -\frac{121}{3}$$

5. Nadi ekstreme f-je $z = e^{2x}(x+y^2+2y)$. Rj: $z_{\min}\left(\frac{1}{2}, -1\right) = -\frac{1}{2}e$

6. Nadi ekstreme f-je $z = x^4 + y^4 - 2x^2$.

Rj: $\frac{\partial z}{\partial x} = 4x^3 - 4x$

$\frac{\partial z}{\partial y} = 4y^3$

$$\begin{array}{c} 4x^3 - 4x = 0 \text{ i.g.} \\ 4y^3 = 0 \text{ i.g.} \\ \hline x^3 - x = 0 \\ y^3 = 0 \end{array} \quad \begin{array}{c} x(x^2 - 1) = 0 \\ y^3 = 0 \end{array} \quad \begin{array}{c} x(x-1)(x+1) = 0 \\ y^3 = 0 \end{array}$$

Stacionarne tačke f-je su $M_1(-1, 0)$, $M_2(0, 0)$; $M_3(1, 0)$,

$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 4$

za $M_1(-1, 0)$, $A=8$, $B=0$, $C=0$

$\frac{\partial^2 z}{\partial x \partial y} = 0$

$D=0$ ispitujemo ponovno f-je u okolini tačke $M_1(-1, 0)$

$\frac{\partial^2 z}{\partial y^2} = 12y^2$

$\Delta z = z(-1+\varepsilon, 0+\omega) - z(-1, 0) =$

$= (-1+\varepsilon)^4 + \omega^4 - 2(-1+\varepsilon)^2 - [(-1)^4 + 0^4 - 2(-1)^2]$

$= 1-4\varepsilon+6\varepsilon^2-4\varepsilon^3+\varepsilon^4+\omega^4-2(1-2\varepsilon+\varepsilon^2)-(1-2)$

$= \underline{1-4\varepsilon+6\varepsilon^2} - \underline{4\varepsilon^3} + \underline{\varepsilon^4} + \underline{\omega^4} - \underline{2+4\varepsilon-2\varepsilon^2+1}$

$= \varepsilon^4 - 4\varepsilon^3 + 4\varepsilon^2 + \omega^4 = \varepsilon^2(\varepsilon^2 - 4\varepsilon + 4) + \omega^4$

$= \varepsilon^2(\varepsilon-2)^2 + \omega^4 \geq 0 \text{ za } \forall \varepsilon; \forall \omega$

f-ja ima minimum u tački $M_1(-1, 0)$, $z_{\min} = -1$

za $M_2(0, 0)$, $A=-4$, $B=0$, $C=0$, $D=AC-B^2=0$

ispitujemo ponovno f-je u okolini tačke

$\Delta z = z(0+\varepsilon, 0+\omega) - z(0, 0) = \varepsilon^4 + \omega^4 - 2\varepsilon^2 = \varepsilon^2(\varepsilon^2 - 2) + \omega^4$

$\varepsilon=0: \Delta z = \omega^4$

$\omega=0: \Delta z = \varepsilon^2(\varepsilon^2 - 2) \Rightarrow \Delta z < 0 \text{ za } \varepsilon^2 < 2$

$\Delta z > 0 \text{ za } \varepsilon^2 > 2$

u tački M_2

Pravilo f-je je prouzročivo znaka par f-ja nemaju ekstreum!

za $M_3(1, 0)$, $A=8$, $B=0$, $C=0$, $D=AC-B^2=0$ ispitujemo ponovno f-je u okolini tačke

$\Delta z = z(1+\varepsilon, 0+\omega) - z(1, 0) = (1+\varepsilon)^4 + \omega^4 - 2(1+\varepsilon)^2 - (1-2)$

$= \underline{1+4\varepsilon+6\varepsilon^2+4\varepsilon^3+4\varepsilon^4} + \underline{\omega^4} - \underline{2+4\varepsilon-2\varepsilon^2+1} = \varepsilon^4 + 4\varepsilon^3 + 4\varepsilon^2 + \omega^4$

$\Delta z = \varepsilon^2(\varepsilon+2)^2 + \omega^4 \geq 0 \text{ za } \forall \varepsilon; \forall \omega$ f-ja z u tački M_3 ima min

$z_{\min} = -1$

(7.)^v Naći sve ekstreme f-je $z = x^4 + y^4 - x^2 - 2xy - y^2$.

$$R_j: z_{\min}(-1, -1) = -2 \\ z_{\min}(1, 1) = -2$$

(8.)^v Naći ekstreme f-je

$$z = -2x^2 + 4x^2y^2 - 2y^2. \quad R_j: z_{\max}(0, 0) = 0$$

(9.)^v Naći ekstreme f-je $z = \frac{2}{3}x^3 - 5xy + \frac{5}{2}y^2 + 8x - 5y.$

Uсловni ekstremi f-ja daju prouzročivih

Ako trebamo naći ekstrem f-je $z = f(x, y)$ tako da x, y zadovoljavaju neki uslov $g(x, y) = 0$ tada tražimo ekstrem Lagranžove f-je $F(x, y) = f(x, y) + \lambda g(x, y)$ (λ je Lagranžov multiplicator).

$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = 0$$

sistem.

rečavajući sistem dobijeno neke
stacionarne tačke npr. $M_1(x_1, y_1)$

$$A = \frac{\partial^2 F}{\partial x^2}(x_1, y_1) \quad d^2F(x_1, y_1) = F''_{xx}(x_1, y_1) dx^2$$

$$B = \frac{\partial^2 F}{\partial x \partial y}(x_1, y_1) \quad + 2F''_{xy}(x_1, y_1) dx dy$$

$$C = \frac{\partial^2 F}{\partial y^2}(x_1, y_1) \quad + F''_{yy}(x_1, y_1) dy^2$$

$$d^2F(x_1, y_1) > 0, \quad z_{\min}(x_1, y_1) \\ d^2F(x_1, y_1) < 0, \quad z_{\max}(x_1, y_1)$$

(10.) Naći ekstreme f-je $z = 6 - 4x - 3y$ uz uslov $x^2 + y^2 = 1$.

$$R_j: F(x, y) = 6 - 4x - 3y + \lambda(x^2 + y^2 - 1) = 0$$

$$\frac{\partial F}{\partial x} = -4 + 2\lambda x$$

$$2\lambda x - 4 = 0$$

$$\frac{\partial F}{\partial y} = -3 + 2\lambda y$$

$$2\lambda y - 3 = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 1$$

$$\begin{matrix} 2\lambda x = 4 \\ 2\lambda y = 3 \\ x^2 + y^2 = 1 \end{matrix}$$

$$x = \frac{2}{\lambda}$$

$$4\lambda^2 = 25$$

$$y = \frac{3}{2\lambda}$$

$$\lambda_1, 2 = \pm \frac{5}{2}$$

$$x^2 + y^2 = 1$$

$$\frac{4}{\lambda^2} + \frac{9}{4\lambda^2} = 1$$

$$\frac{25}{4\lambda^2} = 1$$

$$\lambda_1 = -\frac{5}{2} \Rightarrow x_1 = -\frac{4}{5}; \quad y_1 = \frac{3}{2 \cdot (-\frac{5}{2})} = -\frac{3}{5}$$

Stacionarne tačke

$$\text{sa } M(-\frac{4}{5}, -\frac{3}{5}) \text{ za } \lambda = -\frac{5}{2}$$

$$\lambda_2 = \frac{5}{2} \Rightarrow x_2 = \frac{2}{\frac{5}{2}} = \frac{4}{5}; \quad y_2 = \frac{3}{2 \cdot \frac{5}{2}} = \frac{3}{5}$$

$$\text{i } N(\frac{4}{5}, \frac{3}{5}) \text{ za } \lambda = \frac{5}{2}.$$

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$\frac{\partial^2 F}{\partial y^2} = 2\lambda$$

$$\text{za } M\left(-\frac{4}{5}, -\frac{3}{5}\right), \lambda = -\frac{5}{2}$$

$$A = -5, B = 0, C = 5, D = AC - B^2 = 25 > 0$$

f-ja ima ekstrem, $A < 0$ f-ja ima maksimum.

$$z_{\max}\left(-\frac{4}{5}, -\frac{3}{5}\right) = 6 - 4\left(-\frac{4}{5}\right) - 3\left(-\frac{3}{5}\right) = \frac{30 + 16 + 9}{5} = \frac{55}{5} = 11$$

$$\text{za } N\left(\frac{4}{5}, \frac{3}{5}\right), \lambda = \frac{5}{2}, A = 5, B = 0, C = 5, D = AC - B^2 = 25 > 0$$

f-ja ima ekstrem u tački N , $A > 0$ f-ja ima minimum

$$z_{\min}\left(\frac{4}{5}, \frac{3}{5}\right) = 6 - 4 \cdot \frac{4}{5} - 3 \cdot \frac{3}{5} = \frac{30 - 16 - 9}{5} = \frac{5}{5} = 1$$

2. Nadi uslovne ekstreme f-je $z = y + 2x + 3$ uz uslov $x^2 - 6x + y + 5 = 0$.

$$\text{Rj: } F(x, y) = 2x + y + 3 + \lambda(x^2 - 6x + y + 5)$$

$$-x = -3 - 1$$

$$\frac{\partial F}{\partial x} = 2 + 2\lambda x - 6\lambda$$

$$2\lambda x - 6\lambda + 2 = 0 \quad | :2$$

$$x = 4$$

$$\frac{\partial F}{\partial y} = 1 + \lambda$$

$$\begin{aligned} 1 + 1 &= 0 \\ x^2 - 6x + y + 5 &= 0 \end{aligned}$$

$$x^2 - 6x + y + 5 = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2 - 6x + y + 5$$

$$\lambda x = 3\lambda - 1$$

$$16 - 24 + y + 5 = 0$$

$$x^2 - 6x + y + 5 = 0$$

$$y = 3$$

Tačka $M(4, 3)$ je stacionarna tačka, za $\lambda = -1$

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$M(4, 3), \lambda = -1$$

$$A = -2, B = 0, C = 0 \Rightarrow D = AC - B^2 = 0$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$d^2 F = \frac{\partial^2 F}{\partial x^2} dx^2 + \frac{\partial^2 F}{\partial x \partial y} dy^2 + \frac{\partial^2 F}{\partial y^2} dy^2$$

$$\frac{\partial^2 F}{\partial y^2} = 0$$

$$d^2 F = 2\lambda dx^2 \Rightarrow d^2 F = -2 dx^2 < 0$$

U tački $M(4, 3)$ f-ja ima maksimum, $z_{\max}(4, 3) = 3 + 8 + 3 = 14$

3. Odrediti ekstreme f-je $z = x^2 + y^2$ uz uslov $\frac{x}{2} + \frac{y}{3} = 1$.

$$\text{Rj: } z_{\min}\left(\frac{18}{13}, \frac{12}{13}\right) = \frac{36}{13}, \lambda = -\frac{72}{13}$$

4. Nadi uslovne ekstreme f-je $z = \ln(x+y)$, ako je $x^2 + 2y^2 = 4$.

$$\text{Rj: } z_{\max}\left(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) = \ln\left(3\sqrt{\frac{2}{3}}\right), \lambda = -\frac{1}{8}$$

Beskonačni redovi

Neka je $\{a_n\}$ niz realnih brojeva. Formirajmo novi niz $\{S_n\}$ na sljedeći način: $S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k$ ($n=1, 2, \dots$).

Uređen par $(\{a_n\}, \{S_n\})$ zovemo beskonačan red.

Broj S_n zovemo n -ta parcijalna suma reda.

Za red kažemo da konvergira (KV) ako niz $\{S_n\}$ konvergira.

Red divergira (DV) ako niz $\{S_n\}$ divergira.

Sljedeće simbole čemo koristiti da označimo definisani red: $a_1 + a_2 + \dots + a_n + \dots$, $a_1 + a_2 + a_3 + \dots$, $\sum_{k=1}^n a_k$.

1) Dokazati da red $\sum_{n=1}^{\infty} \frac{1}{n^2-n}$ konvergira i nadi njegovu sumu.

Rj. Posmatraćemo niz parcijalnih suma $S_n = \sum_{k=2}^n \frac{1}{k^2-k}$. Ako $\{S_n\}$ teži konacnom broju kad $n \rightarrow \infty$ tada red KV.

$$\frac{1}{n^2-n} = \frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1} / \cdot n(n-1)$$

$$1 = A(n-1) + Bn \quad a_n = \frac{1}{n^2-n} = -\frac{1}{n} + \frac{1}{n-1} = \frac{1}{n-1} - \frac{1}{n}$$

$$n: A+B=0 \quad A=-1 \quad B=1$$

$$n^o: -A=1$$

$$S_n = a_1 + a_2 + \dots + a_n$$

$$S_n = \frac{1}{2^2-2} + \frac{1}{3^2-3} + \frac{1}{4^2-4} + \dots + \frac{1}{(n-1)^2-(n-1)} + \frac{1}{n^2-n} =$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \cancel{\frac{1}{n-2}} - \frac{1}{n-1} + \cancel{\frac{1}{n-1}} - \frac{1}{n} =$$

$$= 1 - \frac{1}{n} \quad \text{tj. } S_n = 1 - \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} S_n = 1 \Rightarrow \text{red konvergira}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2-n} = 1 \quad \text{suma reda}$$

Broj kojem niz parcijalnih suma KV nazivamo

suma reda

2. Dokazati da red $\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \dots + \frac{1}{(5n-3)(5n+2)} + \dots$ konvergira i naći njegovu sumu.

Rj:

$$\frac{1}{(5n-3)(5n+2)} = \frac{A}{5n-3} + \frac{B}{5n+2} \quad | \cdot (5n-3)(5n+2)$$

$$1 = A(5n+2) + B(5n-3)$$

$$25B = -5$$

$$n^1: 5A + 5B = 0 \quad | \cdot 2 \quad 10A + 10B = 0$$

$$B = -\frac{1}{5} \Rightarrow A = \frac{1}{5}$$

$$n^o: \underline{2A - 3B = 1} \quad | \cdot 5 \quad \underline{-10A - 15B = 5}$$

$$\frac{1}{(5n-3)(5n+2)} = \frac{\frac{1}{5}}{5n-3} + \frac{-\frac{1}{5}}{5n+2} = \frac{1}{5} \left(\frac{1}{5n-3} - \frac{1}{5n+2} \right) = a_n$$

$$S_n = \frac{1}{5} \left[\frac{1}{2} - \frac{1}{7} + \frac{1}{7} - \frac{1}{12} + \dots + \frac{1}{5(n-1)-3} - \frac{1}{5(n-1)+2} + \frac{1}{5n-3} - \frac{1}{5n+2} \right]$$

$$S_n = \frac{1}{5} \left(\frac{1}{2} - \frac{1}{5n+2} \right), \quad \lim_{n \rightarrow \infty} S_n = \frac{1}{10} \Rightarrow \text{red KV}$$

suma reda je

$$\sum_{n=1}^{\infty} \frac{1}{(5n-3)(5n+2)} = \frac{1}{10}$$

3. Dokazati da red $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ konvergira i naći njegovu sumu.

$$Rj: \frac{1}{n(n+1)(n+2)} = \frac{A}{n(n+1)} + \frac{B}{(n+1)(n+2)} \quad | \cdot n(n+1)(n+2)$$

$$1 = A(n+2) + Bn$$

$$A = \frac{1}{2} \Rightarrow B = -\frac{1}{2}$$

$$n^1: A + B = 0$$

$$n^o: \underline{2A = 1}$$

$$\frac{1}{n(n+1)(n+2)} = \frac{\frac{1}{2}}{n(n+1)} - \frac{\frac{1}{2}}{(n+1)(n+2)}$$

$$S_n = a_1 + a_2 + \dots + a_n = \frac{1}{2} \left(\frac{1}{1 \cdot 2} - \cancel{\frac{1}{2 \cdot 3}} + \cancel{\frac{1}{2 \cdot 3}} - \cancel{\frac{1}{3 \cdot 4}} + \dots + \cancel{\frac{1}{n(n+1)}} - \frac{1}{(n+1)(n+2)} \right)$$

$$S_n = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right), \quad \lim_{n \rightarrow \infty} S_n = \frac{1}{4} \quad \text{red konvergira}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4} \quad \text{suma reda}$$

4.) Dokazati da red $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ konvergira i nadi njegovu sumu.

Rj: $a_n = \frac{1}{(2n-1)(2n+1)}$ opći član reda, $a_1 = \frac{1}{1 \cdot 3}, a_2 = \frac{1}{3 \cdot 5}, a_3 = \frac{1}{5 \cdot 7}$

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} \quad | \cdot (2n-1)(2n+1)$$

$$1 = A(2n+1) + B(2n-1)$$

$$\begin{aligned} n^1: \quad 2A + 2B &= 0 \\ A - B &= 1 \end{aligned} \quad \begin{aligned} 2A + 2B &= 0 \\ + 2A - 2B &= 2 \end{aligned} \quad \begin{aligned} 4A &= 2 \\ A &= \frac{1}{2} \end{aligned} \Rightarrow B = -\frac{1}{2}$$

$$\frac{1}{(2n-1)(2n+1)} = \frac{\frac{1}{2}}{2n-1} + \frac{-\frac{1}{2}}{2n+1} = \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = a_n$$

$$S_n = a_1 + a_2 + \dots + a_n = \frac{1}{2} \left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1} \right)$$

$$S_n = \frac{1}{2} \left(1 - \frac{1}{2n+1} \right), \quad \lim_{n \rightarrow \infty} S_n = \frac{1}{2} \Rightarrow \text{red KV}$$

suma reda je $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$

5.) Dokazati da red $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \dots + \frac{1}{n^2+2n} + \dots$ konvergira i nadi njegovu sumu.

Rj: $\frac{3}{4}$

6.) Dokazati da red $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$ konvergira i nadi njegovu sumu.

7.) Dokazati da red $\sum_{n=1}^{\infty} \frac{n}{2^n}$ konvergira i nadi njegovu sumu.

Uputa: $S_n = \dots \quad \left. \begin{array}{l} \\ \end{array} \right\} - \Rightarrow \frac{1}{2} S_n = \frac{1}{2} \cdot \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} - \frac{n}{2^{n+1}} \dots$

Teorema (potreban uslov za konvergenciju reda)

$$\sum_{n=1}^{\infty} a_n \text{ KV} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

što je ekvivalentno sa $a_n \neq 0 \ (n \rightarrow \infty) \Rightarrow \sum_{n=1}^{\infty} a_n \text{ DV}$

(8.) Ispitati konvergenciju redova

a) $\sum_{n=1}^{\infty} \frac{3n+4}{5n-1}$ $\text{Rj. } \lim_{n \rightarrow \infty} \frac{3n+4}{5n-1} : n = \lim_{n \rightarrow \infty} \frac{3 + \frac{4}{n}}{5 - \frac{1}{n}} = \frac{3}{5} \neq 0$

red DV

b) $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ $\text{Rj. } \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1 \neq 0 \Rightarrow \text{red DV}$

c) $\sum_{n=1}^{\infty} \frac{1}{n}$ $\text{Rj. } \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ na osnovu ovog rezultata
ne možemo više zaključiti

$$a_n = \frac{1}{n}, \quad a_{2^m} = \frac{1}{2^m}, \quad a_{n+1} + a_{n+2} + \dots + a_{n+p} = \underbrace{\frac{1}{2^m+1} + \dots + \frac{1}{2^m+2^m}}_{\geq \frac{1}{2^m+2^m}} \geq \frac{2^m}{2^m+2^m} = \frac{1}{2}$$

pa Karijev uslov nije zadovoljen kad $\epsilon < \frac{1}{2}$

red $\sum_{n=1}^{\infty} \frac{1}{n}$ DV (ovo je harmonički red)

Karijev - uslov - za - redove: Red $\sum a_n$ KV akko

$\forall \epsilon > 0 \ \exists N \ (n > N \Rightarrow |a_{n+1} + \dots + a_{n+p}| < \epsilon \text{ za } p=1,2,\dots)$

d) $\sum_{n=1}^{\infty} \frac{5+3n^2}{4n+12n^2}$

f) $\sum_{n=1}^{\infty} \sin n \alpha, \quad 0 < \alpha < \pi$

e) $\sum_{n=1}^{\infty} \frac{3n!}{7n! - n^3}$

Rj: $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sin n \alpha$

ova: limes ne postoji (stalno oscilira)

$\Rightarrow \text{red DV}$

Testovi za konvergenciju redova sa pozitivnim članovima

I Test upoređivanja (kriterij poređenja)

a) $a_n > 0, b_n > 0, a_n \leq b_n \quad \forall n > N$

$$\sum_{n=1}^{\infty} b_n \text{ KV} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ KV}$$

$$\sum_{n=1}^{\infty} a_n \text{ DV} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ DV}$$

b) $a_n > 0, b_n > 0, \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l \neq 0 \Rightarrow \sum a_n : \sum b_n$
istovremeno su KV ili DV

1) Ispitati konvergenciju redova:

a) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ R.j. $\ln x > 1$ akko $x > e$ ($e = 1$)
 $\ln n > 1 \quad (n=3, 4, 5, \dots)$
 $\frac{\ln n}{n} > \frac{1}{n}$, kako $\sum_{n=1}^{\infty} \frac{1}{n}$ DV $\Rightarrow \sum_{n=2}^{\infty} \frac{\ln n}{n}$ DV

$\left[\sum_{n=0}^{\infty} q^n$ geometrički red, KV (ako je $|q| < 1$)
DV (ako je $|q| \geq 1$)

b) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$ R.j. $\frac{1}{3^n} = \frac{1}{3^n}$

$$\frac{1}{n \cdot 3^n} \leq \frac{1}{3^n} \quad (n=1, 2, \dots)$$

kako $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ KV (geom. red) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$ KV

c) $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ hiperharmonijski red, $\alpha \in \mathbb{R}$

Rj: $\alpha \leq 1 \Rightarrow n^\alpha \leq n \Rightarrow \frac{1}{n^\alpha} \geq \frac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ DV} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ DV } (\alpha \leq 1)$$

$$n^2 = n^2$$

$$n^2 - n < n^2, n=1, 2, \dots$$

$$\frac{1}{n^2-n} > \frac{1}{n^2} \text{ kako red } \sum_{n=1}^{\infty} \frac{1}{n^2-n} \text{ KV (zadatok broj 1 iz Beckovih red.)}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ KV}$$

$$\alpha \geq 2 \Rightarrow n^\alpha \geq n^2 \Rightarrow \frac{1}{n^\alpha} \leq \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ KV} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ KV } (\alpha \geq 2)$$

d) $\sum_{n=1}^{\infty} n \sin \frac{1}{n^3}$

e) $\sum_{n=1}^{\infty} \frac{1}{\ln n}$

f) $\sum_{n=1}^{\infty} 2^n \sin \frac{1}{3^n}$

g) $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

Rj: $\lim_{n \rightarrow \infty} \frac{\frac{n}{n^2+3}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+3} \underset{1:n^2}{\underset{1:n^2}{\sim}} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{3}{n^2}} = 1$

$$\sum \frac{1}{n} \text{ DV} \Rightarrow \sum \frac{n}{n^2+3} \text{ DV}$$

Napomena: Ako je $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ pišemo $a_n \sim b_n$ ($n \rightarrow \infty$).

Cita se: "a_n asymptotski jednako sa b_n".

h) $\sum_{n=1}^{\infty} \frac{5n^2}{5n^3+7}$

i) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

j) $\sum_{n=1}^{\infty} \frac{1}{5^n - n}$

II D'Alembertov kriterij

$g = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$, za red $\sum_{n=1}^{\infty} a_n$, $a_n > 0$

1° $g < 1$, red KV

2° $g > 1$, red DV

3° $g = 1$, ne možemo višta zaključiti

1. Ispitati konvergenciju redova:

$$a) \sum_{n=1}^{\infty} \frac{2^n}{n^p}, p \in \mathbb{R} \quad \text{rj: } g = \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{(n+1)^p}}{\frac{2^n}{n^p}} = \lim_{n \rightarrow \infty} \frac{2 \cdot 2 \cdot n^p}{2^n \cdot (n+1)^p} \\ = 2 \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^p = 2 \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n} \right)^p} = 2$$

$g > 1 \Rightarrow \text{red DV}$

$$b) \sum_{n=1}^{\infty} \frac{n^p}{n!}, p \in \mathbb{R} \quad \text{rj: } g = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^p}{(n+1)!}}{\frac{n^p}{n!}} = \lim_{n \rightarrow \infty} \frac{n! \cdot (n+1)^p}{n! \cdot (n+1) \cdot n^p} \\ = \lim_{n \rightarrow \infty} \frac{1}{n+1} \cdot \left(\frac{n+1}{n} \right)^p = \lim_{n \rightarrow \infty} \frac{1}{n+1} \cdot \left(1 + \frac{1}{n} \right)^p = 0$$

$g < 1 \Rightarrow \text{red KV}$

$$c) \sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdot 10 \cdots (3n+1)}{2 \cdot 6 \cdot 10 \cdots (4n-2)} \quad \text{rj: } g = \lim_{n \rightarrow \infty} \frac{\frac{4 \cdot 7 \cdot 10 \cdots (3(n+1)+1)}{2 \cdot 6 \cdot 10 \cdots (4(n+1)-2)}}{\frac{4 \cdot 7 \cdot 10 \cdots (3n+1)}{2 \cdot 6 \cdot 10 \cdots (4n-2)}} \\ = \lim_{n \rightarrow \infty} \frac{3n+4}{4n+2} = \frac{3}{4} < 1$$

$g < 1 \Rightarrow \text{red KV}$

$$d) \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$e) \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2^{n+1}}$$

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots 2n, \quad (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

$$f) \sum_{n=1}^{\infty} \frac{n}{(\sqrt{3})^n}$$

III Cauchy-jev kriterij

$g = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$, za red $\sum_{n=1}^{\infty} a_n$, $a_n > 0$

- 1° $g < 1$, red KV
- 2° $g > 1$, red DV
- 3° $g = 1$, neodlučno

1. Ispitati konvergenciju redova:

a) $\sum_{n=1}^{\infty} \frac{n}{a^n}$, $a > 1$ $\text{Rj. } g = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{a^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{a} = \frac{1}{a} < 1$

(od ranije znamo da $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$) $g < 1 \Rightarrow \text{red KV}$

b) $\sum_{n=2}^{\infty} \frac{1}{\ln^n n}$ $\text{Rj. } g = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{\ln^n n}} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0$
 $g < 1 \Rightarrow \text{red KV}$

c) $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{n^2+n+1} \right)^{n^2}$ $\text{Rj. } g = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n^2+1}{n^2+n+1} \right)^{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{n^2+1}{n^2+n+1} \right)^n = 1^\infty$

1^∞ - povravamo se na broj e

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{n^2+n+1-n}{n^2+n+1} \right)^n &= \lim_{n \rightarrow \infty} \left(1 - \frac{n}{n^2+n+1} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n^2+n+1}{-n}} \right)^n \\ &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{\frac{n^2+n+1}{-n}} \right)^{\frac{n^2+n+1}{-n}} \right]^{\frac{n}{\frac{n^2+n+1}{-n}}} = e^{\lim_{n \rightarrow \infty} \frac{-n^2}{n^2+n+1} \frac{1}{1:n^2}} = e^{-1} = \frac{1}{e} < 1 \end{aligned}$$

$g < 1 \Rightarrow \text{red KV}$

d) $\sum_{n=1}^{\infty} \frac{n^2}{(2+\frac{1}{n})^n}$

e) $\sum_{n=1}^{\infty} \frac{n^n}{a^{n^2}}$, $a > 0$

f) $\sum_{n=1}^{\infty} \left(\frac{2n^2+4n+5}{2n^2+n-1} \right)^{n^2+4n}$

Početni uslovi

Rješenje oblika $\varphi(x, y, c) = 0$ diferencijalne jednačine $y' = f(x, y)$ zovemo opšte rješenje diferencijalne jednačine. Ako u opštem rješenju konstanta c dobije neku određenu vrijednost, dobijamo partikularno rješenje diferencijalne jednačine.

- (2.) Naci ono rješenje diferencijalne jednačine $y' = y$ koje zadovoljava uvjete $y=1$ za $x=0$.

Rj. $y' = y$ $\int \frac{dy}{y} = \int dx$ $y(0) = 1$
 $\frac{dy}{dx} = y$ $1 \cdot \frac{dy}{y}$ $ce^0 = 1$
 $\frac{dy}{y} = dx$ $\ln|y| = x + C$ $C = 1$
 $y = e^{x+C}$ $y = e^x$ partikularno
 $y = C e^x$ opšte rješenje rješenje

- (3.) Naci opšte rješenje diferencijalne jednačine $y' = x^2 - 2$.

Rj. $\frac{dy}{dx} = x^2 - 2$ $dy = (x^2 - 2) dx$ $\int dy = \int (x^2 - 2) dx$
 $y = \frac{x^3}{3} - 2x + C$ opšte rješenje

Diferencijalne jednačine - prvo reda su:

1. diferencijalne jednačine sa razdvojenim promjenjivim

2. homogene diferencijalne jednačine $y' = f(\frac{y}{x})$

uvodimo smjeru $\frac{y}{x} = u$

3. diferenc. jedn. koje se svede na homogene

$y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$, ako je $a_1b_2 - a_2b_1 = 0$
 uvodimo smjeru $a_1x + b_1y = u$

ako je $a_1b_2 - a_2b_1 \neq 0$ uvodimo smjeru $x = u + d$, $y = v + \beta$

4. linearna diferencijalna jednačina $y' + p(x)y = g(x)$

uvodimo smjeru $y = u \cdot v$

5. Bernulijeva diferenc. jednač. $y' + p(x)y = g(x) \cdot y^n$, nek $n \neq 0, n \neq 1$
 uvodimo smjeru $y = u \cdot v$

6. Lagranžova diferencijalna jednačina $y = xf(y') + g(y')$
uvodimo supjene $y' = p$, $x = uv$

Diferencijalne jednačine sa razdvojenim promjenljivim

su oblika: $y' = f(x) \cdot g(y)$

1. Riješiti diferencijalnu jednačinu $xy' = y - xy \sin x$.

$$R_j: xy' = y - xy \sin x$$

$$xy' = y(1 - x \sin x) \quad | : x \quad (x \neq 0)$$

$$y' = y \cdot \frac{1 - x \sin x}{x} \quad \begin{array}{l} \text{ovo je dif.} \\ \text{jedn. sa razdv.} \end{array}$$

$$\frac{dy}{dx} = y \cdot \frac{1 - x \sin x}{x} \quad | \cdot \frac{dx}{y} \quad \begin{array}{l} \text{promj.} \\ | \cdot \frac{dx}{Y} \end{array}$$

$$\frac{dy}{y} = \left(\frac{1}{x} - \sin x \right) dx \quad |||$$

$$\int \frac{dy}{y} = \int \frac{1}{x} dx - \int \sin x dx$$

$$\ln|y| = \ln|x| + \cos x + \ln C$$

$$\ln|y| = \ln|x \cdot C| + \ln e^{\cos x}$$

ISPITNI ZADATAK

$y = Cx e^{\cos x}$ opšte rješenje dif. jedn.

2. Riješiti diferencijalnu jednačinu

$$(xy^2 + 3x) dx + (2x^2 y - 5y) dy = 0.$$

$$R_j: (2x^2 y - 5y) dy = -(xy^2 + 3x) dx$$

$$y(2x^2 - 5) dy = -x(y^2 + 3) dx$$

$$\int \frac{y}{y^2 + 3} dy = \left| \begin{array}{l} y^2 = t \\ 2y dy = dt \end{array} \right| = \frac{1}{2} \int \frac{dt}{t+3}$$

$$= \frac{1}{2} \ln|y^2 + 3|$$

$$\frac{y}{y^2 + 3} dy = \frac{-x}{2x^2 - 5} dx \quad \begin{array}{l} \text{ovo je dif.} \\ \text{jedn. sa} \\ \text{razdv. prom.} \end{array}$$

$$\int \frac{x}{2x^2 - 5} dx = \left| \begin{array}{l} 2x^2 = t \\ 4x dx = dt \\ x dx = \frac{1}{2} dt \end{array} \right| = \frac{1}{4} \int \frac{dt}{t-5}$$

$$= \frac{1}{4} \ln|2x^2 - 5|$$

$$\int \frac{y}{y^2 + 3} dy = - \int \frac{x}{2x^2 - 5} dx$$

$$\frac{1}{2} \ln|y^2 + 3| = -\frac{1}{4} \ln|2x^2 - 5| + \ln C_1 \quad | \cdot 4$$

$$\ln|y^2 + 3|^2 = \ln|C \cdot (2x^2 - 5)^{-1}|$$

$$(y^2 + 3)^2 = \frac{C}{2x^2 - 5}$$

opšte rješenje dif. jedn.

3. Riješiti diferencijalnu jednačinu

$$3y'(x^2 - 1) - 2x y = 0$$

$$R_j: y^3 = C(x^2 - 1) \quad \begin{array}{l} \text{opšte} \\ \text{rješ.} \\ \text{dif. jedn.} \end{array}$$

Homogene diferencijalne jednačine

su oblika $y' = f\left(\frac{y}{x}\right)$, uvodimo smjeru $\frac{y}{x} = u \Rightarrow y = ux$, $y' = u'x + u$

(1) Riješiti diferencijalnu jednačinu $xy' + y = -x$.

$$Rj: xy' + y = -x \quad | :x (x \neq 0)$$

$$y' + \frac{y}{x} = -1$$

$$y' = -1 - \frac{y}{x} \quad \text{ovo je hom. dif. jedn.}$$

$$\text{uvodimo smjeru } u = \frac{y}{x}$$

$$y = ux, \quad y' = u'x + u$$

$$u'x + u = -1 - u$$

$$2u + 1 = \left(\frac{c_1}{x}\right)^2$$

$$2u = \frac{c}{x^2} - 1 \Rightarrow 2\frac{y}{x} = \frac{c}{x^2} - 1 \Rightarrow 2y = \frac{c}{x^2} - x \quad t.j. \quad y = \frac{c}{x^2} - \frac{x}{2}$$

(2) Nadi partikularno rješenje diferencijalne jednačine $xy' = y(1 + \ln y - \ln x)$ tako da zadovoljava uslov $y(1) = e$.

$$Rj: xy' = y\left(1 + \ln \frac{y}{x}\right) \quad | :x$$

$$y' = \frac{y}{x}\left(1 + \ln \frac{y}{x}\right) \quad \text{ovo je hom. dif. jedn.}$$

$$u = \frac{y}{x} \Rightarrow y = ux, \quad y' = u'x + u$$

$$u'x + u = u(1 + \ln u)$$

$$u'x = u \ln u, \quad u' = \frac{du}{dx}$$

$$\frac{du}{u \ln u} = \frac{dx}{x} \quad |||$$

$$\frac{du}{u} \times = -1 - 2u \quad | \cdot \frac{dx}{(-1-2u) \cdot x}$$

$$\frac{du}{-1-2u} = \frac{dx}{x}$$

$$\frac{du}{2u+1} = -\frac{dx}{x} \quad |||$$

$$\begin{aligned} 2u &= t \\ 2du &= dt \\ du &= \frac{1}{2}dt \end{aligned}$$

$$\frac{1}{2} \ln|2u+1| = -\ln|x| + \ln|C_1| \quad | \cdot 2$$

$$\ln|2u+1| = 2 \ln|\frac{C_1}{x}|$$

opšte rješenje
diferenc. jed.

$$\int \frac{du}{u \ln u} = \left| \begin{array}{l} \ln u = t \\ \frac{du}{u} = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| = \ln \ln u$$

$$\ln \ln u = \ln x + \ln C$$

$$\ln u = xC \Rightarrow u = e^{cx}$$

$$y = x e^{cx} \quad \text{opšte rješenje
dif. jedn.}$$

$$\left. \begin{array}{l} y(1) = e \\ y(1) = 1 \cdot e^{c \cdot 1} \end{array} \right\} \Rightarrow e^c = e \Rightarrow c = 1$$

$$y = x e^x \quad \text{partikularno rješenje dif. jedn.}$$

(3) Nadi opšte rješenje dif. jednačine $xy' = x e^{\frac{y}{x}} + y$.

$$Rj: y = -x \ln \ln \frac{c}{x}$$

Diferencijalne jednačine koje se rade na homogene

su oblika $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

ako je $a_1b_2 - a_2b_1 = 0$ uvodimo smjeru $x = u + \lambda$ i dobijamo dif. jedn. sa razduojevim pravjenjivim.

ako je $a_1b_2 - a_2b_1 \neq 0$ uvodimo smjeru $x = u + \lambda$, $y = v + \beta$ gdje i β dobijamo iz sistema $a_1\lambda + b_1\beta + c_1 = 0$, $a_2\lambda + b_2\beta + c_2 = 0$.

1. Riješiti diferencijalnu jednačinu $(x-2y+1)y' = 2x-y+1$.

Rj: $y' = \frac{2x-y+1}{x-2y+1}$, $a_1b_2 - a_2b_1 \neq 0 \Rightarrow x = u + \lambda$, $y = v + \beta$

$$\begin{cases} 2\lambda - \beta + 1 = 0 \\ \lambda - 2\beta + 1 = 0 \end{cases} \Rightarrow \lambda = -\frac{1}{3}, \beta = \frac{1}{3} \quad \begin{array}{l} x = u - \frac{1}{3} \\ y = v + \frac{1}{3} \end{array} \quad y' = v'$$

$$v' = \frac{2(u - \frac{1}{3}) - (v + \frac{1}{3}) + 1}{(u - \frac{1}{3}) - 2(v + \frac{1}{3}) + 1}$$

$$z'u + z = \frac{2-z}{1-2z}$$

$$z'u = \frac{2(z^2 - z + 1)}{1-2z}, \quad z' = \frac{dz}{du}$$

$$\frac{1-2z}{z^2 - z + 1} dz = 2 \frac{du}{u} \quad //$$

$$-\ln(z^2 - z + 1) = 2 \ln u + \ln C_1$$

$$\text{smjera } \frac{v}{u} = z, \quad v = uz \quad v' = z'u + z$$

$$\ln \frac{1}{z^2 - z + 1} = \ln C_1 u^2$$

$$1 = C_1 u^2 (z^2 - z + 1)$$

$$C = (y - \frac{1}{3})^2 - (x + \frac{1}{3})(y - \frac{1}{3}) + (x + \frac{1}{3})^2$$

$$1 = C_1 u^2 (\frac{v^2}{u^2} - \frac{v}{u} + 1) \quad /: C_1$$

opr̄ite rješenje
diferenc. jednač.

$$C = v^2 - uv + u^2$$

2. Riješiti diferencijalnu jednačinu $(2x+y+1)y' = 4x+2y+3$. Rj: $\ln C x^{16} (8x+4y+5) = 4(2x+y+1)$

3. Riješiti diferencijalnu jednačinu

$$(2x-4y+6)dx + (x+y-3)dy = 0 \quad Rj: (y-2x)^3 = C(y-x-1)^2$$

opr̄ite rješenje

Linearna diferencijalna jednačina

su oblika $y' + p(x) \cdot y = q(x)$. uvodimo smjeru $y = uv$.

1. Riješiti diferencijalnu jednačinu $(1+x^2)y' = x(2y+1)$.

$$Rj.: (1+x^2)y' = 2x y + x$$

$$(1+x^2)y' - 2x y = x \quad |:(1+x^2)$$

$$y' - \frac{2x}{1+x^2} y = \frac{x}{1+x^2} \quad \begin{array}{l} \text{ovo je} \\ \text{lin. dif.} \\ \text{jedn.} \end{array}$$

$$y = uv, \quad y' = u'v + u \cdot v'$$

uvjetimo smjeru

$$u' \cdot v + u \cdot v' - \frac{2x}{1+x^2} uv = \frac{x}{1+x^2}$$

$$u' \cdot v + u \cdot \left(v' - \frac{2x}{1+x^2} v \right) = \frac{x}{1+x^2}$$

ovaj dio iz jednačine
sa o da bi našli v'

a)

$$v' - \frac{2x}{1+x^2} v = 0, \quad v' = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{2x}{1+x^2} v$$

$$\frac{dv}{v} = \frac{2x}{1+x^2} dx \quad ||$$

$$b) \quad u' v = \frac{x}{1+x^2}$$

$$u' (1+x^2) = \frac{x}{1+x^2}, \quad u' = \frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{x}{(1+x^2)^2}, \quad du = \frac{x}{(1+x^2)^2} dx \quad ||$$

$$\int du = \int \frac{x}{(1+x^2)^2} dx \quad \begin{array}{l} 1+x^2 = t \\ 2x dx = dt \\ x dx = \frac{1}{2} dt \end{array}$$

$$u = -\frac{1}{2(1+x^2)} + C$$

$$Y = u \cdot v = \left[-\frac{1}{2(1+x^2)} + C \right] (1+x^2)$$

$$Y = C(1+x^2) - \frac{1}{2} \quad \begin{array}{l} \text{opr̄te rješenje} \\ \text{diferencijalne} \\ \text{jednačine} \end{array}$$

2.

Riješiti diferencijalnu jednačinu ako je $y(1) = -1$.

$$Rj.: \quad y = \frac{x}{x+1} (x + \ln|x| + C) \quad \begin{array}{l} \text{opr̄te rješenje} \\ \text{diferenc. jedn.} \end{array}$$

$$xy' - \frac{y}{x+1} = x$$

$$y = \frac{x}{x+1} (x + \ln|x| - 3)$$

partikularno rješenje
diferenc. jedn.

3.

Riješiti diferencijalnu jednačinu $y' + y \cos x = 0,5 \sin 2x$

$$Rj.: \quad y = 1 - \sin x + C e^{-\sin x} \quad \begin{array}{l} \text{opr̄te} \\ \text{rješenje} \\ \text{dif. jedn.} \end{array}$$

Bernulijeva diferencijalna jednačina

su oblika $y' + p(x)y = q(x)y^n$, $n \in \mathbb{Q}$, $n \neq 0$ i $n \neq 1$

uvodimo smjenu $y = uv$ ove dif. jedn. rješavamo u isti način
kao što smo rješavali linearne dif. jedn.

1) Riješiti diferencijalnu jednačinu $xy' - x^2\sqrt{y} = 4y$.

$$Rj: xy' - 4y = x^2\sqrt{y} \quad | : x$$

$$b) u'v = x\sqrt{uv}$$

$$y' - \frac{4}{x}y = x\sqrt{y} \quad \text{ovo je Bern. dif. jedn.}$$

$$\text{smjena } y = uv$$

$$y' = u'v + uv'$$

$$u'v + uv' - \frac{4}{x}uv = x\sqrt{uv}$$

$$u'v + u(v' - v\frac{4}{x}) = x\sqrt{uv}$$

$\stackrel{=0}{(du bi smo našli v)}$

$$a) v' - v\frac{4}{x} = 0 \Rightarrow v' = v\frac{4}{x}$$

$$v' = \frac{dv}{dx}, \quad \frac{dv}{v} = \frac{4}{x}dx \quad //$$

$$\int \frac{dv}{v} = \int \frac{4}{x}dx \Rightarrow \ln|v| = 4\ln|x|$$

$$v = x^4$$

$$\int \frac{du}{\sqrt{u}} = \int \frac{dx}{x} \Rightarrow 2\sqrt{u} = \ln x + C$$

$$\sqrt{u} = \frac{\ln x + C}{2}$$

$$u = \frac{(\ln x + C)^2}{4}$$

$$y = uv = \frac{(\ln x + C)^2}{4} \cdot x^4$$

$$y = \frac{x^4}{4} (\ln x + C)^2$$

opršte
rješenje
difer. jedn.

2) Naci partikularno rješenje diferencijalne jednačine $y' = xy^3 - y$ koje prolazi kroz tačku $A(0, 1)$.

$$Rj: y^{-2} = e^{2x} \left[e^{-2x} \left(x + \frac{1}{2} \right) + C \right]$$

opršte
rješenje
dif. jedn.

$$y^{-2} = \frac{1}{2} e^{2x} + x + \frac{1}{2}$$

partikul. rješ. dif. jedn.

3) Riješiti diferencijalnu jednačinu

$$(1-x^2)y' = xy + x^2y^2$$

$$Rj: y = \frac{c}{\sqrt{1-x^2}} - 1$$

Lagranžova diferencijalna jednačina

su oblika $y = x f(y') + g(y')$ uvodimo smjenu $y' = p$, $x = uv$

1. riješiti diferencijalnu jednačinu $y + xy' = 4\sqrt{y'}$.

Rj. $y = x \cdot (-y') + 4\sqrt{y'}$ ovo je Lagr. dif. jedn.

$$uv + u(v' + \frac{v}{2p}) = \frac{1}{p\sqrt{p}} = 0$$

uvodimo smjenu $y' = p$

$$y = -xp + 4\sqrt{p} \quad / \frac{d}{dx}$$

$$y' = -p - x \cdot p' + 4 \cdot \frac{1}{2\sqrt{p}} \cdot p', \quad y' = p$$

$$a) v' + \frac{v}{2p} = 0, \quad \frac{dv}{dp} = -\frac{v}{2p}$$

$$\frac{dv}{v} = -\frac{1}{2} \frac{dp}{p} \quad //$$

$$2p = p'(-x + \frac{2}{\sqrt{p}}), \quad p' = \frac{dp}{dx}$$

$$\ln|v| = -\frac{1}{2} \ln|p|$$

$$\frac{1}{p'} = \frac{dx}{dp} = x', \quad \frac{1}{p'} = \frac{-x + \frac{2}{\sqrt{p}}}{2p}$$

$$v = p^{-\frac{1}{2}} = \frac{1}{\sqrt{p}}$$

$$x' = -\frac{x}{2p} + \frac{1}{p\sqrt{p}}$$

$$b) u \cdot \frac{1}{\sqrt{p}} = \frac{1}{p\sqrt{p}} \quad | \cdot \sqrt{p} \Rightarrow u = \frac{1}{p}$$

$$x' + \frac{x}{2p} = \frac{1}{p\sqrt{p}} \quad \text{ovo je linear. dij. jedn.}$$

$$u = \ln|p| + C$$

$$x = uv = \frac{\ln|p| + C}{\sqrt{p}} \quad (*)$$

uvodimo smjenu $x = uv$,
 $x' = u'v + uv'$

$$y = -xp + 4\sqrt{p} = -p \frac{c + \ln|p|}{\sqrt{p}} + 4\sqrt{p}$$

$$u'v + uv' + \frac{uv}{2p} = \frac{1}{p\sqrt{p}}$$

$$y = \sqrt{p} (4 - c - \ln|p|) \quad (*) \text{ i } (**)$$

2. riješiti diferencijalnu jednačinu u parametarskom obliku

$$y'(2x-y) = y$$

Rj. $\left. \begin{array}{l} x = \frac{2}{3}p + \frac{c}{p^2} \\ y = 2xp - p^2 \end{array} \right\}$ opšte rješ. dif. jedn.

3. Nadi rješenje diferencijalne jednačine $y = xy' - 2 - y'$ koje prolazi kroz tačku $A(2,5)$.

Rj. $y = xc - 2 - c$ opšte rješenje

$$y = 7x - 9 \quad \text{partikularno rješenje}$$