TABLICA DERIVIRANJA ELEMENTARNIH FUNKCIJA

1. $y = C = \text{konstanta} \quad \Rightarrow \quad y' = 0$

2. $y = x^n, n \in \mathbf{R} \quad \Rightarrow \quad y' = n \cdot x^{n-1}$ (posebno: $y = x \;\Rightarrow\; y' = 1$)

3. $y = a^x \quad \Rightarrow \quad y' = a^x \cdot \ln a$ (posebno: $y = e^x \;\Rightarrow\; y' = e^x$)

4. $y = \log_a x \quad \Rightarrow \quad y' = \frac{1}{x} \cdot \frac{1}{\ln a}$ (posebno: $y = \ln x \;\Rightarrow\; y' = \frac{1}{x}$)

5. $y = \sin x \quad \Rightarrow \quad y' = \cos x$

6. $y = \cos x \quad \Rightarrow \quad y' = -\sin x$

7. $y = tgx \quad \Rightarrow \quad y' = \frac{1}{\cos^2 x}$

8. $y = ctgx \quad \Rightarrow \quad y' = \frac{-1}{\sin^2 x}$

9. $y = \arcsin x \quad \Rightarrow \quad y' = \frac{1}{\sqrt{1-x^2}}$

10. $y = \arccos x \quad \Rightarrow \quad y' = -\frac{1}{\sqrt{1-x^2}}$

11. $y = arctgx \quad \Rightarrow \quad y' = \frac{1}{1+x^2}$

12. $y = arcctgx \quad \Rightarrow \quad y' = -\frac{1}{1+x^2}$

PRAVILA DERIVIRANJA

1. $y = f(x) \pm g(x) \quad \Rightarrow \quad y' = f'(x) \pm g'(x)$

2. $y = f(x) \cdot g(x) \quad \Rightarrow \quad y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

(posebno: $y = C \cdot f(x) \;\Rightarrow\; y' = C \cdot f'(x)$)

3. $y = \frac{f(x)}{g(x)} \quad \Rightarrow \quad y' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$

4. $y = (g \circ f)(x) = g(f(x)) \quad \Rightarrow \quad y' = g'(f(x)) \cdot f'(x) = \frac{dg}{df} \cdot \frac{df}{dx}$

Zadatak 1.

a) $(x^2)' = 2 \cdot x^{2-1} = 2x^1 = 2x$

b) $(x^3)' = 3 \cdot x^{3-1} = 3x^2$

c) $(x^5)' = 5 \cdot x^{5-1} = 5x^4$

d) $(x)' = 1 \cdot x^{1-1} = x^0 = 1$

e) $(x^{-1})' = (-1) \cdot x^{-1-1} = (-1) \cdot x^{-2} = -\dfrac{1}{x^2}$

f) $(x^{-4})' = (-4) \cdot x^{-4-1} = (-4) \cdot x^{-5} = -\dfrac{4}{x^5}$

g) $(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \dfrac{1}{2} x^{\frac{1}{2}-1} = \dfrac{1}{2} x^{-\frac{1}{2}} = \dfrac{1}{2} \cdot \dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

h) $(\sqrt[3]{x^2})' = \left(x^{\frac{2}{3}}\right)' = \dfrac{2}{3} x^{\frac{2}{3}-1} = \dfrac{2}{3} x^{-\frac{1}{3}} = \dfrac{2}{3} \cdot \dfrac{1}{x^{\frac{1}{3}}} = \dfrac{2}{3} \cdot \dfrac{1}{\sqrt[3]{x}} = \dfrac{2}{3\sqrt{x}}$

i) $(5^x)' = 5^x \cdot \ln 5$

j) $\log_5 x = \dfrac{1}{x} \cdot \dfrac{1}{\ln 5}$

k) $\log x = \dfrac{1}{x} \cdot \dfrac{1}{\ln 10}$

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Zadatak 2. Nađite derivacije slijedećih funkcija:

a) $f(x) = 3x - 2$

b) $f(x) = 5x^2 - 3x + 4$

c) $f(x) = \frac{1}{3}x^3 + x^2 - x$

d) $f(x) = \frac{3}{x} - \frac{6}{x^2}$

e) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

Rješenje:

Ove zadatke ćemo rješavati pomoću pravila 1. za derivaciju zbroja, odnosno razlike funkcija te pomoću pravila 3. za derivaciju funkcije pomnožene nekim konstantnim brojem.

a) $f'(x) = (3x-2)' = (3x)' - 2' = 3 \cdot (x)' - 2' = 3 \cdot 1 - 0 = 3$

b) $f'(x) = (5x^2 - 3x + 4)' = 5 \cdot (x^2)' - 3 \cdot (x)' + 4' = 5 \cdot 2x - 3 \cdot 1 + 0 = 10x - 3$

c) $f'(x) = \left(\frac{1}{3}x^3 + x^2 - x\right)' = \frac{1}{3} \cdot (x^3)' + (x^2)' - (x)' = \frac{1}{3} \cdot 3x^2 + 2x - 1 = x^2 + 2x - 1$

d) $f'(x) = \left(\frac{3}{x} - \frac{6}{x^2}\right)' = \left(3 \cdot x^{-1} - 6 \cdot x^{-2}\right)' = 3 \cdot (x^{-1})' - 6 \cdot (x^{-2})' = -\frac{3}{x^2} + \frac{12}{x^3}$

e) $f'(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)' = \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} - \frac{1}{2} \cdot \frac{1}{x^{\frac{3}{2}}} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

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Zadatak 3. Nađite derivacije slijedećih funkcija:

a) $f(x) = (3-4x)(x^2 - 3x + 1)$

b) $f(x) = (x^2 - x + 1)(x^2 + x - 1)$

c) $f(x) = x \cdot \sin x$

d) $f(x) = x^3 \cdot \ln x$

e) $f(x) = (x+1)e^x$

Rješenje:

Ovaj zadatak ćemo riješiti pomoću pravila 3. za derivaciju umnoška funkcija, te pomoću pravila 1.

a) $f'(x) = (3-4x)'(x^2 - 3x + 1) + (3-4x)(x^2 - 3x + 1)'$

$f'(x) = (0-4)\cdot(x^2 - 3x + 1) + (3-4x)\cdot(2x - 3 + 0)$

$f'(x) = -4(x^2 - 3x + 1) + (3-4x)(2x-3)$

$f'(x) = -4x^2 + 12x - 4 + 6x - 9 - 8x^2 + 12x = -12x^2 + 30x - 3$

b) $f'(x) = (x^2 - x + 1)'(x^2 + x - 1) + (x^2 - x + 1)(x^2 + x - 1)'$

$f'(x) = (2x-1)(x^2 + x - 1) + (x^2 - x + 1)(2x+1)$

$f'(x) = 2x^3 + 2x^2 - 2x - x^2 - x + 1 + 2x^3 + x^2 - 2x^2 - x + 2x + 1$

$f'(x) = 4x^3 - 2x + 2$

c) $f'(x) = (x \cdot \sin x)' = (x)' \cdot \sin x + x \cdot (\sin x)' = 1 \cdot \sin x + x \cdot \cos x = \sin x + x \cos x$

d) $f'(x) = (x^3 \cdot \ln x)' = (x^3)' \cdot \ln x + x^3 \cdot (\ln x)' = 3x^2 \cdot \ln x + x^3 \cdot \frac{1}{x} = 3x^2 \cdot \ln x + x^2$

e) $f'(x) = ((x+1)e^x)' = (x+1)' \cdot e^x + (x+1) \cdot (e^x)' = 1 \cdot e^x + (x+1) \cdot e^x$

$$f'(x) = e^x(1+(x+1)) = e^x(2+x)$$

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Zadatak 4. Derivirajte slijedeće funkcije:

a) $f(x) = \dfrac{2x}{3-x}$

b) $f(x) = \dfrac{x+2}{x-2}$

c) $f(x) = \dfrac{x^2-6x+5}{x-3}$

d) $f(x) = \dfrac{\ln x}{x}$

e) $f(x) = \dfrac{\sin x - \cos x}{\sin x + \cos x}$

Rješenje:

Ovaj zadatak ćemo rješavati pomoću pravila 4. za kvocijent dviju funkcija, te pomoću 1. pravila.

a) $$f'(x) = \left(\frac{2x}{3-x}\right)' = \frac{(2x)'\cdot(3-x)-2x\cdot(3-x)'}{(3-x)^2} = \frac{2(3-x)-2x\cdot(-1)}{(3-x)^2}$$

$$f'(x) = \frac{6-2x+2x}{(3-x)^2} = \frac{6}{(3-x)^2}$$

b) $$f'(x) = \left(\frac{x+2}{x-2}\right)' = \frac{(x+2)'(x-2)-(x+2)(x-2)'}{(x-2)^2}$$

$$f'(x) = \frac{1\cdot(x-2)-(x+2)\cdot 1}{(x-2)^2} = \frac{-4}{(x-2)^2}$$

c) $$f'(x)=\left(\frac{x^2-6x+5}{x-3}\right)'=\frac{(x^2-6x+5)'(x-3)-(x^2-6x+5)(x-3)'}{(x-3)^2}$$

$$f'(x)=\frac{(2x-6)(x-3)-(x^2-6x+5)\cdot 1}{(x-3)^2}=\frac{2x^2-6x-6x+18-x^2+6x-5}{(x-3)^2}$$

$$f'(x)=\frac{x^2-6x+13}{(x-3)^2}$$

d) $$f'(x)=\left(\frac{\ln x}{x}\right)'=\frac{(\ln x)'\cdot x-\ln x\cdot(x)'}{x^2}=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}$$

e) $$f'(x)=\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)'$$

$$f'(x)=\frac{(\sin x-\cos x)'(\sin x+\cos x)-(\sin x-\cos x)(\sin x+\cos x)'}{(\sin x+\cos x)^2}$$

$$f'(x)=\frac{(\cos x+\sin x)(\sin x+\cos x)-(\sin x-\cos x)(\cos x-\sin x)}{(\sin x+\cos x)^2}$$

$$f'(x)=\frac{(\cos x+\sin x)^2+(\sin x-\cos x)^2}{(\sin x+\cos x)^2}$$

$$f'(x)=\frac{\cos^2 x+2\sin x\cos x+\sin^2 x+\sin^2 x-2\sin x\cos x+\cos^2 x}{(\sin x+\cos x)^2}$$

$$f'(x)=\frac{2}{(\sin x+\cos x)^2}$$

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Derivaciju složene funkcije računamo formulom

$$f(g(x))'=f'(g(x))\cdot g'(x),$$

što zapisujemo još i na ovaj način: $f(g(x))'=f'(u)\cdot g'(x),\quad u=g(x)$. Ova se formula naziva i **pravilo o ulančanom deriviranju**. Pravilo možemo iskazati riječima: složena se

funkcija derivira tako da se derivacija "vanjske" funkcije pomnoži s derivacijom "unutarnje". Treba naglasiti da je poredak deriviranja suprotan poretku računanja vrijednosti složene funkcije, kad prvo računamo vrijednost unutarnje funkcije.

Zadatak 5. Odredite derivaciju funkcije $f(x) = (x-5)^4$.

Rješenje:

Ovdje je $y = f(x) = u^4, u = g(x) = x - 5$. Po formuli imamo:

$$\left((x-5)^4\right)' = f'(u) \cdot g'(x) = \left(u^4\right)' \cdot (x-5)' = 4u^3 \cdot 1 = 4u^3 = 4(x-5)^3$$

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ne ispisujući funkcije koje treba derivirati:

$\left((x-5)^4\right)' =$ (deriviramo potenciju i množimo derivacijom argumenta)

$$\left((x-5)^4\right)' = 4(x-5)^3 \cdot (x-5)' = 4(x-5)^3 \cdot 1 = 4(x-5)^3$$

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Zadatak 6. Derivirajte slijedeće funkcije:

a) $f(x) = (x+2)^3$

b) $f(x) = (2x-1)^7$

c) $f(x) = (5x^2 - 3x + 1)^2$

d) $f(x) = \sqrt{x^2 + 1}$

e) $f(x) = \sqrt[4]{(3x+2)^3}$

Rješenje:

a) $f'(x) = \left((x+2)^3\right)' = 3(x+2)^2 \cdot (x+2)' = 3(x+2)^2$

b) $f'(x) = \left((2x-1)^7\right)' = 7(2x-1)^6 \cdot (2x-1)' = 7(2x-1)^6 \cdot 2 = 14(2x-1)^6$

c) $f'(x) = \left((5x^2 - 3x + 1)^2\right)' = 2(5x^2 - 3x + 1)^1 \cdot (5x^2 - 3x + 1)' = 2(5x^2 - 3x + 1)(10x - 3)$

d) $f'(x) = \left(\sqrt{x^2+1}\right)' = \left(\left(x^2+1\right)^{\frac{1}{2}}\right)' = \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \cdot \left(x^2+1\right)' = \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$

e) $f'(x) = \left(\sqrt[4]{(3x+2)^3}\right)' = \left((3x+2)^{\frac{3}{4}}\right)' = \frac{3}{4}(3x+2)^{-\frac{1}{4}} \cdot (3x+2)' = \frac{3}{4} \cdot \frac{1}{\sqrt[4]{3x+2}} \cdot 3 = \frac{9}{4 \cdot \sqrt[4]{3x+2}}$

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Zadatak 7. Derivirajte slijedeće funkcije:

a) $f(x) = \sin 2x$

b) $f(x) = \sin^2 x$

c) $f(x) = \cos(5x-1)$

d) $f(x) = \cos^3(2x+1)$

e) $f(x) = \log_5(x-1)$

f) $f(x) = \ln(\sin x)$

g) $f(x) = e^{-x^2+2}$

Rješenje:

a) $f'(x) = (\sin 2x)' = \cos 2x \cdot (2x)' = \cos 2x \cdot 2 = 2\cos 2x$

b) $f'(x) = \left(\sin^2 x\right)' = \left((\sin x)^2\right)' = 2(\sin x)^1 \cdot (\sin x)' = 2\sin x \cdot \cos x$

c) $f'(x) = \left(\cos(5x-1)\right)' = -\sin(5x-1) \cdot (5x-1)' = -\sin(5x-1) \cdot 5 = -5\sin(5x-1)$

d) $f'(x) = \left(\cos^3(2x+1)\right)' = 3\cos^2(2x+1)\cdot(-\sin(2x+1))\cdot 2 = -6\cos^2(2x+1)\sin(2x+1)$

e) $f'(x) = \left(\log_5(x-1)\right)' = \dfrac{1}{x-1}\cdot\dfrac{1}{\ln 5}\cdot(x-1)' = \dfrac{1}{(x-1)\ln 5}$

f) $f'(x) = \left(\ln(\sin x)\right)' = \dfrac{1}{\sin x}\cdot(\sin x)' = \dfrac{1}{\sin x}\cdot\cos x = ctgx$

g) $f'(x) = \left(e^{-x^2+2}\right)' = e^{-x^2+2}\cdot(-x^2+2)' = e^{-x^2+2}\cdot(-2x)$

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Zadatak 8. Derivirajte slijedeće funkcije:

a) $f(x) = (x-1)^2(x+1)^2$

b) $f(x) = (2x+3)^2(3-2x)^3$

c) $f(x) = (x-1)\sqrt{x^2+1}$

Rješenje:

a) $f'(x) = \left((x-1)^2\right)'(x+1)^2 + (x-1)^2\left((x+1)^2\right)'$

$f'(x) = 2(x-1)\cdot(x-1)'\cdot(x+1)^2 + (x-1)^2\cdot 2(x+1)\cdot(x+1)'$

$f'(x) = 2(x-1)(x+1)^2 + 2(x-1)^2(x+1) = 2(x-1)(x+1)\left[(x+1)+(x-1)\right]$

$f'(x) = 2(x-1)(x+1)\cdot 2x = 4x(x-1)(x+1)$

b) $f'(x) = \left((2x+3)^2\right)'(3-2x)^3 + (2x+3)^2\left((3-2x)^3\right)'$

$f'(x) = 2(2x+3)^1\cdot(2x+3)'\cdot(3-2x)^3 + (2x+3)^2\cdot 3(3-2x)^2\cdot(3-2x)'$

$f'(x) = 2(2x+3)\cdot 2\cdot(3-2x)^3 + (2x+3)^2\cdot 3(3-2x)^2\cdot(-2)$

$f'(x) = 4(2x+3)(3-2x)^3 - 6(2x+3)^2(3-2x)^2$

$$f'(x) = 2(2x+3)(3-2x)^2\left[2(3-2x)-3(2x+3)\right]$$

$$f'(x) = 2(2x+3)(3-2x)^2(6-4x-6x-9)$$

$$f'(x) = 2(2x+3)(3-2x)^2(-3-10x)$$

c) $$f'(x) = (x-1)'\sqrt{x^2+1} + (x-1)\left(\left(x^2+1\right)^{\frac{1}{2}}\right)'$$

$$f'(x) = 1\cdot\sqrt{x^2+1} + (x-1)\cdot\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot(x^2+1)' = \sqrt{x^2+1} + \frac{(x-1)}{2\sqrt{x^2+1}}\cdot 2x$$

$$f'(x) = \sqrt{x^2+1} + \frac{x(x-1)}{\sqrt{x^2+1}} = \frac{(x^2+1)+x^2-x}{\sqrt{x^2+1}}$$

$$f'(x) = \frac{2x^2-x+1}{\sqrt{x^2+1}}$$

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Zadatak 9. Derivirajte funkcije:

a) $y = \dfrac{\sqrt{x+1}}{\sqrt{x-1}}$

b) $y = \dfrac{\sin x}{\cos^2 x}$

c) $y = \ln\dfrac{1+x^2}{1-x^2}$

d) $y = x^2\cdot\ln x$

e) $y = e^x + e^{-x}$

Rješenje:

a) $$y = \frac{\sqrt{x+1}}{\sqrt{x-1}} = \sqrt{\frac{x+1}{x-1}} = \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}\left(\frac{x+1}{x-1}\right)^{-\frac{1}{2}} \cdot \left(\frac{x+1}{x-1}\right)' = \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} \cdot \frac{(x+1)'(x-1)-(x+1)(x-1)'}{(x+1)^2}$$

$$y' = \frac{1}{2}\sqrt{\frac{x-1}{x+1}} \cdot \frac{x-1-x-1}{(x-1)^2} = \frac{1}{2} \cdot \frac{\sqrt{x-1}}{\sqrt{x+1}} \cdot \frac{-2}{(x-1)^2}$$

$$y' = \frac{-\sqrt{x-1}}{(x-1)^2 \cdot \sqrt{x+1}} = \frac{-1}{\sqrt{(x-1)^3} \cdot \sqrt{x+1}}$$

b) $$y' = \frac{(\sin x)' \cos^2 x - \sin x \cdot (\cos^2 x)'}{\cos^4 x} = \frac{\cos x \cdot \cos^2 x - \sin x \cdot 2\cos x \cdot (-\sin x)}{\cos^4 x}$$

$$y' = \frac{\cos x\left[\cos^2 x + 2\sin^2 x\right]}{\cos^4 x} = \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x}$$

c) $$y' = \frac{1}{\frac{1+x^2}{1-x^2}} \cdot \left(\frac{1+x^2}{1-x^2}\right)' = \frac{1-x^2}{1+x^2} \cdot \frac{2x(1-x^2)-(1+x^2)\cdot(-2x)}{(1-x^2)^2}$$

$$y' = \frac{1}{1+x^2} \cdot \frac{2x-2x^3+2x+2x^3}{1-x^2} = \frac{4x}{1-x^4}$$

d) $$y' = (x^2)' \ln x + x^2 \cdot (\ln x)' = 2x \cdot \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x = x(2\ln x + 1)$$

e) $$y' = (e^x)' + (e^{-x})' = e^x + e^{-x} \cdot (-x)' = e^x - e^{-x}$$

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Derivacija drugog reda (druga derivacija) funkcije $y = f(x)$ je derivacija prve derivacije, tj. $y'' = \left[f'(x)\right]'$.

Derivacija trećeg reda (treća derivacija) funkcije $y = f(x)$ je derivacija druge derivacije, tj. $y''' = \left[f''(x)\right]'$.

Analogno se definiraju i sve ostale derivacije do *n*-tog reda.

Zadatak 10. Nađite druge derivacije slijedećih funkcija:

a) $y = 3x^4 - 5x^3 + 2x^2 - x$

b) $y = (2x+5)^3$

c) $y = \dfrac{1}{x-1}$

d) $y = \cos^2 x$

e) $y = e^{-x^2}$

Rješenje:

a) $y' = 12x^3 - 15x^2 + 4x - 1$

$y'' = 36x^2 - 30x + 4$

b) $y' = 3(2x+5)^2 \cdot (2x+5)' = 3(2x+5)^2 \cdot 2 = 6(2x+5)^2$

$y'' = 6 \cdot \left[(2x+5)^2\right]' = 6 \cdot 2(2x+5)^1 \cdot (2x+5)' = 12(2x+5) \cdot 2 = 24(2x+5)$

c) $y = \dfrac{1}{x-1} = (x-1)^{-1}$

$y' = -1(x-1)^{-2} \cdot (x-1)' = -1(x-1)^{-2}$

$y'' = 2(x-1)^{-3} \cdot (x-1)' = 2(x-1)^{-3} = \dfrac{2}{(x-1)^3}$

d) $y' = 2\cos x \cdot (\cos x)' = 2\cos x \cdot (-\sin x) = -2\sin x \cos x$

$y'' = -2 \cdot (\sin x \cos x)' = -2\left[(\sin x)' \cos x + \sin x (\cos x)'\right]$

$y'' = -2(\cos x \cos x + \sin x(-\sin x)) = -2(\cos^2 x - \sin^2 x)$

e) $y' = e^{-x^2} \cdot (-x^2)' = e^{-x^2} \cdot (-2x) = -2xe^{-x^2}$

$y'' = -2\left[(x)' e^{-x^2} + x\left(e^{-x^2}\right)'\right] = -2(e^{-x^2} + x \cdot (-2xe^{-x^2})) = -2e^{-x^2}(1 - 2x^2)$

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