# ELEKTRODINAMIKA

17.10.2006. g.

prof. S. Brant
- IV. kat (FO)
- IFS - soba 113 (Nataša

literatura:
Nayfeh, Brussel : "Electricity and magnetism"
Berkeley 2
Supek : "Teorijska fizika"

Nada Brković : Zbirka zadataka, za srednju školu → OSNOVE (znati!)

## Vektori i koordinatni sustavi

1. $c$ – veličine koje imaju iznos – skalari
2. $\vec{a}$ – veličine koje imaju iznos i smjer – vektori

$|\vec{a}|$, $\vec{a}_0$, $\vec{a}$

$$\vec{a} = |\vec{a}|\vec{a}_0$$

3. 3D – 3×3 matrica – TENZOR

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

– sve fizikalne veličine su tenzori!

– simetrični – svi "pametni" tenzori u fizici

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \rightarrow \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

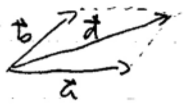
dijagonalna matrica

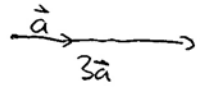
– dielektrična konstanta – $\varepsilon$ – tenzor

komutativna grupa { grupa {
$\vec{a} + \vec{b} = \vec{d}$
$\vec{a} + (\vec{b} + \vec{d}) = (\vec{a} + \vec{b}) + \vec{d}$
$\vec{a}, -\vec{a}, \ \vec{a} + (-\vec{a}) = \vec{0}$
$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$c\vec{a} = \vec{b}$

} V.P. (vektorski prostor)





$\rightarrow$ funkcije su vektori! (VAŽNO!)

$$V.P. + \vec{a}\cdot\vec{b} = c \rightarrow \text{UNITARNI PROSTOR}$$

SKALARNI PRODUKT: $f(x)\cdot g(x) = \int f^*(x)\, g(x)\, dx$

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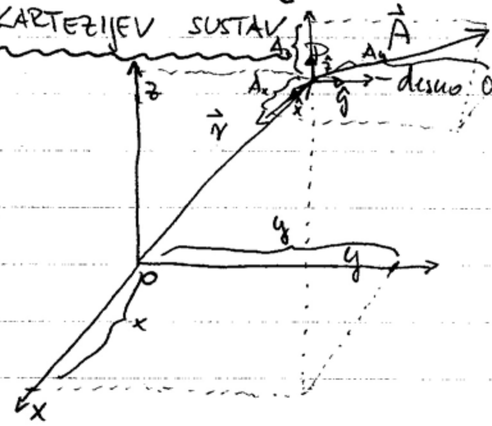
## KARTEZIJEV SUSTAV



desno orijentiran sustav

$\vec{A}$ - vektor

$P(\ldots,\ldots,\ldots)$

$P(x, y, z)$

$$\vec{A}(x,y,z) = \_\_\_ + \_\_\_ + \_\_\_$$

$$\vec{A}(x,y,z) = A_x\underbrace{(x,y,z)}\hat{x} + A_y\underbrace{(x,y,z)}\hat{y} + A_z\underbrace{(x,y,z)}\hat{z}$$

podrazumijeva se

$\hat{x}, \hat{y}, \hat{z}$ - definiraju se u točki, a ne u ishodištu

$\rightarrow$ jedinični vektori

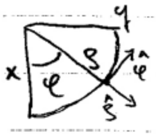
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\dot{\vec{r}} = \dot{x}\hat{x} + x\dot{\hat{x}} + \ldots$$

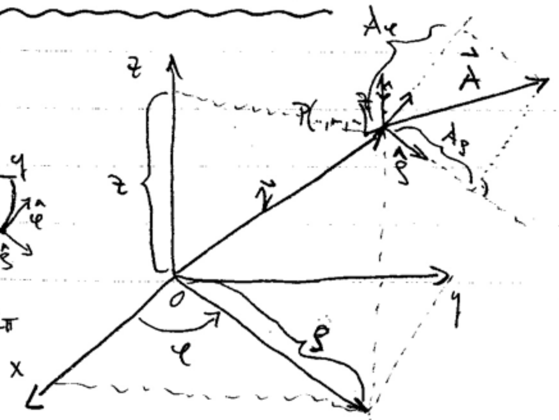
$$\frac{d\hat{x}}{dt} = 0 \Rightarrow \text{u Kart. sustavu}$$

$$|\vec{A}| = A_x^2 + A_y^2 + A_z^2$$

## CILINDRIČNI SUSTAV



$0 \le \varphi < 2\pi$



$$\vec{A}(\rho,\varphi,z) = \_\_\_ + \_\_\_ + \_\_\_$$

$$\vec{A}(\rho,\varphi,z) = A_\rho(\rho,\varphi,z)\hat{\rho} + A_\varphi(\rho,\varphi,z)\hat{\varphi} + A_z(\rho,\varphi,z)\hat{z}$$

- na osi $z$ ne može se definirati $\rho$ i $\varphi$ komponenta

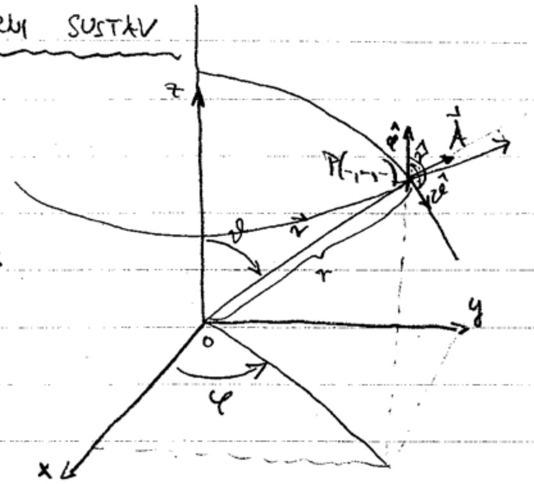
$$\vec{r} = \rho\cdot\hat{\rho} + z\hat{z}$$

$$\hat{z} = \hat{z}$$
$$\hat{\rho} = \hat{x}\cos\varphi + \hat{y}\sin\varphi$$
$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

## SFERNI SUSTAV

$P(r, \vartheta, \varphi)$

$\hat{e}_i,\ i = 1,2,3$

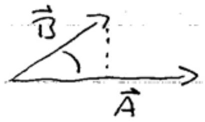



$$0 \le \vartheta \le \pi$$
$$0 \le \varphi < 2\pi$$

$$\vec{A} = A_r\hat{r} + A_\vartheta\hat{\vartheta} + A_\varphi\hat{\varphi}$$

$$\hat{r} = \hat{x}\sin\vartheta\cos\varphi + \hat{y}\cos\vartheta\sin\varphi + \hat{z}\cos\vartheta$$
$$\hat{\vartheta} = \hat{x}\sin\vartheta\cos\varphi + \hat{y}\cos\vartheta\sin\varphi - \hat{z}\sin\varphi$$
$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

$$\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\cos(\vec{A},\vec{B})$$




$$|\vec{A}| \neq 0$$
$$|\vec{B}| \neq 0$$
$$\vec{A}\cdot\vec{B} = 0 \implies \vec{A}\perp\vec{B}$$

$\hat{e}_i$ $\quad (\hat{e}_1, \hat{e}_2, e_3)$ $\qquad \hat{e}_i\cdot\hat{e}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$(\hat{x}, \hat{y}, \hat{z})$

$(\hat{\rho}, \hat{\varphi}, \hat{z})$ $\qquad \hat{e}_i\cdot\hat{e}_j = \delta_{ij}$

$(\hat{r}, \hat{\vartheta}, \hat{\varphi})$

$$\vec{A} = A_r\hat{r} + A_\vartheta\hat{\vartheta} + A_\varphi\hat{\varphi}\ ;\ \vec{B} = B_r\hat{r} + B_\vartheta\hat{\vartheta} + B_\varphi\hat{\varphi}$$

$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ , $\quad \underline{\vec{A}\cdot\vec{B} = A_rB_r + A_\vartheta B_\vartheta + A_\varphi B_\varphi}$

$$\vec{A}\times\vec{B} = |\vec{A}|\,|\vec{B}|\sin(\vec{A},\vec{B})\,\hat{u} \qquad \hat{u}\perp\vec{A},\ \hat{u}\perp\vec{B}$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \left(A_y B_z - A_z B_y\right)\hat{x} + \dots$$

$$\underbrace{\hat{e}_i \times \hat{e}_j = \hat{e}_k}\quad \{i, j, k\} \quad \{1,2,3\}$$

$$\left.\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & -1 \end{array}\right\} \longrightarrow \text{– ciklički poredak –}$$

$$\hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3) = 1$$

1. $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B}$
2. $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

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1D $\quad \int_C \vec{A} \cdot d\vec{r}$

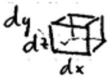

2D $\quad \int_S \vec{A} \cdot \underbrace{d\vec{a}}_{d\vec{s}}$

area $\quad d\vec{a} = da\,\hat{u}$

surface

3D $\quad \int_V f\, dV$

$$\iiint f(x,y,z)\underbrace{dx\,dy\,dz}$$

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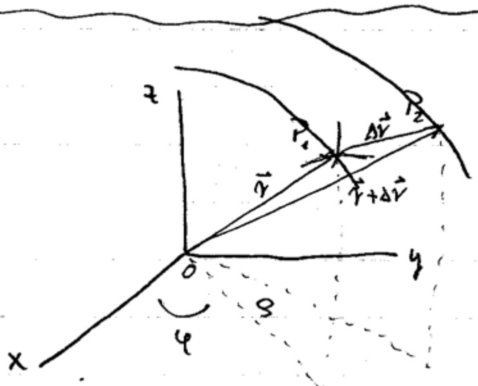
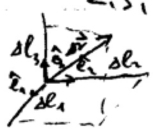



$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\Delta\vec{r} = \Delta x\hat{x} + \Delta y\hat{y} + \Delta z\hat{z}$

$$\boxed{d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}} \qquad \text{K.S.}$$

K.S.




$\Delta l_1 = \Delta x$

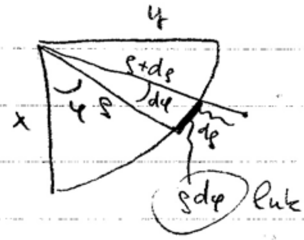
$\Delta l_2 = \Delta y$

$\Delta l_3 = \Delta z$

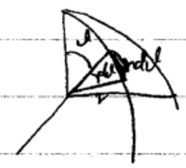
3

C.S. $d\vec{r} = \underbrace{\quad}_{dl_1}\hat{\varrho} + \underbrace{\quad}_{dl_2}\hat{\varphi} + \underbrace{\quad}_{dl_3}\hat{z}$

$P_1(\varrho, \varphi, z)$



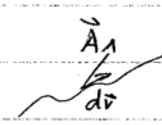
$$\boxed{d\vec{r} = d\varrho\,\hat{\varrho} + \varrho\, d\varphi\,\hat{\varphi} + dz\,\hat{z}}$$



S.S. $d\vec{r} = \underbrace{\quad}_{dl_1}\hat{r} + \underbrace{\quad}_{dl_2}\hat{\vartheta} + \underbrace{\quad}_{dl_3}\hat{\varphi}$

$$\boxed{d\vec{r} = dr\,\hat{r} + r\,d\vartheta\,\hat{\vartheta} + r\sin\vartheta\, d\varphi\,\hat{\varphi}}$$

$\int_C \vec{A}\,d\vec{r}$



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$d\vec{a} = da\,\hat{u}$

$d\vec{s} = ds\,\hat{u}$

$\int_S \vec{A}\,d\vec{s}$

$ds = dl_i\,dl_j \qquad dl_k = 0 \qquad i,j,k = 1,2,3$

K.S. $ds = dx\,dy \qquad z = \text{const.}$

$dy\,dz \qquad x = \text{const.}$

$dx\,dz \qquad y = \text{const.}$

C.S. $ds = \varrho\, d\varrho\, d\varphi \qquad z = \text{const.}$

$d\varrho\, dz \qquad \varphi = \text{const.}$

$\varrho\, d\varphi\, dz \qquad \varrho = \text{const.}$

S.S. $ds = r^2 \sin\vartheta\, d\vartheta\, d\varphi \qquad r = \text{const.}$ — VRLO ČESTI !

$r\sin\vartheta\, dr\, d\varphi \qquad \vartheta = \text{const.}$

$r\,dr\,d\vartheta \qquad \varphi = \text{const.}$

$$\int_V f \, dV \qquad Q = \int \rho(\vec{r})\, dV$$

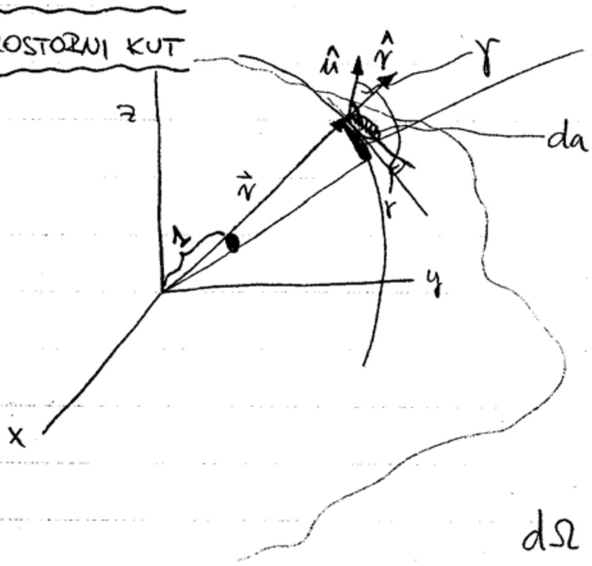
po volumenu

K.S. $dV = dx\,dy\,dz$

C.S. $dV = \rho\, d\rho\, d\varphi\, dz$

S.S. $dV = r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$

## PROSTORNI KUT




→ kod sfere $\hat{n} = \hat{r}$ (samo kod nje)

$r = |\vec{r}| \qquad d\vec{a} = da\,\hat{n}$

dif. s. nast., $r = \text{const.}$

$$d\Omega = \frac{d\vec{a}\cdot\hat{r}}{r^2} = \frac{da\,\hat{n}\cdot\hat{r}}{r^2} = \frac{da\cos\gamma}{r^2}$$

→ podijelili sa $r^2$ da bi dobili da $r$ bude 1 (jedinična sfera), da ne ovisi o $r$

$$d\Omega = \frac{\cancel{r^2}\sin\vartheta\, d\vartheta\, d\varphi}{\cancel{r^2}}$$

$$\boxed{d\Omega = \sin\vartheta\, d\vartheta\, d\varphi}$$

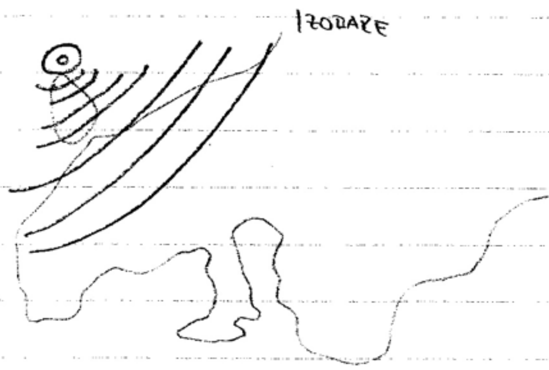
element površine jedinične sfere

diferencijal prostornog kuta

$$\int_0^{\pi}\int_0^{2\pi} \sin\vartheta\, d\vartheta\, d\varphi = 2\pi \int_0^{\pi} \sin\vartheta\, d\vartheta = 4\pi$$

$x = \cos\vartheta \qquad dx = -\sin\vartheta\, d\vartheta$

$$-\int_1^{-1} dx = +\int_{-1}^{1} dx = 2$$




Skalarnim poljem

$\underline{f(\vec{r})}$ ← površina S

$\nabla f(\vec{r})$ vekt. polje

gradijent f-je

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$$f(\vec{r}) \to f(x,y,z)$$

$$d\vec{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

$$\longrightarrow df = dx\frac{\partial f}{\partial x} + dy\frac{\partial f}{\partial y} + dz\frac{\partial f}{\partial z}$$

$$df = (\ldots)\cdot d\vec{r}$$

vektor — Usmjerena derivacija u 3 smjera

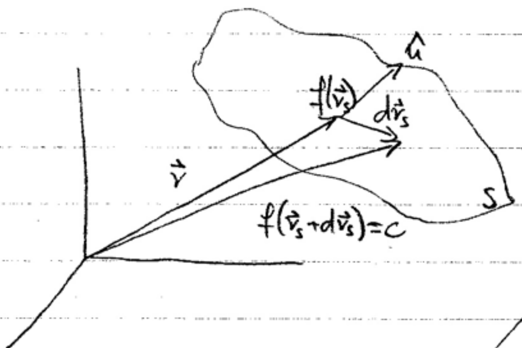
$$\boxed{df = \nabla f \cdot d\vec{r}}$$

$\nabla f$ – gradijent f-je $f$

dif. operator i vektorski

$\nabla, \vec{\nabla}$, grad, $\vec{\text{grad}}$

nabla




$$f(\vec{r}_s) = C$$

$$df = 0 \Big|_{\vec{r} = \vec{r}_s}$$

$$\underset{=0}{df} = (\nabla f)\cdot d\vec{r}$$

$$\nabla f = \hat{n}\frac{df}{dn}$$

$$\vec{E} = -\nabla\phi$$

$$\nabla f = \hat{x}(\nabla f)_x + \hat{y}(\nabla f)_y + \hat{z}(\nabla f)_z$$

$$\rightarrow \quad df = dx(\nabla f)_x + dy(\nabla f)_y + dz(\nabla f)_z$$

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$\Rightarrow \boxed{\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$ – VEKTOR i DIFERENCIJALNI OPERATOR

C.S. $f(\varrho, \varphi, z)$

$$\rightarrow df = \frac{\partial f}{\partial \varrho}d\varrho + \frac{\partial f}{\partial \varphi}d\varphi + \frac{\partial f}{\partial z}dz$$

$$d\vec{r} = \hat{\varrho}\,d\varrho + \hat{\varphi}\,\varrho\,d\varphi + \hat{z}\,dz$$

$$\frac{\partial f}{\partial \varphi} = (\nabla f)_\varphi\,\varrho$$

$$(\nabla f)_\varphi = \frac{1}{\varrho}\frac{\partial f}{\partial \varphi}$$

$$\nabla f = \hat{\varrho}(\nabla f)_\varrho + \hat{\varphi}(\nabla f)_\varphi + \hat{z}(\nabla f)_z$$

$$\rightarrow \quad df = d\varrho(\nabla f)_\varrho + d\varphi(\nabla f)_\varphi\varrho + dz(\nabla f)_z$$

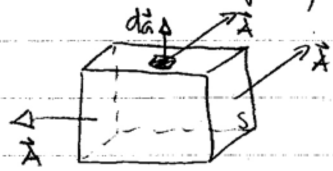
$$\nabla f = \hat{\varrho}\frac{\partial f}{\partial \varrho} + \hat{\varphi}\frac{1}{\varrho}\frac{\partial f}{\partial \varphi} + \hat{z}\frac{\partial f}{\partial z}$$

$$\Rightarrow \boxed{\nabla = \hat{\varrho}\frac{\partial}{\partial \varrho} + \hat{\varphi}\frac{1}{\varrho}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}}$$

S.S $$\Rightarrow \boxed{\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\vartheta}\frac{1}{r}\frac{\partial}{\partial \vartheta} + \hat{\varphi}\frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}}$$

# DIVERGENCIJA

$\nabla\cdot\vec{A}(\vec{r})$ – divergencija polja A




$\oint_S \vec{A}\cdot d\vec{a}$

razlika fluksa u toj kocki


$\vec{A}\cdot d\vec{a} = \vec{A}\cdot\hat{u}\, da$

fluks toka $= \int_S \vec{A}\cdot d\vec{a}$

$$\boxed{\nabla\cdot\vec{A}(\vec{r}) = \lim_{\Delta V\to 0}\frac{\oint_S \vec{A}\cdot d\vec{a}}{\Delta V}}$$

→ izvori/ponori vektora $\vec{A}$ u točki $\vec{r}$

S.S.

ŠALABAHTER

| | $\hat{r}$ | $\hat{\vartheta}$ | $\hat{\varphi}$ |
|---|---|---|---|
| $\frac{\partial}{\partial r}$ | 0 | 0 | 0 |
| $\frac{\partial}{\partial \vartheta}$ | $\hat{\vartheta}$ | $-\hat{r}$ | 0 |
| $\frac{\partial}{\partial \varphi}$ | $\hat{\varphi}\sin\vartheta$ | $\hat{\varphi}\cos\vartheta$ | $-[\hat{r}\sin\vartheta+\hat{\vartheta}\cos\vartheta]$ |

$$\nabla\cdot\vec{A} = \Big[\underbrace{\hat{r}\frac{\partial}{\partial r}} + \hat{\vartheta}\frac{1}{r}\frac{\partial}{\partial\vartheta} + \hat{\varphi}\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\Big]\cdot\Big[\underbrace{A_r\hat{r}} + \hat{\vartheta}A_\vartheta + \hat{\varphi}A_\varphi\Big]$$

$$\boxed{\hat{r}\frac{\partial}{\partial r}\cdot\hat{r}A_r} = \hat{r}\Big[\hat{r}\frac{\partial A_r}{\partial r} + \frac{\partial\hat{r}}{\partial r}A_r\Big] =$$

$$= \underbrace{\hat{r}\cdot\hat{r}}_{1}\frac{\partial A_r}{\partial r} + \cancel{\hat{r}\cdot\underbrace{\boxed{\frac{\partial\hat{r}}{\partial r}}}_{0}A_r} = \frac{\partial A_r}{\partial r}$$

D.Z. OSTATAK!

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\vartheta}\Big[\frac{\partial}{\partial\vartheta}(\sin\vartheta A_\vartheta) + \frac{\partial A_\varphi}{\partial\varphi}\Big]}$$

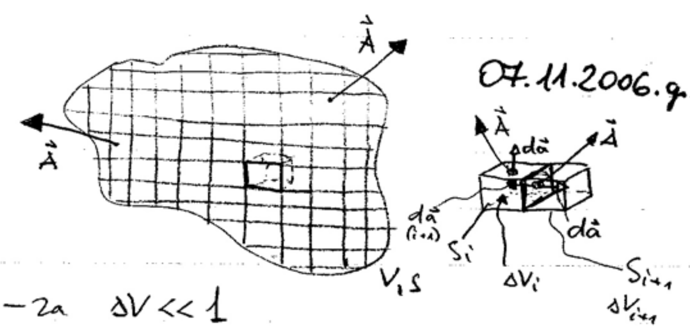
C.S. $\frac{\partial\hat{\rho}}{\partial\varphi} = \hat{\varphi}$ ; $\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{\rho}$

$$\boxed{\nabla\cdot\vec{A} = \underbrace{\frac{\partial A_\rho}{\partial\rho} + \frac{1}{\rho}A_\rho}_{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}}$$

K.S. $$\boxed{\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

## GAUSS-ov TM

07.11.2006.g.




$$\nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{A} \cdot d\vec{a}}{\Delta V}$$

$$(\nabla \cdot \vec{A}) \Delta V \cong \oint_S \vec{A} \cdot d\vec{a} \qquad - \text{za } \Delta V << 1$$

$$\sum_{i=1}^{N} (\nabla \cdot \vec{A}) \Delta V_i = \sum_{i=1}^{N} \underbrace{\oint_{S_i} \vec{A} \cdot d\vec{a}} \qquad - \text{za } \Delta V <<< 1$$

što manji to bolje vrijedi formula

$$\Delta V_i \to 0$$
$$N \to \infty$$

$$\Rightarrow \boxed{\int_V (\nabla \cdot \vec{A}) \cdot dV = \oint_S \vec{A} \cdot d\vec{a}}$$

po vanjskoj plohi

## ROTACIJA

$$\nabla \times \vec{A} = \lim_{\Delta V \to 0} \frac{\oint_S d\vec{a} \times \vec{A}}{\Delta V}$$

$\vdots$

K.K.S.
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \ldots$$

C.K.S.
$$\nabla \times \vec{A} = \hat{\rho} \left[ \frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right] + \hat{\varphi} \left[ \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \hat{z} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\varphi) - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \varphi} \right]$$

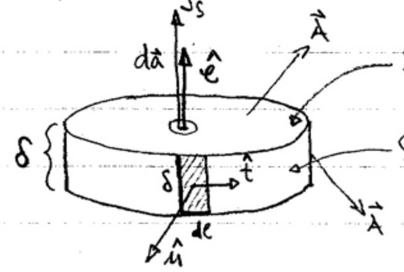
S.K.S.
$$\nabla \times \vec{A} = \vec{r} \; \frac{1}{r \sin\vartheta} \left[ \frac{\partial}{\partial \vartheta} (A_\varphi \sin\vartheta) - \frac{\partial}{\partial \varphi} A_\vartheta \right] +$$
$$+ \hat{\vartheta} \left[ \frac{1}{r \sin\vartheta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r} (r A_\varphi) \right] +$$
$$+ \hat{\varphi} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r A_\vartheta) - \frac{1}{r} \frac{\partial A_r}{\partial \vartheta} \right]$$

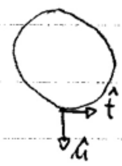
# STOKES-ov TM

$$(\nabla\times\vec{A})\,\Delta V \cong \oint_S d\vec{a}\times\vec{A}$$

$\Delta V << 1$

Sve je jako tanko!!!



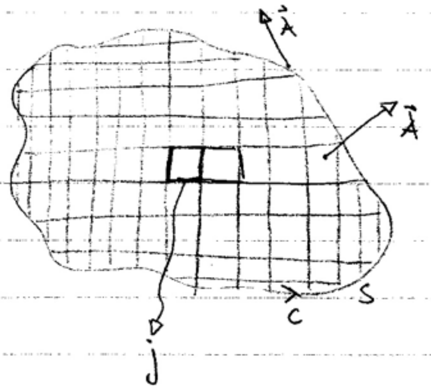

$$\hat{e}\cdot(\nabla\times\vec{A}) = \frac{1}{S_e\,\delta}\underbrace{\oint_S \hat{e}\cdot(d\vec{a}\times\vec{A})}$$

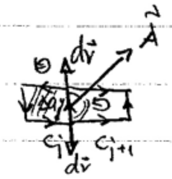
$$d\vec{a} = \delta\, dl\, \hat{u}$$

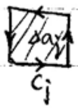
$$\oint_{S_b}(\hat{e}\times d\vec{a})\cdot\vec{A} = \oint_{S_b}\underbrace{(\hat{e}\times\hat{u})}_{\hat{t}}\cdot\vec{A}\,\delta\, dl = \oint_C d\vec{r}\cdot\vec{A}\cdot\delta$$

$$dl\cdot\hat{t} = d\vec{r}$$

$$S_e\,\hat{e}\cdot(\nabla\times\vec{A}) \cong \oint_C d\vec{r}\cdot\vec{A}$$




$$\underbrace{\Delta a_j\,\hat{e}_j}_{\Delta\vec{a}_j}(\nabla\times\vec{A}) \cong \oint_{c_j} d\vec{r}\cdot\vec{A}$$

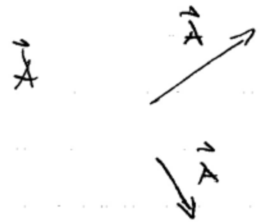
$$\sum_{j=1}^{N}(\nabla\times\vec{A})\cdot\Delta\vec{a}_j = \sum_{j=1}^{N}\oint_{c_j}\vec{A}\cdot d\vec{r}$$

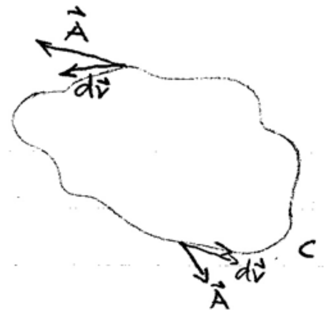
$\Delta a_j \to 0$

$N \to \infty$

$$\boxed{\oint_S(\nabla\times\vec{A})\cdot d\vec{a} = \oint_C \vec{A}\cdot d\vec{r}}$$

po rubu vanjskane




$\oint_c \vec{A}\cdot d\vec{v} = 0$ može se dogoditi i ako je $\vec{A} \neq 0$.

$$\boxed{\oint_c \vec{A}\cdot d\vec{v} = 0} \quad \forall c \Rightarrow \vec{A}$$ (konzervativno polje)

konzervativna (sila) vektor

KONZERVANTIVNO VEKTORSKO POLJE

$$\int_0^{2\pi} \sin x \, dx = 0$$

$$\int_a^b f(x)\,dx = 0 \quad \forall (a,b) \Rightarrow f(x) = 0$$

$$\oint_S (\nabla\times\vec{A})\cdot d\vec{a} = 0 \quad \forall S \Rightarrow \boxed{\nabla\times\vec{A} = 0} \Rightarrow \boxed{\vec{A} = \nabla\cdot f}$$

D.Z. $\rightarrow$ $\boxed{\nabla\times(\nabla\cdot f) = 0}$ – provjeriti!

– za bilo koju f-ju f (bilo koji k.s.)

$\boxed{\nabla\cdot(\nabla\times\vec{A}) = 0}$ – provjeriti!

---

$$\nabla(f+g) = \nabla f + \nabla g \qquad \nabla(fg) = (\nabla f)g + f(\nabla g)$$

$$\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B} \qquad \nabla\cdot(f\vec{A}) = (\nabla f)\cdot\vec{A} + f(\nabla\cdot\vec{A})$$

$$\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B} \qquad \nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f(\nabla\times\vec{A})$$

---

$$\nabla(\vec{A}\cdot\vec{B}) = \underline{(\vec{B}\cdot\nabla)\vec{A}} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B})$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - (\nabla\times\vec{B})\cdot\vec{A}$$

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\cdot\vec{A} + \vec{A}(\nabla\cdot\vec{B}) - (\vec{A}\cdot\nabla)\vec{B} - \vec{B}(\nabla\cdot\vec{A})$$

$\boxed{7}$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \qquad \vec{A}f = A_x f\hat{x} + A_y f\hat{y} + A_z f\hat{z}$$

$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \qquad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \qquad \vec{B}\cdot\nabla = B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}$$

$$\vec{A}(\nabla\cdot\vec{B}) = A_x\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)\hat{x} + A_y\left( \;-\|-\; \right)\hat{y} + A_z\left( \;-\|-\; \right)\hat{z}$$

$$(\vec{B}\cdot\nabla)\vec{A} = \left(B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}\right)\hat{x} + A_x\left(B_x\underset{0}{\underset{\|}{\frac{\partial\hat{x}}{\partial x}}} + B_y\underset{0}{\underset{\|}{\frac{\partial\hat{x}}{\partial y}}} + B_z\underset{0}{\underset{\|}{\frac{\partial\hat{x}}{\partial z}}}\right) + \left(B_x\frac{\partial A_y}{\partial x} + B_y\frac{\partial A_y}{\partial y} + B_z\frac{\partial A_y}{\partial z}\right)\hat{y} + \left(B_x\frac{\partial A_z}{\partial x} + B_y\frac{\partial A_z}{\partial y} + B_z\frac{\partial A_z}{\partial z}\right)\hat{z}$$

**14.11.2006.g.**

$\nabla$

$\nabla f$

$\nabla\cdot\vec{A}$

$\nabla\times\vec{A}$

$\rightarrow \nabla\cdot\nabla = \Delta$ Laplaceov operator

$\rightarrow \cancel{\nabla\times\nabla} \equiv 0$

K.K.S. $\nabla^2 = \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

C.K.S. $\Delta = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial\varphi^2} + \dfrac{\partial^2}{\partial z^2}$

S.K.S. $\Delta = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{1}{\sin\vartheta}\dfrac{\partial}{\partial\vartheta}\left(\sin\vartheta\dfrac{\partial}{\partial\vartheta}\right) + \dfrac{1}{r^2\sin^2\vartheta}\dfrac{\partial^2}{\partial\varphi^2}$

$$E = -\nabla\phi \Rightarrow \Delta\phi = 0 \quad \text{Laplaceova jdba}$$

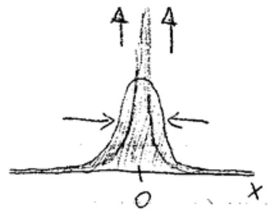
$(x,y,z)$
$(\rho,\varphi,z)$
$(r,\vartheta,\varphi)$
$\vdots$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi(x,y,z) = 0$$

$$\frac{\partial^2\phi(x,y,z)}{\partial x^2} + \frac{\partial^2\phi(x,y,z)}{\partial y^2} + \frac{\partial^2\phi(x,y,z)}{\partial z^2} = 0$$

$$\phi(x,y,z) = X(x)Y(y)Z(z)$$

$\downarrow \quad \downarrow \quad \downarrow$



$$\delta(x) = \begin{cases} 0 & \text{svuda osim u } x=0 \\ \infty & x=0 \end{cases}$$

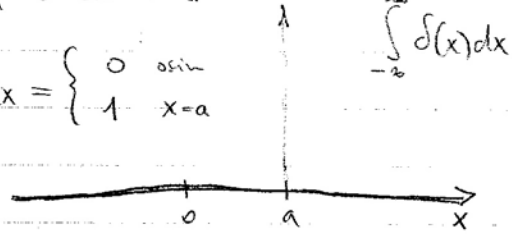
$\delta(\vec{r})$ Dirac (delta f-ja)

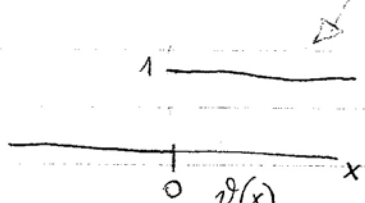
$$\int_a^b \delta(x)\,dx = \begin{cases} 0 & \text{osim ako} \\ 1 & x=0 \text{ unutar } (a,b) \end{cases}$$

$$\delta(x-a) = \begin{cases} 0 & \\ \infty & \text{osim } x=a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\int \delta(x-a)\,dx = \begin{cases} 0 & \text{osim} \\ 1 & x=a \end{cases}$$





$\vartheta(x)$

Heaviside step f-ja

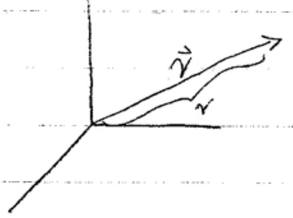
$$\int f(x)\,\delta(x-a)\,dx = \begin{cases} 0 & \text{osim} \\ f(a) & x, a \text{ unutar} \end{cases}$$

$\delta(x)$ je derivacija $\vartheta(x)$

$$\int f(x)\,\delta'(x-a)\,dx = \begin{cases} 0 & \text{osim} \\ -f'(a) & x, a \text{ unutar} \end{cases}$$

---

$\boxed{r \neq 0}$ $$\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) \stackrel{\nabla}{=} \underbrace{(\nabla \cdot \vec{r})}_{3} \cdot \frac{1}{r^3} + \left(\nabla \cdot \frac{1}{r^3}\right) \cdot \vec{r} = \frac{3}{r^3} - 3\frac{1}{r^4}\underbrace{\hat{r} \cdot r\hat{r}}_{1} = 0$$



$$\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = \infty \qquad r = 0$$

$$\frac{r\hat{r}}{r^3} = \frac{1}{r^2}\hat{r}$$

$$\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = \underline{\qquad}\,\delta(\vec{r})$$

$d\vec{a} = da \cdot \hat{u}$ ($\hat{u}$: $\hat{r}$ – za kuglu)

$$\int_V \nabla \cdot \left(\frac{\vec{r}}{r^3}\right) dv = \oint_S \frac{\vec{r}}{r^3} \cdot \hat{r}\,da$$

$$= \oint_S \frac{r\hat{r}}{r^3} \cdot \hat{r}\,da$$

$$= \frac{1}{R^2} \underbrace{\oint da}_{4R^2\pi} = 4\pi$$

$$\boxed{\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 4\pi\,\delta(\vec{r})}$$

$$\boxed{\frac{\vec{r}}{r^3} = -\nabla\left(\frac{1}{r}\right)}$$ lako za provjeriti

$$\nabla\cdot\left(-\nabla\left(\frac{1}{r}\right)\right) = 4\pi\,\delta(\vec{r})$$

$$\boxed{\Delta\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r})}$$

# ELEKTROSTATIKA

## Naboj

(1) Naboj je skalar, broj. Imamo dvije vrste naboja. Pozitivan i negativan

(2) Priroda je neutralna. Materijali su neutralni po prirodi.
Naboj se ne može stvoriti ni uništiti → zakon o sačuvanju naboja. Samo selimo naboje.

(3) Simetrija naboja (charge symmetry)

$e^-$ $e^+$

$\pi^+$ $\pi^-$

$p^+$ $p^-$

(4). Naboj ne može biti proizvoljno mali.
Postoje jedinični naboji. Naboj je kvantiziran.
$e, 2e, 3e, \ldots$

(5) Kvarkovi → $\frac{1}{3}e$ → flavour: $\underbrace{u \quad d}_{}$ 1. generacija kvarkova; $s \quad c$ 2. generacija

u – UP, d – DOWN, s – STRANGE, c – CHARME

3. generacija: t – TOP, b – BOTTOM

$e, \nu$ (electron, neutrino)
sami za sebe čestice

$\gamma$ – foton (sam za sebe)

Sve su čestice građene od ovih kvarkova osim $e, \nu, \gamma$

$$m(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$ — masa mirovanja

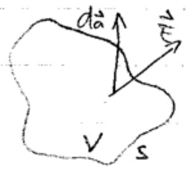
$\gamma, \nu$ – nemaju masu mirovanja pa zato putuju brzinom c (ili jako jako malu masu mirovanja)

– sve čestice koje imaju masu mirovanja imaju naboj

– NABOJ – svojstvo koje je povezano, usko, sa masom mirovanja materije

KLASIČNA ELEKTRODINAMIKA $\rightarrow$ $q$ (naboj) !!!!! (postoji i imamo ga, ne zanima ju odkuda on)

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q$$




1. – –
2. – –
3. —
4. —

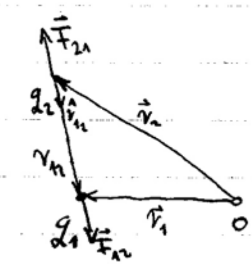
4 Maxwellove jdbe (sve se mogu objasniti)

$\vec{F}$ – krenuti od sile (krenuti od eksperimenta da se objasni fizika, definiranje)

$\vec{F} \rightsquigarrow \vec{E}$ $\quad \vec{F} = q\vec{E}$

Amperov zakon $\rightarrow$ $\vec{H}$ (magnetizirajuće polje)

$\vec{F} = q\vec{v}\times\vec{B}$ $\hookrightarrow$ polje!




eksperimentalno utvrđeno! **21.11.2006.g.**

$$\vec{F}_{12} = k\frac{q_1\cdot q_2}{r_{12}^2}\hat{r}_{12}$$

$$\vec{F}_{21} = -\vec{F}_{12}$$ – ovo sve vrijedi ako su naboji točkasti

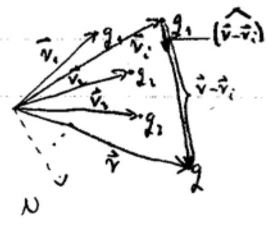
$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$

$r_{12} = |\vec{r}_{12}|$

$\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$

① – sila djeluje na spojnici dva tijela

② – ova sila zadovoljava princip supraopozicije (superpozicije)




$$\vec{F}_q(\vec{r}) = k\frac{q q_i}{|\vec{r}-\vec{r}_i|^2}(\widehat{\vec{r}-\vec{r}_i}) \Big/ \frac{|\vec{r}-\vec{r}_i|}{|\vec{r}-\vec{r}_i|}$$

$$\boxed{\vec{F}_q(\vec{r}) = k\sum_{i=1}^{N}\frac{q q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i)}$$

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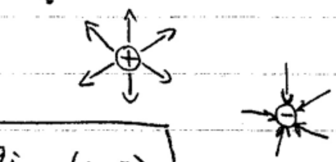
$$\boxed{\vec{F}_q(\vec{r}) = q \sum_{i=1}^{N} k \frac{q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i)}$$

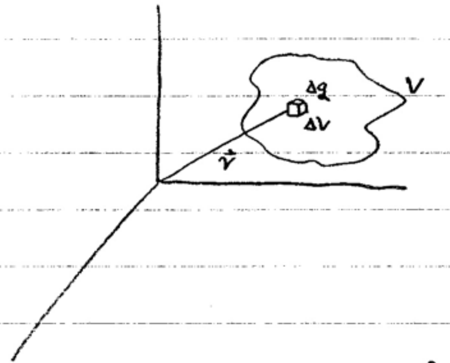
$\varepsilon_0$ - permitivnost vakuuma



SI $\quad k = \frac{1}{4\pi\varepsilon_0} \equiv 10^{-7}c^2 \simeq 9\times10^9\ Nm^2/C^2$

$$\boxed{\vec{F}_q(\vec{r}) = q\,\vec{E}(\vec{r})} \qquad \boxed{\vec{E}(\vec{r}) = k\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i)}$$

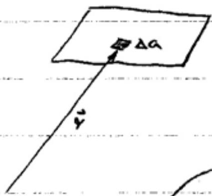
---




– mali volumen ΔV

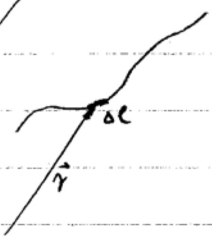
volumna gustoća naboja

$$\lim_{\Delta V\to 0}\frac{\Delta q}{\Delta V} = \boxed{\frac{dq(\vec{r})}{dV} = \rho(\vec{r})}$$




površinska gustoća naboja

$$\lim_{\Delta a\to 0}\frac{\Delta q}{\Delta a} = \boxed{\frac{dq(\vec{r})}{da} = \sigma(\vec{r})}$$




linijska gustoća naboja

$$\lim_{\Delta l\to 0}\frac{\Delta q}{\Delta l} = \boxed{\frac{dq(\vec{r})}{dl} = \lambda(\vec{r})}$$

---

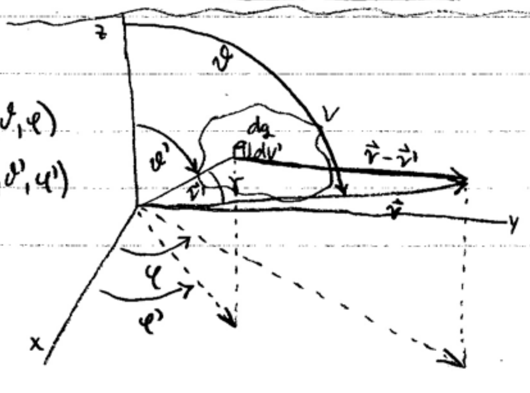
$$\int_V dq(\vec{r}) = \int_V \rho(\vec{r})\,dV \quad \Rightarrow \quad \boxed{Q = \int_V \rho(\vec{r})\,dV}$$

$$\boxed{Q = \int_S \sigma(\vec{r})\,da} \qquad \boxed{Q = \int_C \lambda(\vec{r})\,dl}$$

---

$\vec{r} \sim (r,\vartheta,\varphi)$

$\vec{r}' \sim (r',\vartheta',\varphi')$




$$\int_V d\vec{E}(\vec{r}) = k\int_V \frac{dq(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

$$\boxed{\vec{E}(\vec{r}) = k\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')\,dV'}$$

$dV' = r'^2 \sin\vartheta' dr' d\vartheta' d\varphi'$

$$\frac{1}{|\vec r-\vec r'|}=\frac{1}{\sqrt{r^2-2rr'\cos\gamma+r'^2}}$$

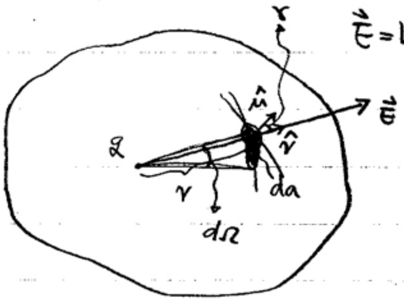
$r=|\vec r|$

$r'=|\vec r'|$

$$\cos\gamma=\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos(\varphi-\varphi')$$

# GAUSSOV ZAKON

$$\vec E=k\frac{q}{r^2}\hat r$$

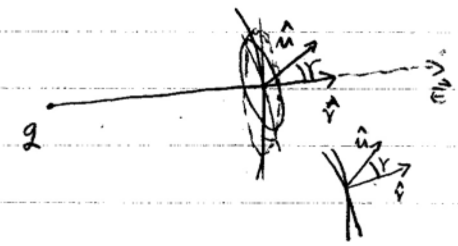



$d\vec a=da\,\hat u$

$$\vec E\cdot d\vec a=\vec E\cdot\hat u\cdot da=kq\frac{\overbrace{\hat r\cdot\hat u}^{\cos\gamma}}{r^2}da$$

$$\cos\gamma\,da=r^2d\Omega$$

$$\oint_S\vec E\cdot\hat u\cdot da=kq\overbrace{\oint d\Omega}^{4\pi}$$




$$\oint_S\vec E\cdot\hat u\cdot da=kq4\pi$$

$dq=\varrho dV$

$$\oint_S\vec E\cdot\hat u\cdot da=4\pi k\overbrace{\sum_{i=1}^{n}q_i}^{\int dq}\to Q$$

$$\boxed{\oint_S\vec E\cdot\hat u\cdot da=4\pi k\int_V\varrho(\vec r)dV}$$ Gaussov zakon

$$\oint_S\vec E\cdot d\vec a=4\pi k\int_V\varrho(\vec r)dV$$

G.T.

cgs : $k=1 \rightsquigarrow 4\pi$

SI : $k=\frac{1}{4\pi\varepsilon_0}\rightsquigarrow\frac{1}{\varepsilon_0}$

$$\int_V\nabla\cdot\vec E\,dV-4\pi k\int_V\varrho dV=0$$

$\vec E(\vec r),\ \varrho(\vec r)$

$$\int_V\underbrace{(\nabla\cdot\vec E-4\pi k\varrho)}_{=0}dV=0\ ,\ \forall V$$

$$\boxed{\nabla\cdot\vec E=4\pi k\varrho}\rightsquigarrow \text{SI}\ \boxed{\nabla\vec E=\frac{1}{\varepsilon_0}\varrho}$$

diferencijalni oblik Gaussovog zakona

vrijedi i kod $\vec E(\vec r,t),\ \varrho(\vec r,t)$!

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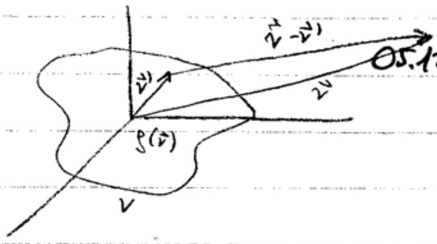
05.12.2006.

$$\vec{E}(\vec{r}) = k\int_V \rho(\vec{r}\,')\boxed{\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}}dv'$$




$$\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = -\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

$$|\vec{r}-\vec{r}\,'| = \sqrt{|x-x'|^2+|y-y'|^2+|z-z'|^2}$$

$$\frac{\partial}{\partial x}$$

$$\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

$(-1) = (-)$

$(-1)^\ell = (-)^\ell$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial x} + \hat{z}\frac{\partial}{\partial z} \qquad \nabla' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}$$

$$\vec{E}(\vec{r}) = k\int_V \rho(\vec{r}\,')(-)\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)dv'$$

$$\boxed{\vec{E}(\vec{r}) = -\nabla\underbrace{\left[k\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dv'\right]}_{f(\vec{r}) \Rightarrow \Phi(\vec{r})}}$$

$$k = \begin{cases} 1 & \text{G.c.g.s.} \\ \frac{1}{4\pi\varepsilon_0} & \text{S.I.} \end{cases}$$

$$\boxed{\Phi(\vec{r}) = k\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dv'}$$ POTENCIJAL ELEKTROSTATIČKOG POLJA
(Coulombov potencijal) (skalarni potencijal)

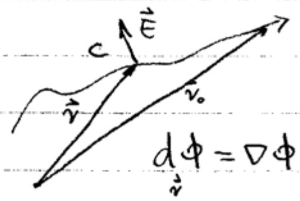
$$\boxed{\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r})}$$

$$\nabla\times(\nabla f) \equiv 0 \qquad \nabla\times(\underbrace{(-)\nabla\Phi}_{=\vec{E}}) \equiv 0$$

$$\Rightarrow \boxed{\nabla\times\vec{E} = 0}$$ → ZA STATIKU!

Stokes

$$\oint_C \vec{E}\cdot d\vec{r} = 0$$




$$d\Phi = \nabla\Phi\cdot d\vec{r}$$

$$\int_{\vec{r}_0}^{\vec{r}} d\Phi = \int_{\vec{r}_0}^{\vec{r}} \underbrace{\nabla\Phi}_{-\vec{E}}\cdot d\vec{r} \Rightarrow \Phi(\vec{r}) - \underbrace{\Phi(\vec{r}_0)}_{\to 0} = -\int_{\vec{r}_0}^{\vec{r}}\vec{E}\cdot d\vec{r}$$

$\vec{r}_0 \to \infty$

$$\Rightarrow \boxed{\Phi(\vec{r}) = -\int_\infty^{\vec{r}}\vec{E}\cdot d\vec{r}}$$

$$\nabla\cdot\vec{E} = 4\pi k \rho$$
$$\vec{E} = -\nabla\phi$$

$$-4\pi k \to -\frac{1}{\varepsilon_0}$$

$$\Rightarrow \underbrace{\nabla\cdot(-\nabla\phi)} = 4\pi k\rho$$

$$\boxed{\Delta\phi(\vec{r}) = -4\pi k\rho(\vec{r})}$$ Poissonova jdba

$$\boxed{\Delta\phi = 0}$$ Laplaceova jdba

- potencijal je određen do na konstantu:

$$\to \underbrace{\phi' = \phi + C} \to$$ gauge transformacija

$$\vec{E} = -\nabla\phi' \Rightarrow \vec{E} = -\nabla(\phi + C) = \underbrace{-\nabla\phi}_{\vec{E}} - \underbrace{\nabla C}_{=0} \qquad \boxed{\nabla C \equiv 0}$$

gauge = baždarenje

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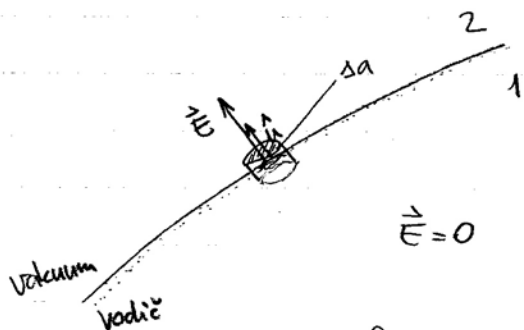



1, 2 - mediji!

$$\underbrace{\oint_S \vec{E}\cdot d\vec{a} = \vec{E}\cdot\Delta\vec{a} = \vec{E}\cdot\hat{u}\,\cancel{\Delta a}} \overset{G.Z.}{=} 4\pi k Q$$

$$\vec{E} = 0$$

$$= 4\pi k\sigma\,\cancel{\Delta a}$$

$$Q = \sigma\cdot\Delta a$$

$$\Rightarrow \vec{E}\cdot\hat{u} = \underbrace{4\pi k}_{\frac{1}{\varepsilon_0}}\sigma$$

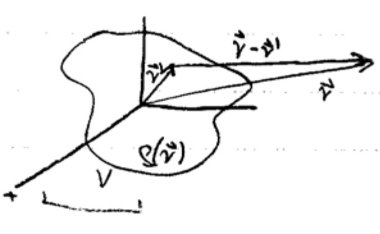
$$\boxed{|\vec{E}| = \frac{\sigma}{\varepsilon_0}}$$

$$\Rightarrow \boxed{\vec{E}_2\cdot\hat{u} - \vec{E}_1\cdot\hat{u} = 4\pi k\sigma}$$ - kod dva medija!

---

razvijanje u red!

$$\phi(\vec{r}) = k\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' =$$




$$= \underline{\qquad} + \underline{\qquad} + \underline{\qquad} + \underline{\qquad}$$

MULTIPOLNI RAZVOJ

# 11

oktupolni

$$\Phi(\vec{r}) = \underbrace{k\frac{Q}{r}}_{} + \underbrace{k\frac{\vec{p}\cdot\vec{r}}{r^3}}_{} + \underbrace{\quad(\because)\quad}_{} + \underbrace{\qquad}_{}$$

monopolni — dipolni — kvadrupolni → tenzor!

potencijal dipola

$r = |\vec{r}|$

$r' = |\vec{r}'|$

· distribuciju aproksimiram na točkasti naboj

distribucija sa 4 pola (aproksimacija)

$\vec{p}$ - dipolni moment

$\frac{r'}{r}$ → kritična veličina pri razvoju u red!

$$\frac{r'}{r} < 1 \qquad r' << r$$

to je uvjet!

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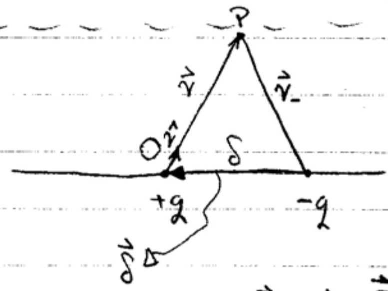



$$\Phi(\vec{r}) = \Phi_q(\vec{r}) + \Phi_{-q}(\vec{r}) =$$

$$= k\frac{q}{r} - k\frac{q}{r_-} = kq\left(\frac{1}{r} - \frac{1}{|\vec{r}+\vec{\delta}|}\right)$$

$r = |\vec{r}|$

$r_- = |\vec{r}_-|$

$\delta = |\vec{\delta}|$

$\hat{r} = \frac{\vec{r}}{r}$

$$\vec{r}_- = \vec{r} + \vec{\delta}$$

$\delta << r$

$$\frac{1}{|\vec{r}+\vec{\delta}|} = \underbrace{\frac{1}{r}\left(1 - \frac{\vec{r}\cdot\vec{\delta}}{r^2}\right)}_{-\frac{1}{r}+\frac{\vec{r}\cdot\vec{\delta}}{r^3}} + \frac{1}{2r}\left[3\left(\frac{\hat{r}\vec{\delta}}{r}\right)^2 - \left(\frac{\delta}{r}\right)^2\right] + \dots$$

$$\boxed{\Phi(\vec{r}) = kq\frac{1}{r}\underbrace{\frac{\hat{r}\vec{\delta}}{r^2}}_{\frac{\delta}{r}}} - kq\frac{1}{2r}\cancel{\left[\;\dots\;\right]}_{\left(\frac{\delta}{r}\right)^2} + \cancel{\dots \left(\frac{\delta}{r}\right)^3 + \dots}$$

zbog $\delta << r$

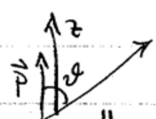
$$\boxed{\vec{p} = q\vec{\delta}}$$

DIPOLNI MOMENT

(svojstvo distribucije, a ne točke promatranja)

$$\boxed{\Phi(\vec{r}) = k\frac{\vec{p}\cdot\vec{r}}{r^3}}$$

$$\vec{E} = -\nabla\Phi$$

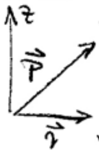



$p = |\vec{p}|$

⇓

Doma / D.Z.

$$\boxed{\vec{E} = k\frac{p}{r^3}\left[2\cos\vartheta\,\hat{r} + \sin\vartheta\,\hat{\vartheta}\right]}$$




$$\boxed{\vec{E} = k\frac{1}{r^3}\left[3(\hat{r}\vec{p})\hat{r} - \vec{p}\right]}$$

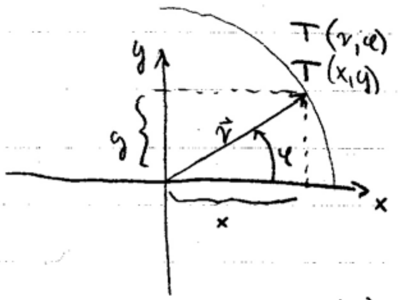
→ polje dipolnog tipa!

$$U = -\vec{p}\,\vec{E} \qquad \vec{F} = (\vec{p}\cdot\nabla)\vec{E} \qquad \vec{N} = \vec{p}\times\vec{E} + \vec{r}\times\vec{F}$$

poten. energ. — sila na dipol — zakretni moment

(orjentira dipol u smjeru polja)

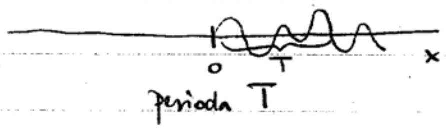
12.12.2006.g.




$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

primjer: $f(\vec{r}) = x^2 + 2y = r^2\cos^2\varphi + 2r\sin\varphi$

$\boxed{r=1}$ $\underline{f(\vec{r}) = \cos^2\varphi + 2\sin\varphi}$ (PRIMJER)




$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega x) + b_n\sin(n\omega x)\right]$$

Fourierov red

$$\omega = \frac{2\pi}{T} \qquad a_k = \frac{2}{T}\int_0^T f(x)\cos(k\omega x)\,dx$$

$$b_k = \frac{2}{T}\int_0^T f(x)\sin(k\omega x)\,dx$$

$\underline{T = 2\pi} \Rightarrow \omega = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx$$

sin i cos čine skup potpunih f-ja na segmentu $2\pi$ !

sin i cos u potpuni skup!

$$\int_0^{2\pi}\cos^2 x\,dx = \pi \qquad \int_0^{2\pi}\sin^2 x\,dx = \pi$$

$$\int_0^{2\pi}\sin x\cos x\,dx = 0$$

$$f \cdot g = \int f^* g \, dx \quad - \text{u 1. varijabli}$$

$$\pi = \cos x \cdot \cos x \rightarrow \int_0^{2\pi} \cos^2 x \, dx = \pi$$
$$\pi = \sin x \cdot \sin x \rightarrow \int_0^{2\pi} \sin^2 x \, dx = \pi$$
$$0 = \sin x \cdot \cos x \rightarrow \int_0^{2\pi} \sin x \cos x = 0$$

baza!

$$\left(\frac{1}{\sqrt{\pi}}\cos x\right)\left(\frac{1}{\sqrt{\pi}}\cos x\right) = 1$$
$$\left(\frac{1}{\sqrt{\pi}}\sin x\right)\cdot\left(\frac{1}{\sqrt{\pi}}\sin x\right) = 1$$
$$\left(\frac{\sin x}{\sqrt{\pi}}\right)\cdot\left(\frac{\cos x}{\sqrt{\pi}}\right) = 0$$

– ortonormirana baza!

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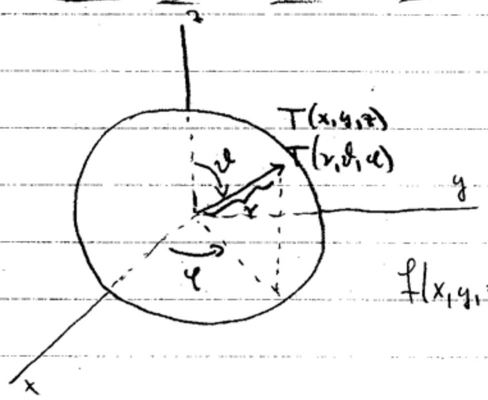


$f(x,y,z)$

$$\overset{1}{\downarrow}$$
$$x = r \sin\vartheta \cos\varphi$$
$$y = r \sin\vartheta \sin\varphi$$
$$z = r \cos\vartheta$$

$$f(\vartheta,\varphi) = \sum_k A_k Y_k(\vartheta,\varphi)$$

? da li analogno mogu to naći kao u 2D prostoru (kao kod sin i cos)

odgovor je da može!

→ to su KUGLINE FJE (specijalna klasa polinoma)

$$Y_k \cdot Y_{k'} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta \, d\vartheta \, Y_k^*(\vartheta,\varphi) Y_{k'}(\vartheta,\varphi) = \delta_{kk'}$$

nije jedan index, već dva kod kuglinih fja:

$$\rightarrow f(\vartheta,\varphi) = \sum_{l,m} A_l^m Y_l^m(\vartheta,\varphi)$$

$$\rightarrow Y_l^m \cdot Y_{l'}^{m'} = \delta_{ll'}\delta_{mm'}$$

$$\Delta \phi(r,\vartheta,\varphi) = 0 \qquad \text{Laplaceova jdba}$$

$$\left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \left[\frac{1}{r^2}\frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]\right\}\phi(r,\vartheta,\varphi) = 0$$

pretpostavka: $\phi(r,\vartheta,\varphi) = R(r)\,Y(\vartheta,\varphi)$

$$Y\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + R\left[\frac{1}{r^2}\cdots\right]Y = 0 \quad \Big/ \frac{r^2}{RY}$$

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)} + \underbrace{\frac{1}{Y}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]Y} = 0$$

KRITIČNI ARGUMENT

$f(r)$ ili $C_1$ → $g(\vartheta,\varphi)$ ili $C_2$ $= 0$

$$\underbrace{f(r)}_{k} + \underbrace{g(\vartheta,\varphi)}_{-k} = 0 \quad \forall r,\vartheta,\varphi$$

jedini način da ovo bude zadovoljeno: $\lambda \qquad -\lambda$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda$$

OVO NAS ZANIMA SADA!

$$\frac{1}{Y}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = -\lambda$$

$$\boxed{\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0}$$

KVANTNA: $\psi_{n\ell m}(r,\vartheta,\varphi) = \_\_\, R_{n,\ell}(r)\,Y_\ell^m(\vartheta,\varphi)$

$$\mu_z = \frac{e\hbar}{2mc} L_z\, m \quad \text{magnetski kvantni broj}$$

$\mu_B$

$$\hat{L}^2\psi(\_) = L^2 RY = R\underbrace{L^2 Y}_{\ell(\ell+1)\hbar^2 Y}$$

ORBITALNI KVANTNI BROJ

$$L^2 = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

$$L_z Y_\ell^m = m\hbar\, Y_\ell^m$$

$$\vec{m} \sim \vec{L}$$

$$\vec{m} = \frac{e}{2mc}\vec{L}$$

$\underline{O\,v = a\,v}$ eigenvector $v$, eigenvalue $a$

$$H\psi = \boxed{E}\,\psi$$

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$$Y(\vartheta,\varphi)=P(\vartheta)\,\Phi(\varphi)$$

$$\Phi\,\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial P}{\partial\vartheta}\right)+\frac{P}{\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2}+\lambda P\Phi=0\quad\Big/\ \frac{\sin^2\vartheta}{P\Phi}$$

$$\underbrace{\frac{\sin\vartheta}{P}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial P}{\partial\vartheta}\right)+\lambda\sin^2\vartheta}_{m^2}+\underbrace{\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}}_{-m^2}=0$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=-m^2$$

$$\frac{d^2\Phi}{d\varphi^2}=-m^2\Phi\Rightarrow\quad\frac{d^2\Phi}{d\varphi^2}+m^2\Phi=0$$

$$\boxed{\Phi\sim e^{\pm im\varphi}}$$ rješenje!

$$\Phi(\varphi)=\Phi(\varphi+2\pi)$$

$$e^{im\varphi}=e^{im(\varphi+2\pi)}$$

$$e^{im\varphi}=\cancel{e^{im\varphi}}+e^{im2\pi}$$

$$e^{2\pi im}=1$$

$$\boxed{m\in\mathbb{Z}}$$ VAŽNO! NB!

$$\boxed{\frac{\sin\vartheta}{P}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP}{d\vartheta}\right)+\lambda\sin^2\vartheta=m^2}$$

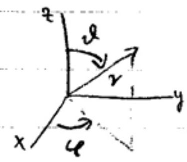
$$Y(\vartheta,\varphi)=P(\vartheta)\underbrace{\Phi(\varphi)}_{e^{im\varphi}}$$

19.12.2006.g.

može biti bilo što (za sada)

$$\lambda=l(l+1)$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP}{d\vartheta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\vartheta}\right]P=0$$




$$x=\cos\vartheta$$

$$\sin\vartheta=\sqrt{1-x^2}$$

$$\sin^2\vartheta=1-x^2$$

$$\frac{d}{d\vartheta}=\frac{d}{dx}\frac{dx}{d\vartheta}\qquad\frac{d}{d\vartheta}=-\sin\vartheta\frac{d}{dx}$$

$$\frac{d^2}{d\vartheta^2}=\sin^2\vartheta\frac{d^2}{dx^2}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0$$

$m \in \mathbb{Z}$

$\dots -3,-2,-1, 0, 1, 2, 3, \dots$

$x = \pm 1$ – problematično!

$\to \vartheta = 0, \pi$

tako se rješava taj problem

!!NB!!

ako je $\boxed{\ell}$ cijeli broj onda se singulariteti u polovima uklanjaju!

(to je argument i u kvantnoj fizici)

→ fizikalni argumenti su ovo...

$\boxed{m=0}$

$$\underline{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\ell(\ell+1)P=0}$$

$$P(x)=\sum_{j=0}^{\infty}a_j x^j$$

$$\frac{dP}{dx}=\sum_{j=0}^{\infty}a_j j x^{j-1}$$

$$\underbrace{\frac{d}{dx}\left[(1-x^2)\sum_j a_j j x^{j-1}\right]}+\ell(\ell+1)\sum_j a_j x^j=0$$

$$\frac{d}{dx}\left(\sum_j a_j j x^{j-1}-\sum_j a_j j x^{j+1}\right)$$

$$\sum_{j=0}^{\infty}a_j j(j-1)x^{j-2}-\sum_{j=0}^{\infty}a_j j(j+1)x^j+\ell(\ell+1)\sum_{j=0}^{\infty}a_j x^j=0$$

$$\underbrace{\sum_{k=2}^{\infty}a_k k(k-1)x^{k-2}}_{k=2\to j}-\sum_{j=0}^{\infty}a_j j(j+1)x^j+\ell(\ell+1)\sum_{j=0}^{\infty}a_j x^j=0$$

$$\sum_{j=0}^{\infty}a_{j+2}(j+2)(j+1)x^j$$

$$\sum_{j=0}^{\infty}\underbrace{\left[a_{j+2}(j+2)(j+1)-a_j\{j(j+1)-\ell(\ell+1)\}\right]}_{=0}x^j=0$$

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$$a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+1)(j+2)} a_j \,, \quad \underline{j_{max} = l}$$

$$\rightarrow \boxed{P_l(x) = \sum_{j=0}^{l} a_j x^j} \text{ polinom kao rješenje}$$

Legendreovi polinomi

(polinom reda $l$)

$$\left.\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \end{aligned}\right\} \text{ konvencija!}$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$\vdots$$

$$\boxed{\int_{-1}^{+1} P_l(x) P_{l'}(x)\, dx = \frac{2}{2l+1} \delta_{ll'}}$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

$$\underbrace{(1-x^2)\frac{d^2 P_l}{dx^2}}_{(1)} - \underbrace{2x\frac{dP_l}{dx}}_{(2)} + \underbrace{l(l+1)P_l}_{(3)} = 0$$

$\frac{d^m}{dx^m}$

(1) $$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^m P_l}{dx^m}\right) - 2mx\frac{d}{dx}\left(\frac{d^m P_l}{dx^m}\right) - m(m-1)\frac{d^m P_l}{dx^m}$$

(2) $$-2\left[x\frac{d}{dx}\left(\frac{d^m P_l}{dx^m}\right) + m\frac{d^m P_l}{dx^m}\right.$$

(3) $$l(l+1)\frac{d^m P_l}{dx^m}$$

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2(m+1)x\frac{d}{dx} + l(l+1) - m(m+1)\right]\frac{d^m P_l}{dx^m} = 0$$

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + l(l+1) - \frac{m^2}{1-x^2}\right]\boxed{\underbrace{(1-x^2)^{\frac{m}{2}}\frac{d^m P_l}{dx^m}}_{P_l^m(x)}} = 0$$

~~generalno~~ rješenje za P !

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + l(l+1) - \frac{m^2}{1-x^2}\right]\boxed{P} = 0$$

→ pridružena Legendreova f-ja (polinom) : $P_l^m(x)$

$$\boxed{P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l}$$

$(-1)^m = (-)^m$

$$\boxed{P_l^{-m}(x) = (-)^m \frac{|l-m|!}{|l+m|!} P_l^m(x)}$$

$$\begin{array}{ll} l & \overbrace{-l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l}^{\overbrace{m}^{2l+1}} \\ 2 & \underbrace{-2, -1, 0, 1, 2}_{5} \end{array}$$

$$\boxed{Y_l^m(\vartheta,\varphi) = \underbrace{P_l^m}_{N \text{ (norma)}}(\vartheta)\,\Phi(\varphi)} \quad \text{- KUGLINA FJA!}$$

$x = \cos\vartheta$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta\, Y_{l'}^{m'*}(\vartheta,\varphi)\, Y_l^m(\vartheta,\varphi) = \delta_{ll'}\delta_{mm'}$$

$$\boxed{Y_l^m(\vartheta,\varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\, P_l^m(\cos\vartheta)\, e^{im\varphi}}$$

$7!! = 1\times3\times5\times7$

$8!! = 2\times4\times6\times8$

dvostruka faktorijela

$$\underline{Y_l^{-m}(\vartheta,\varphi) = (-)^m Y_l^{m*}(\vartheta,\varphi)}$$

| | |
|---|---|
| $l=0$ | s |
| 1 | p |
| 2 | d |
| 3 | f |
| 4 | g |
| 5 | h |
| 6 | i |
| ⋮ | ⋮ |

$e^{i\varphi} = \cos\varphi + i\sin\varphi$

$e^{-i\varphi} = \cos\varphi - i\sin\varphi$

$\psi^*\psi$ gustoća vjerojatnosti

$Y_l^{m*}(\vartheta,\varphi)\, Y_l^m(\vartheta,\varphi)$

$l=0$ $\quad \frac{1}{4\pi}$

$l=1$

$l=0 \qquad Y_0^0(\vartheta,\varphi) = \frac{1}{\sqrt{4\pi}}$

$l=1 \qquad Y_1^1(\vartheta,\varphi) = -\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{i\varphi}$

$$Y_1^0(\vartheta,\varphi) = \sqrt{\frac{3}{4\pi}} \cos\vartheta$$

$$Y_1^{-1}(\vartheta,\varphi) = \sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{-i\varphi}$$

$l=2 \qquad Y_2^2(\vartheta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\vartheta\, e^{2i\varphi}$

$$Y_2^1(\vartheta,\varphi) = -\sqrt{\frac{15}{8\pi}} \sin\vartheta\cos\vartheta\, e^{i\varphi}$$

$$Y_2^0(\vartheta,\varphi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\vartheta - \frac{1}{2}\right)$$

$$Y_2^{-1}(\vartheta,\varphi) = \sqrt{\frac{15}{8\pi}} \sin\vartheta\cos\vartheta\, e^{-i\varphi}$$

$$Y_2^{-2}(\vartheta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\vartheta\, e^{-2i\varphi}$$

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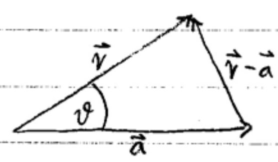
**09.01.2007.**




$$|\vec{r}-\vec{a}| = \sqrt{r^2 - 2ar\cos\vartheta + a^2}$$

$r = |\vec{r}| \qquad a = |\vec{a}|$

$r_< = \min\{r, a\}$

$r_> = \max\{r, a\}$

$$\frac{1}{|\vec{r}-\vec{a}|} = \frac{1}{\sqrt{r_>^2 - 2r_>r_<\cos\vartheta + r_<^2}} = \frac{1}{r_>}\,\frac{1}{\sqrt{1 - 2\frac{r_<}{r_>}\cos\vartheta + \left(\frac{r_<}{r_>}\right)^2}}$$

$$\to \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

$|t|<1$

u knjizi!

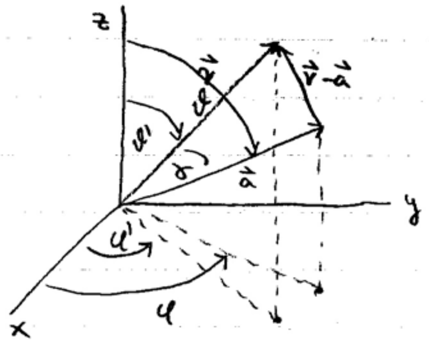
f-ja izvodnica za Laugrangeove polinome

$x = \cos\vartheta$

$t = \frac{r_<}{r_>}$

KORISNA RELACIJA!

$$\frac{1}{|\vec{r}-\vec{a}|} = \frac{1}{r_>}\sum_{l=0}^{\infty} P_l(\cos\vartheta)\left(\frac{r_<}{r_>}\right)^l = \sum_{l=0}^{\infty} P_l(\cos\vartheta)\,\frac{r_<^l}{r_>^{l+1}}$$




: 5 str. dokaza! :)

kompleksno konjugiran

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^{m*}(\vartheta',\varphi')\,Y_l^m(\vartheta,\varphi)$$

preko kuteva u S.K.S.

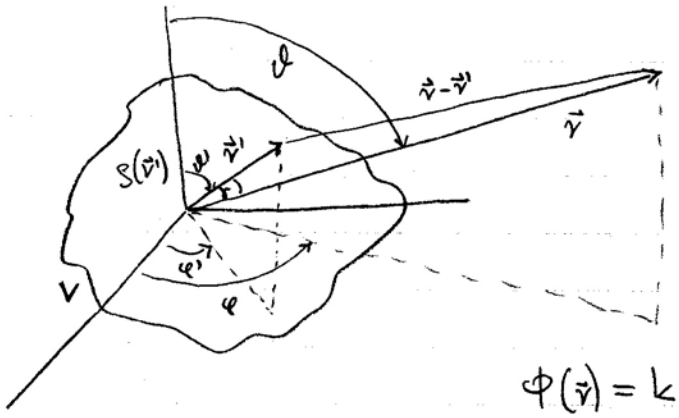
→ ADICIONI TM za KUGLINE FJE

$$\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi')$$

$\vec{a} \rightsquigarrow \vec{r}'$ ; $a \rightsquigarrow r'$

$$\boxed{\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,\frac{r_<^l}{r_>^{l+1}}\,Y_l^{m*}(\vartheta',\varphi')\,Y_l^m(\vartheta,\varphi)}$$

– SVUGDJE GA IMA!




$\vec{r}_> = \vec{r}$ $\quad r_> = r$

$\vec{r}_< = \vec{r}'$ $\quad r_< = r'$

$\vec{r}' \rightsquigarrow (r', \vartheta', \varphi')$ ←

$\vec{r} \rightsquigarrow (r, \vartheta, \varphi)$

pod kojim su povezane $\vec{r}$ i $\vec{r}'$
→ integrali teško izbjeći

$$\Phi(\vec{r}) = k\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dv'$$

$$\Phi(\vec{r}) = k\int \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,\frac{r'^l}{r^{l+1}}\,Y_l^{m*}(\vartheta',\varphi')\,Y_l^m(\vartheta,\varphi)\,\rho(\vec{r}')\,dv'$$

$$\boxed{\Phi(\vec{r}) = k\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,q_l^m\,\frac{1}{r^{l+1}}\,Y_l^m(\vartheta,\varphi)}$$

MULTIPOLNI RAZVOJ ELEKTROSTATIČNOG POTENCIJALA u SFERNOM SUSTAVU

– napravili smo separaciju varijabli!

$$\boxed{q_l^m = \int Y_l^{m*}(\vartheta',\varphi')\,r'^l\,\rho(\vec{r}')\,dv'}$$

MULTIPOLNI MOMENT u SFERNOM SUSTAVU

$$\sim \frac{1}{r}\,\frac{1}{r^l}\,r'^l \sim \frac{1}{r}\left(\frac{r'}{r}\right)^l$$

$$\frac{r'}{r} \ll 1 \qquad \underline{r' \ll r}$$ konvergira red jako brzo!

K.k.s.

$$\phi(\vec r) = \underbrace{k\,\frac{Q}{r}}_{\text{MONOPOLNI ČLAN POTENCIJALA (RAZVOJA)}} + \underbrace{k\,\frac{\vec p\cdot\vec r}{r^3}}_{\text{DIPOLNI ČLAN}} + \underbrace{\frac{1}{2}k\sum_{i,j=1}^{3} Q_{ij}\,\frac{x_i x_j}{r^5}}_{\text{KVADRUPOLNI ČLAN}} + \dots$$

$l=0$

$m=0$

$$q_0^0 = \int \underbrace{Y_0^{0*}(\vartheta',\varphi')}_{\frac{1}{\sqrt{4\pi}}}\,\underbrace{r'^0}_{1}\,\rho(\vec r')\,dv' = \frac{1}{\sqrt{4\pi}}\underbrace{\int \rho(\vec r')\,dv'}_{Q}$$

$$\frac{\cancel{4\pi}}{\underbrace{2\cdot 0+1}_{\cancel{4\pi}}}\cdot\frac{1}{\sqrt{4\pi}}\,Q\cdot\frac{1}{r}\,\underbrace{Y_0^0(\vartheta,\varphi)}_{\frac{1}{\sqrt{4\pi}}}$$

$$q_0^0 = \frac{1}{\sqrt{4\pi}}\,Q \qquad\qquad q_l^{m*} = (-)^m q_l^{-m}$$

$l=1$

$$q_1^1 = -\sqrt{\frac{3}{8\pi}}\int \sin\vartheta'(\cos\varphi' - i\sin\varphi')\,r'\rho(\vec r')\,dv' =$$
$$= -\sqrt{\frac{3}{8\pi}}\int (x'-iy')\,\rho(\vec r')\,dv' = -\sqrt{\frac{3}{8\pi}}\,(p_x - ip_y)$$
$$q_1^0 = \sqrt{\frac{3}{4\pi}}\int \cos\vartheta'\,r'\rho(\vec r')\,dv' = \sqrt{\frac{3}{4\pi}}\int z'\rho(\vec r')\,dv' =$$
$$= \sqrt{\frac{3}{4\pi}}\,p_z$$
$$q_1^{-1} = +\sqrt{\frac{3}{8\pi}}\,(p_x + ip_y)$$


$$\int x'\rho(\vec r')\,dv' = p_x$$
$$\int y'\rho(\vec r')\,dv' = p_y$$

$$\boxed{\vec p = \int \vec r'\,\rho(\vec r')\,dv'}$$ DIPOLNI MOMENT def.

D.Z. ostatak raspetljati + za $l=2$ OBAVEZNO!

$l=2$

$$q_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int (x'-iy')^2\rho(\vec r')\,dv' = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\,(Q_{11} - 2iQ_{12} - Q_{22})$$

$$q_2^1 = -\sqrt{\frac{15}{8\pi}}\int z'(x'-iy')\,\rho(\vec r')\,dv' = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\,(Q_{13} - iQ_{23})$$

$$q_2^0 = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\int (3z'^2 - r'^2)\,\rho(\vec r')\,dv' = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,Q_{33}$$

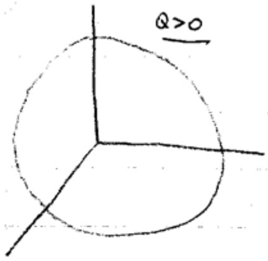
$$q_2^{-1} \qquad = \frac{1}{3}\sqrt{\frac{15}{8\pi}}\,(Q_{13} + iQ_{23})$$

$$q_2^{-2}$$

$r'^2 = x'^2 + y'^2 + z'^2$

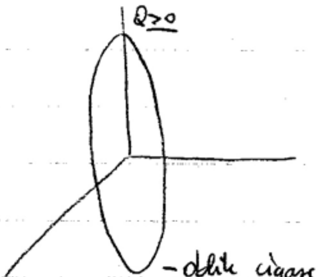
$x_1' \to x' \quad x_2' \to y' \quad x_3' \to z'$

$$\boxed{Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\,\rho(\vec r')\,dv'}$$ TENZOR KVADRUPOLNOG MOMENTA $\sim \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ $\qquad \mathrm{Tr}\,Q_{ij} = 0$



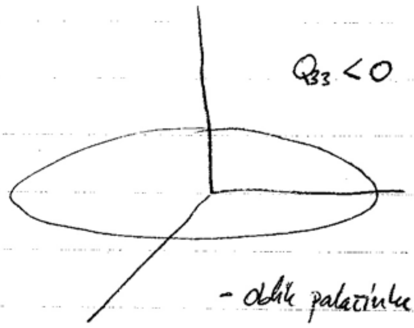

homogeno nabijena sfera, pozitivan naboj

$$Q_{33} = \int \big(3z'^2 \underbrace{- x'^2 - y'^2 - z'^2}_{2z'^2 - x'^2 - y'^2}\big)\, \rho(\vec{r}\,')\, dv' = 0$$




$Q_{33} > 0$

- oblik cigare
- prolate distribution



$Q_{33} < 0$

- oblik palačinke
- oblate distribution

**16. 01. 2007.**

## Green(ove f-je)

$$\vec{E} = E(\vec{r}) \qquad \phi = \phi(\vec{r})$$

$$\nabla \cdot \vec{E} = 4\pi k \rho \qquad \rho = \rho(\vec{r})$$

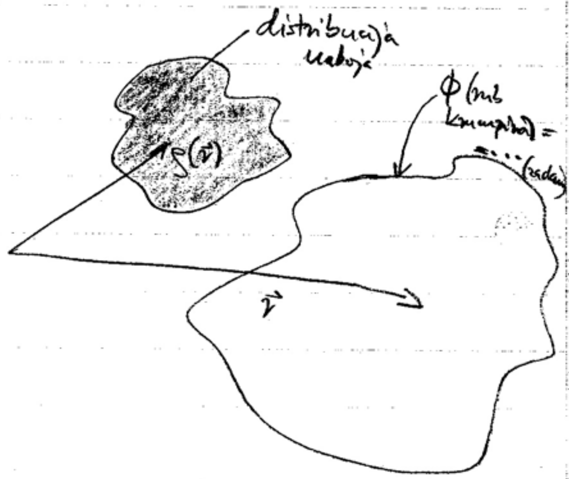
$$\vec{E} = -\nabla \phi$$

$$-\nabla \cdot (\nabla \phi) = 4\pi k \rho$$

$$\boxed{\Delta \phi = -4\pi k \rho}$$ Poissonova jdba

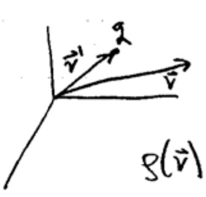
- imaju rubne uvjete ove dif. jdbe

$$\rho = 0 \implies \boxed{\Delta \phi = 0}$$ Laplaceova jdba




$$\phi(\vec{r}) = 4\pi k \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, dv'$$

- rubni uvjet ugurati u integral i riješiti integralnu jdbu
  - → Greenove f-je (metoda)
  - → pojednostavlja problem ako je geometrija poznata




$$\rho(\vec{r}) = q\, \delta(\vec{r} - \vec{r}\,')$$

? $q = \int \rho(\vec{r})\, dv = q \underbrace{\int \delta(\vec{r} - \vec{r}\,')\, dV}_{1}$

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$$\int_V \nabla\cdot\vec{A}\cdot dv = \oint_S \vec{A}\cdot\hat{n}\,ds,$$

$$\vec{A} = \phi\nabla\psi$$

$$\nabla\cdot\vec{A} = \nabla(\phi\nabla\psi) = \phi\Delta\psi + (\nabla\phi)\cdot(\nabla\psi)$$

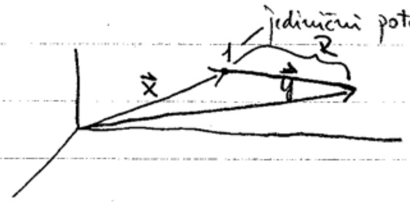
$$\vec{A}\cdot\hat{n} = \phi\underbrace{\nabla\psi}_{\frac{\partial\psi}{\partial n}\hat{n}}\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$ - normala gleda iz volumena van

$$\boxed{\int_V(\phi\Delta\psi + \nabla\phi\nabla\psi)\,dV = \oint_S \phi\frac{\partial\psi}{\partial n}\,ds}$$ 1. Greenov indentitet

$$-\int_V(\psi\Delta\phi + \nabla\psi\nabla\phi)\,dV = -\oint_S \psi\frac{\partial\phi}{\partial n}\,ds$$ +

$$\boxed{\int_V(\phi\Delta\psi - \psi\Delta\phi)\,dV = \oint_S\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds}$$ 2. Greenov indentited (Greenov teorem)

$\underline{\underline{k=1}}$ (za sada)




$\phi \rightsquigarrow \phi$ potencijal

$\psi \rightsquigarrow \frac{1}{R} = \frac{1}{|\vec{x}-\vec{y}|}$ $\quad R = |\vec{x}-\vec{y}|$

$$\Delta\phi(\vec{x}) = -4\pi\rho(\vec{x})$$

$$\Delta\frac{1}{|\vec{y}-\vec{x}|} = -4\pi\delta(\vec{y}-\vec{x})$$ (izvedi smo prije)

$$\int f(x)\,\delta(x-a)\,dx = f(a)$$

$$\int_V \Big[\underbrace{-4\pi\,\phi(\vec{x})\,\delta(\vec{y}-\vec{x})}_{-4\pi\,\phi(\vec{y})} + \underbrace{\frac{4\pi\rho(\vec{x})}{R}}_{4\pi\int\frac{\rho(\vec{x})}{R}dV}\Big]\,\underbrace{dV}_{(\vec{x})} = \oint_S \Big[\phi(\vec{x})\frac{\partial}{\partial n}\Big(\frac{1}{R}\Big) - \frac{1}{R}\frac{\partial\phi(\vec{x})}{\partial n}\Big]\,dS$$

derivacija potencijala na rubu

potencijal na rubu

$$\boxed{\phi(\vec{y}) = \int_V \frac{\rho(\vec{x})}{|\vec{y}-\vec{x}|}\,dV + \frac{1}{4\pi}\oint_S\Big[\underbrace{\frac{1}{R}\boxed{\frac{\partial\phi}{\partial n}}}_{\sim\frac{1}{R}\ \frac{1}{R^2}} - \phi\,\frac{\partial}{\partial n}\Big(\frac{1}{R}\Big)\Big]\,\underbrace{dS}_{R^2 d\Omega}}$$

$R\to\infty \quad \phi\to 0$

- PROBLEM - preodređeni integral
  - treba izbaciti derivaciju $\phi$ ili $\phi$ (potencijal) da izađe van da bi riješili taj problem

---

- treba samo dobro izabrati $\psi$ i stvar je riješena
- ključna je delta f-ja u svemu ovome!!!
- $\rightarrow \psi$ postaje tada Greenova f-ja (nakon izbora za danu geometriju)
- to je generalna ideja

$$\psi \rightsquigarrow G(\vec{y},\vec{x})$$

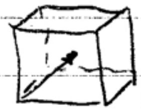
$$\Delta G(\vec{y},\vec{x}) = -4\pi\,\delta(\vec{y}-\vec{x})$$

$$\Delta\frac{1}{|\vec{y}-\vec{x}|} = -4\pi\,\delta(\vec{y}-\vec{x})$$

$$\longrightarrow\quad G(\vec{y},\vec{x}) = \frac{1}{|\vec{y}-\vec{x}|} + \underbrace{F(\vec{y},\vec{x})}_{\Delta F(\vec{y},\vec{x})=0}$$

$$\int\Big[-4\pi\,\phi(\vec{x})\,\delta(\vec{y}-\vec{x}) + \underbrace{4\pi\,G(\vec{y},\vec{x})\,\rho(\vec{x})}_{4\pi\int\rho(\vec{x})G(\vec{y},\vec{x})dV}\Big]dV = \oint_S\Big[\phi(\vec{x})\frac{\partial}{\partial n}G(\vec{y},\vec{x}) - G(\vec{y},\vec{x})\frac{\partial\phi}{\partial n}\Big]dS$$

$$\boxed{\phi(\vec{y}) = \int_V \rho(\vec{x})\, G(\vec{y},\vec{x})\, \underbrace{dV}_{d^3x} + \frac{1}{4\pi}\oint_S \left[G(\vec{y},\vec{x})\frac{\partial \phi(\vec{x})}{\partial n} - \phi(\vec{x})\frac{\partial G(\vec{y},\vec{x})}{\partial n}\right] \underbrace{dS}_{d^2x}}$$



$\phi$(rub kocke)

na rubu S $\phi$ – Dirichlet (uvjet)

$$G(\vec{y},\vec{x}) = G(\vec{x},\vec{y}) \quad \text{– Greenova f-ja}$$

$$G_D(\vec{y},\vec{x}) = 0 \quad \text{ako je } \vec{x} \text{ na rubu od } S$$

$$\boxed{\phi(\vec{y}) = \int_V \rho(\vec{x})\, G_D(\vec{y},\vec{x})\, dV - \frac{1}{4\pi}\oint_S \phi(\vec{x}) \frac{\partial G_D(\vec{y},\vec{x})}{\partial n}\, dS}$$

na rubu S $\frac{\partial \phi}{\partial n}$ – Neumann (rubni uvjet)

$dV \to d^3x$

$$\frac{\partial G_N(\vec{y},\vec{x})}{\partial n} = 0 \quad \text{za } \vec{x} \text{ na rubu } S$$

$$\int_V \Delta G_N(\vec{y},\vec{x})\, d^3x = \underbrace{-4\pi \int \underbrace{\delta(y^3 - x^3)\, d^3x}_{1}}_{-4\pi} \quad \Big/ \int d^3x$$

$$\int_V \nabla\cdot(\nabla G_N)\, d^3x$$

$$\oint_S \nabla G_N \cdot \hat{n}\, dS = \underbrace{\oint_S \frac{\partial G_N}{\partial n} \underbrace{\hat{n}\cdot\hat{n}}_{1}\, dS}_{0}$$

$$\Rightarrow \underline{0 = -4\pi} \quad \text{– loša ideja!}$$

$$\frac{\partial G_N(\vec{y},\vec{x})}{\partial n} = -\frac{4\pi}{S} \quad \text{– ovo je pravilno!}$$

$$-\frac{4\pi}{S} \underbrace{\oint_S dS}_{S} = -4\pi \quad \checkmark$$

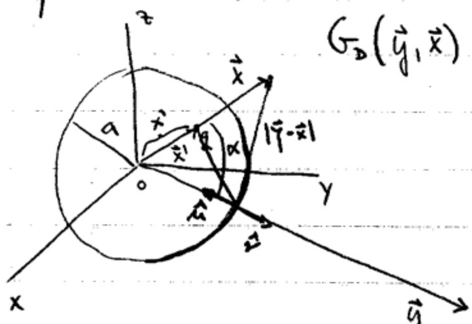
$$\Phi(\vec{y}) = \int_V \rho(\vec{x})\, G_N(\vec{y},\vec{x})\, dV + \frac{1}{4\pi}\oint_S G_N(\vec{y},\vec{x}) \frac{\partial \phi}{\partial n} dS - \underbrace{\frac{1}{4\pi}\left(-\frac{4\pi}{S}\right)\oint_S \phi(\vec{x}) dS}_{\underbrace{\frac{1}{S}\oint_S \phi(\vec{x}) dS}_{\langle \phi(\vec{x}) \rangle_S}}$$

Prosječni potencijal na površini S

23.01.2007.g.

→ Nađite Greenovu fju za Dirichletov problem za kuglu radiusa a.

$G_D(\vec{y},\vec{x})$




$$G_D(\vec{y},\vec{x}) = 0, \quad |\vec{y}| = a$$
$$\vec{y} = \vec{\xi}$$

$$G_D(\vec{y},\vec{x}) = \frac{1}{|\vec{y}-\vec{x}|} + \frac{q}{|\vec{y}-\vec{x}'|}$$

$$\frac{1}{|\vec{\xi}-\vec{x}|} + \frac{q}{|\vec{\xi}-\vec{x}'|} = 0$$

$$\frac{1}{\sqrt{a^2+x^2-2ax\cos\alpha}} + \frac{q}{\sqrt{a^2+x'^2-2ax'\cos\alpha}} = 0$$

$$\frac{1}{x\sqrt{1+\left(\frac{a}{x}\right)^2-2\left(\frac{a}{x}\cos\alpha\right)}} + \frac{q}{a\sqrt{1+\left(\frac{x'}{a}\right)^2-2\left(\frac{x'}{a}\right)\cos\alpha}} = 0$$

$$\Rightarrow \frac{q}{a} = -\frac{1}{x} \qquad \frac{x'}{a} = \frac{a}{x} \qquad \hat{x} = \frac{\vec{x}}{x}$$

$$\boxed{q = -\frac{a}{x}} \qquad \boxed{x' = \frac{a^2}{x}}$$

$$\boxed{G_D(\vec{y},\vec{x}) = \frac{1}{|\vec{y}-\vec{x}|} + \frac{-a}{x\left|\vec{y}-\frac{a^2}{x^2}\vec{x}\right|}}$$

Izračunajte potencijal izvan sfere radiusa a, ako je zadan potencijal na sferi ... bla bla.

$$\phi(r,\vartheta,\varphi) = ?$$

$\phi(a,\vartheta,\varphi) = $ zadano

$\Delta\phi(r,\vartheta,\varphi) = 0$ i na taj način je rješivo

$$\phi = \sum_l \sum_m \left(A_l r^l + B_l \frac{1}{r^{l+1}}\right) Y_l^m$$

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$$\Phi(\vec{y}) = \int_V \cancel{\rho(\vec{x})}\, G_D(\vec{y},\vec{x})\, d^3x - \frac{1}{4\pi}\oint_S \Phi(\vec{x}) \frac{\partial G_D(\vec{y},\vec{x})}{\partial n}\, ds$$

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$$

nema naboja izvana

$$\Rightarrow \Phi(\vec{y}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}) \frac{\partial G_D(\vec{y},\vec{x})}{\partial n}\, \underset{\vec{x} \to \vec{r}}{ds} \quad \to ds = \underset{a^2}{r^2} \sin\vartheta\, d\vartheta\, d\varphi$$

$\vec{x} \to \vec{r}$ $\qquad$ $r \to a$

(na kraju, nakon integrala)

$$G_D(\vec{y},\vec{r}) = \frac{1}{\sqrt{y^2+r^2-2yr\cos\alpha}} - \frac{a}{r\sqrt{y^2+\frac{a^4}{r^2}-2\frac{a^2}{r}y\cos\alpha}}$$

$$\frac{\partial}{\partial r}$$

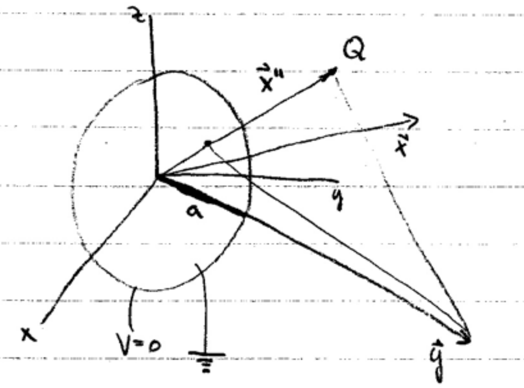
$$\left.\frac{\partial G_D}{\partial r}\right|_{r=a} = \frac{y^2-a^2}{a(y^2+a^2-2ay\cos\alpha)^{3/2}}$$

SI $\frac{1}{4\pi\varepsilon_0}$

$$\boxed{\underbrace{\Phi(\vec{y})}_{\Phi(r_y,\vartheta_y,\varphi_y)} = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \Phi(a,\vartheta,\varphi) \frac{a(y^2-a^2)}{(y^2+a^2-2ay\cos\alpha)^{3/2}} \sin\vartheta\, d\vartheta\, d\varphi}$$

$$\cos\alpha = \cos\vartheta_y\cos\vartheta + \sin\vartheta_y\sin\vartheta\cos(\varphi_y-\varphi)$$

→ težan integral za analitičko rješavanje




$$\rho(\vec{x}) = Q\,\delta(\vec{x}-\vec{x}'')$$

$$\Phi(a) = 0$$

$$\Phi(\vec{y}) = \int_V \rho(\vec{x})\, G_D(\vec{y},\vec{x})\, d^3x$$

$$\Phi(\vec{y}) = Q\int_V \delta(\vec{x}-\vec{x}'')\, \underline{G_D(\vec{y},\vec{x})}\, d^3x$$

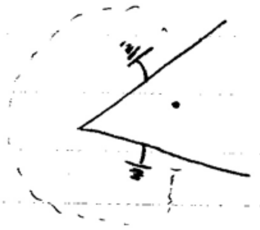
$$\int \delta(\vec{x}-\vec{x}'') f(x'')\, dx = f(x)$$

$$\phi(\vec y) = Q\, G_D(\vec y, \vec x'')$$

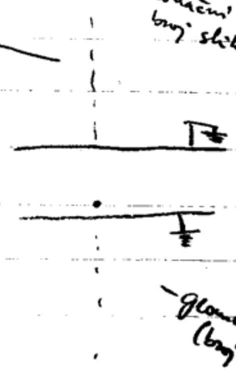
$$\phi(\vec y) = \underbrace{\frac{Q}{|\vec y - \vec x''|}}_{\frac{1}{4\pi\varepsilon_0}} - \underbrace{\frac{aQ}{x''\left|\vec y - \frac{a^2}{x''^2}\vec x''\right|}}_{\frac{1}{4\pi\varepsilon_0}}$$

potencijal naboja vani

potencijal naboja slike unutar sfere



beskonačni broj slika



– geomet. red (broj slika)

# ELEKTROSTATIKA u SREDSTVU

zanemarivo (najčešće)

$$\phi = \cancel{k\frac{\overset{0}{Q}}{r}} + \boxed{k\frac{\vec p\cdot\vec r}{r^3}} + k\sum Q_{ij}\frac{x_i x_j}{r^5} + \dots$$

materijal je neutralan

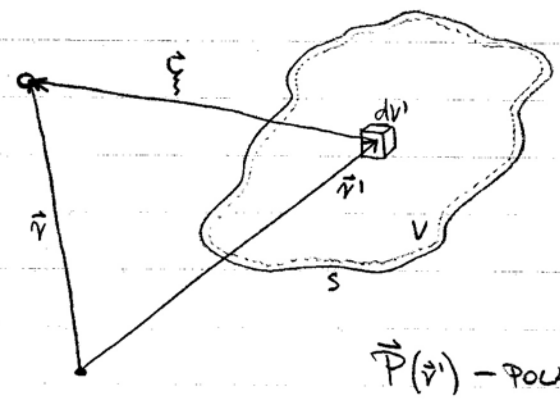
vodeći član u materijalima

$$\rho(\vec r') = \frac{dq(\vec r')}{dV'}$$

$\frac{r'}{r} \ll 1$ $\quad r' \ll r$ $\quad$ brzo konvergira pod ovim uvjetom

$d\vec p(\vec r') = \quad dV'$

$\vec p$

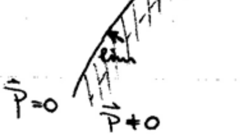


$$\frac{d\vec p(\vec r')}{dV'} = \vec P(\vec r')$$

ključno kod materijala

– na rubu materijala nagli skok u polarizaciji! (PROBLEM)

$\vec P(\vec r')$ – POLARIZACIJA (gustoća dipolnog momenta)



$$\vec p = \int_V \vec P(\vec r')\, dV'$$

⇒ DIELEKTRICI (naziv za te materijale)

$\vec\xi = \vec r - \vec r'$

$\xi = |\vec\xi|$

$$d\phi_p(\vec r) = k\frac{(\vec r - \vec r')\cdot d\vec p}{|\vec r - \vec r'|^3} = k\frac{\vec\xi\cdot d\vec p}{\xi^3} = k\frac{\vec P(\vec r')\cdot\vec\xi}{\xi^3}\,dV'$$

$$\frac{\vec\xi}{\xi^3} = -\nabla\left(\frac{1}{\xi}\right) \qquad \xi = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

oznaka!!

$$\nabla\left(\frac{1}{\xi}\right) = -\nabla'\left(\frac{1}{\xi}\right)$$

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$$d\phi_p(\vec{r}) = -k\vec{P}(\vec{r}\,')\cdot\nabla\left(\frac{1}{\xi}\right)dV'$$

$$\phi_p(\vec{r}) = -k\int_V \vec{P}(\vec{r}\,')\cdot\nabla\left(\frac{1}{\xi}\right)dV'$$

$$\phi_p(\vec{r}) = k\int_{V'} \underbrace{\vec{P}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{\xi}\right)}_{*}dV'$$

$$\nabla'\cdot\left[\frac{\vec{P}(\vec{r}\,')}{\xi}\right] = \frac{\nabla'\cdot\vec{P}(\vec{r}\,')}{\xi} + \underbrace{\vec{P}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{\xi}\right)}_{*}$$

$$\phi_p(\vec{r}) = k\int_{V'} \nabla'\cdot\underbrace{\left[\frac{\vec{P}(\vec{r}\,')}{\xi}\right]}_{\text{Gauss. tm}}dV' - k\int_{V'}\frac{\nabla'\cdot\vec{P}(\vec{r}\,')}{\xi}dV'$$

$$\boxed{\phi_p(\vec{r}) = k\oint_{S'}\frac{\vec{P}(\vec{r}\,')\cdot\hat{u}}{\xi}da' - k\int_{V'}\frac{\nabla'\cdot\vec{P}(\vec{r}\,')}{\xi}dV'}$$

$S' \to S$, $V' \to V$

$$k = \begin{cases} 1 & \text{CGS} \\ \frac{1}{4\pi\varepsilon_0} & \text{SI} \end{cases}$$

$$\phi(\vec{r}) = k\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV' \qquad \phi(\vec{r}) = k\int_S \frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}da'$$

$$\boxed{\sigma_p = \vec{P}(\vec{r}_s\,')\cdot\hat{u}} \qquad \boxed{\rho_p = -\nabla'\cdot\vec{P}(\vec{r}\,')}$$

| vakuum | materijali |
|---|---|
| $\nabla\cdot\vec{E} = 4\pi k\rho$ | $\langle\nabla\cdot\vec{E}\rangle = 4\pi k\langle\rho\rangle$ |
| $\nabla\times\vec{E} = 0$ | $\langle\nabla\cdot\vec{E}\rangle = 0$ |

$$\frac{\langle \nabla \cdot \vec{E} \rangle = \nabla \langle \vec{E} \rangle}{\langle \rho \rangle} \qquad \langle \nabla \times \vec{E} \rangle = \nabla \times \langle \vec{E} \rangle$$

→ potpuno iste jdbe, ali fizika potpuno različita!

– gustoća slobodnog naboja + $\rho_p$ (od polarizacije)
(redistribucija naboja)

– električna indukcija – izvedeno polje (to slijedi u materijalima)
(matematičko pomagalo)
↳ to je konstrukcija!

30.01.2007.

$\langle \vec{E} \rangle$ ?

$\langle \rho \rangle$ ?

f ⇝ free

p ⇝ polarizacija

vakuum : $\rho = \rho_f$

materijal : $\rho = \rho_f + \rho_p \qquad \rho_p = -\nabla \cdot \vec{P} \quad ; \quad \vec{P} = \frac{d\vec{p}}{dV}$

$$\nabla \cdot \vec{E} = 4\pi k \rho$$

$$\underline{\nabla \cdot \vec{E} = 4\pi k \rho_f - 4\pi k \nabla \vec{P}} \quad !!!! \quad \text{OPREZ! PAZI! GRIZE!}$$

SI

$k = \frac{1}{4\pi\varepsilon_0}$

$$\nabla \vec{E} + 4\pi k \nabla \vec{P} = 4\pi k \rho_f$$

$$\nabla \cdot (\vec{E} + 4\pi k \vec{P}) = 4\pi k \rho_f$$

$$\nabla \cdot \left(\frac{\vec{E}}{4\pi k} + \vec{P}\right) = \rho_f$$

$$\nabla \cdot \underbrace{\left(\varepsilon_0 \vec{E} + \vec{P}\right)}_{\vec{D}} = \rho_f$$

$$\boxed{\nabla \vec{D} = \rho_f}$$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

električna indukcija

↳ matemat. konstrukcija

$\underline{\nabla \times \vec{D} \neq 0}$ nije konzervat. polje

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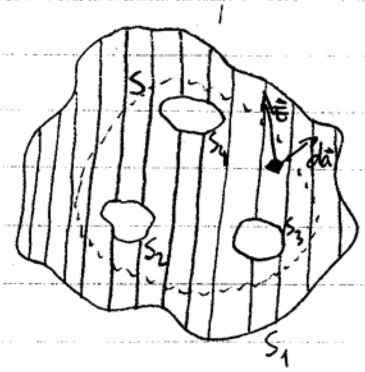
$$\int_V \rho_P\, dv = -\int_V \underbrace{\nabla\cdot\vec P\, dv}_{\oint_S \vec P d\vec a}$$



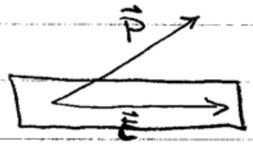

$$\oint_S \vec E\cdot d\vec a = \frac{Q}{\varepsilon_0} \qquad Q = Q_f + Q_P = Q_f + \int_V \rho_P\, dv$$

$$\oint_S \vec E\cdot d\vec a = \frac{Q_f}{\varepsilon} - \frac{1}{\varepsilon_0}\oint_S \vec P\cdot d\vec a$$

Gaussov zakon!

$$\oint_S \left(\vec E + \frac{1}{\varepsilon_0}\vec P\right) d\vec a = \frac{Q_f}{\varepsilon_0} \longrightarrow \oint_S \underbrace{(\varepsilon_0\vec E + \vec P)}\cdot d\vec a = Q_f$$

$$\boxed{\oint_S \vec D\, d\vec a = Q_f}$$




$\vec E$ u direktnoj vezi sa $\vec P$ (kod većine materijala, skoro svi)

$$\vec P \overset{?}{=} f(\vec E) \qquad \begin{aligned} P_x &= f_1(E_x, E_y, E_z) \\ P_y &= f_2(\ -\!\!\shortparallel\!\!-\ ) \\ P_z &= f_3(\ -\!\!\shortparallel\!\!-\ ) \end{aligned}$$

$i = 1 \; x$
$\quad 2 \; y$
$\quad 3 \; z$

pretpostavka: (linearna veza)

$$\begin{aligned} P_1 &= a_{11}E_1 + a_{12}E_2 + a_{13}E_3 \\ P_2 &= a_{21}E_1 + a_{22}E_2 + a_{23}E_3 \\ P_3 &= a_{31}E_1 + a_{32}E_2 + a_{33}E_3 \end{aligned} \qquad P_i = \sum_j a_{ij}E_j$$

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

ili može biti neka druga ovisnost

$$\underbrace{P_i = \sum_j a_{ij}E_j}_{} + \cancel{\sum_{jk} b_{ijk}E_jE_k} + \cancel{\sum E_jE_kE_l} + \dots$$

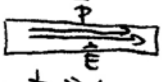
eksperiment kaže da je ovisnost linearna!

simetrična matrica

dijagonalizacija

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \varepsilon_0\chi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

$$\left.\begin{aligned} P_1 &= \varepsilon_0\chi E_1 \\ P_2 &= \varepsilon_0\chi E_2 \\ P_3 &= \varepsilon_0\chi E_3 \end{aligned}\right\} \text{za većinu materijala}$$




najčešće za materijale $\chi$ je isti i iznosi $\boxed{\varepsilon_0\chi}$

$\chi$ – električna suscetibilnost

$$\rightsquigarrow \underline{\vec{P} = \varepsilon_0 \chi \vec{E}}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E}$$

$$\vec{D} = \varepsilon_0 (1+\chi) \vec{E}$$

$$\begin{cases} 1+\chi = K \\ \varepsilon_0 K = \varepsilon \end{cases}$$

u knjigama

$\varepsilon_r$ – relativna permitivnost
– relativna dielektrična konst.

– permitivnost materijala
– dielektrična konst.
! krivi naziv!

$$\boxed{\vec{D} = \varepsilon \vec{E}}$$

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

to je tenzor, a ne konstanta!

pojam dvoloma kod loma svjetlosti pokazatelj da je to tenzor

$$n = \sqrt{K} = \sqrt{\varepsilon_r}$$

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gauss CGS

$$\nabla (\vec{E} + 4\pi k \vec{P}) = 4\pi k \rho_f$$

$$k = 1$$

$$\nabla (\underbrace{\vec{E} + 4\pi \vec{P}}_{\vec{D}}) = 4\pi \rho_f$$

$$\vec{D} = \vec{E} + 4\pi \vec{P}$$

$$\Longrightarrow \boxed{\nabla \vec{D} = 4\pi \rho_f}$$

| | SI | Gauss | jedinice SI | jedinice Gauss |
|---|---|---|---|---|
| duljina | $l$ | $l$ | $1\,m$ | $10^2\,cm$ |
| ⋮ mehanika | | | | |
| el. polje | $\vec{E}$ | $(4\pi\varepsilon_0)^{-1/2} \vec{E}$ | | |

idući put ćemo to napisati!