Cirkulacija i fluks vektorskog polja

Neka je $\vec{v} = (V_x, V_y, V_z)$ dato vektorsko polje.
Cirkulacija vektorskog polja $\vec{v}$ duž krive c je integral

$$C = \int_c \vec{v}\, d\vec{r} = \int_c V_x\, dx + V_y\, dy + V_z\, dz$$

gde je $\vec{r} = (x, y, z)$
$d\vec{r} = (dx, dy, dz)$

Ako je c zatvorena kontura možemo koristiti formulu Stoksa u vektorskom obliku

$$C = \int_c \vec{v}\, d\vec{r} = \iint_S \vec{n} \cdot \operatorname{rot} \vec{v}\, dS = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} dS$$

Fluks (tok, proticanje) vektorskog polja (kroz površ S) je površinski integral

$$\Phi = \iint_S \vec{v} \cdot \vec{n}\, dS = \iint_S (V_x \cos\alpha + V_y \cos\beta + V_z \cos\gamma)\, dS$$

$$= \iint_S V_x\, dy\, dz + V_y\, dx\, dz + V_z\, dx\, dy$$

Ako je S zatvorena površ, fluks polja se može računati pomoću formule Gauss-Ostrogradski

$$\Phi = \iint_S \vec{v} \cdot \vec{n}\, dS = \iiint_\Omega \operatorname{div} \vec{v}\, dx\, dy\, dz = \iiint_\Omega \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx\, dy\, dz$$

gde je Ω oblast u prostoru koja je ograničena površinom S.

(#) Izračunati cirkulaciju vektorskog polja $\vec{v} = -y\vec{i} + x\vec{j} + a\vec{k}$ (a = konstanta) duž kruga $(x-2)^2 + y^2 = 1$, $z = 0$.

Rj.

$\vec{v} = -y\vec{i} + x\vec{j} + a\vec{k}$

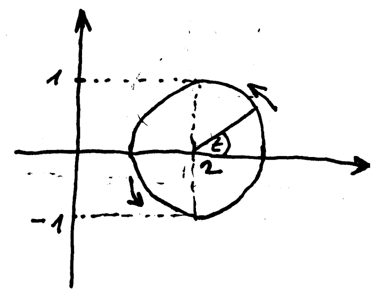
$c: (x-2)^2 + y^2 = 1, \quad z = 0$

$$C = \int_c \vec{v}\, d\vec{r} = \int_c v_x\, dx + v_y\, dy + v_z\, dz$$

cirkulacija polja $\vec{v}$

Imamo krivolinijski integral

$$C = \int_c -y\, dx + x\, dy + a\, dz \quad \text{gdje je} \quad c: \begin{cases} (x-2)^2 + y^2 = 1 \\ z = 0 \end{cases}$$



Parametrizirajmo kružnicu tj. uvedimo smjene

$$\begin{cases} x - 2 = \cos t \\ y = \sin t \\ z = 0 \end{cases} \qquad 0 \le t \le 2\pi \qquad \begin{aligned} dx &= -\sin t\, dt \\ dy &= \cos t\, dt \\ dz &= 0 \end{aligned} \qquad x = 2 + \cos t$$

$$C = \int_0^{2\pi} (-\sin t)(-\sin t)\, dt + (2+\cos t)\cos t\, dt + 0 =$$

$$\int_0^{2\pi} (\sin^2 t + 2\cos t + \cos^2 t)\, dt = \int_0^{2\pi} (1 + 2\cos t)\, dt = (t + 2\sin t)\Big|_0^{2\pi} = 2\pi$$

II način: pomoću Stoksove formule

$$C = \int_c \vec{v}\, d\vec{r} = \iint_S \vec{n} \cdot \operatorname{rot}\vec{v}\, dS = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} dS$$

$$\operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & a \end{vmatrix} = 0\vec{i} + 0\vec{j} + 2\vec{k} = (0, 0, 2)$$

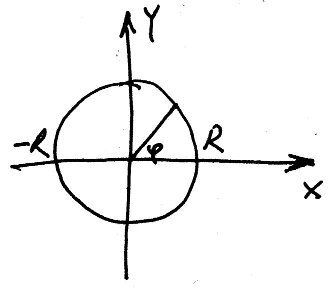
$$C = \iint_S \vec{n} \cdot \operatorname{rot}\vec{v}\, dS = \iint_S 2\cos\gamma\, dS = 2 \underbrace{\iint_S dx\, dy}_{\text{površina kruga}} = 2 \cdot 1^2 \cdot \pi = 2\pi$$

Iz formule Stoksa znamo da je $\cos\gamma\, dS = dx\, dy$

Izračunati cirkulaciju vektorskog polja $\vec{v} = x^2y^3\vec{i} + \vec{j} + z\vec{k}$ duž kružnice c koja je data kao presjek kružnice $x^2+y^2=R^2$ i xOy ravni.

Rj.

$$c: \begin{cases} x^2+y^2=R^2 \\ z=0 \end{cases}$$



$$C = \oint_c \vec{v}\, d\vec{r} = \int_c V_x\,dx + V_y\,dy + V_z\,dz$$

cirkulacija polja $\vec{v}$

I način

Parametrizirajmo kružnicu $\begin{cases} x = R\cos t \\ y = R\sin t \\ z = 0 \end{cases}$ ... ZAVRŠITI ZA VJEŽBU

II način Pomoću formule Stoksa:

$$C = \oint_c \vec{v}\, d\vec{r} = \iint_S \vec{n}\cdot \operatorname{rot}\vec{v}\, dS \qquad \operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = (0,0,-3x^2y^2)$$

$$C = \iint_S (\cos\alpha, \cos\beta, \cos\gamma)\cdot(0,0,-3x^2y^2)\, dS = \iint_S -3x^2y^2\cos\gamma\, dS =$$

$$= -3\iint_S x^2y^2\, dx\, dy \quad \text{gdje je sad } S: \{ x^2+y^2 = R^2$$

Uvodimo polarne koordinate $\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix} \Rightarrow S': \begin{cases} 0\le r\le R \\ 0\le\varphi\le 2\pi \end{cases}$

$dx\,dy = r\,dr\,d\varphi$

$$C = -3\iint_{S'} r^2\cos^2\varphi\, r^2\sin^2\varphi \cdot r\, dr\, d\varphi = -3\int_0^R r^5\left[\int_0^{2\pi}\frac{1}{4}\cdot\underbrace{4\cos^2\varphi\sin^2\varphi}_{(\sin 2\varphi)^2}\, d\varphi\right]dr$$

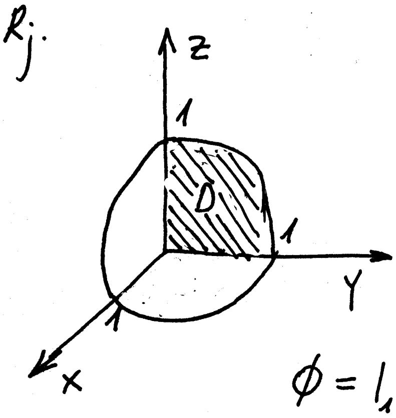
$$= -3\int_0^R r^5\left[\int_0^{2\pi}\frac{1}{4}\sin^2 2\varphi\, d\varphi\right]dr = -\frac{3}{4}\int_0^R r^5\left[\int_0^{2\pi}\frac{1-\cos 4\varphi}{2}\, d\varphi\right]dr$$

$$= -\frac{\cancel{3}^{\,1}}{4}\left[\frac{1}{2}\varphi\Big|_0^{2\pi} - \frac{1}{4}\sin 4\varphi\Big|_0^{2\pi}\right]\cdot\frac{1}{\cancel{6}_{\,2}} r^6\Big|_0^R = -\frac{1}{8}R^6\cdot\pi$$

$$\boxed{\begin{matrix} 1 = \cos^2 2\varphi + \sin^2 2\varphi \\ \cos 4\varphi = \cos^2 2\varphi - \sin^2 2\varphi \end{matrix}}$$

(#) Naći fluks polja $\vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ kroz dio sfere $x^2+y^2+z^2=1$ u I oktantu.

Rj.



I način

$$\Phi = \iint_S \vec{v}\cdot\vec{n}\, dS = \iint_S (v_x \cos\alpha + v_y \cos\beta + v_z \cos\gamma)\, dS$$

$$= \iint_S v_x\, dy\, dz + v_y\, dx\, dz + v_z\, dx\, dy$$

$$\Phi = I_1 + I_2 + I_3 = \iint_S xy\, dy\, dz + \iint_S yz\, dx\, dz + \iint_S zx\, dx\, dy$$

Zbog simetrije $I_1 = I_2 = I_3$ pa je $\Phi = 3I_1$. Računamo samo I_1

$$I_1 = \iint_S xy\, dy\, dz = +\iint_D \sqrt{1-(y^2+z^2)}\; y\, dy\, dz$$ gdje je $D: y^2+z^2 \le 1,\ x \ge 0,\ y \ge 0$

$x^2 = 1-(y^2+z^2)$

$x = \pm\sqrt{1-(y^2+z^2)}$

Vektor normale zaklapa ugao $\alpha \in (0, \frac{\pi}{2})$ sa x-osom. $\cos\alpha > 0$ (u I oktantu).

uzimamo + jer smo u prvom oktantu

Uvodimo polarne koordinate $y = r\cos\varphi$, $z = r\sin\varphi$

$$D': \begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 1 \\ dy\, dz = r\, d\varphi\, dr \end{cases} \qquad y^2+z^2 = r^2$$

$$I_1 = \iint_{D'} r\cos\varphi \sqrt{1-r^2}\cdot r\, d\varphi\, dr = \int_0^1 r^2\sqrt{1-r^2}\left[\int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi\right] dr = \int_0^1 r^2\sqrt{1-r^2}\cdot 1\, dr$$

$$= \left| \begin{matrix} r = \sin t \\ dr = \cos t\, dt \end{matrix} \right| = \int_0^1 \sin^2 t \sqrt{1-\sin^2 t}\, \cos t\, dt = \int_0^1 \sin^2 t \cos^2 t\, dt = \ldots = \frac{3\pi}{16}$$

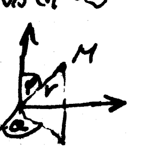
u prethodnom zadatku smo imali slično

II način: Kako je S zatvorena površ možemo primjeniti formulu Gauss-Ostrogradski.

$$\Phi = \iint_S \vec{v}\cdot\vec{n}\, dS = \iiint_\Omega \operatorname{div}\vec{v}\, dx\, dy\, dz = \iiint_\Omega \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\, dy\, dz$$

U našem slučaju $\Phi = \iiint_\Omega (x+y+z)\, dx\, dy\, dz$ gdje je $\Omega: \begin{cases} x^2+y^2+z^2 \le 1 \\ x \ge 0,\ y \ge 0 \\ z \ge 0 \end{cases}$

Uvodimo sferne koordinate



$x = r\sin\varphi\cos\theta$
$y = r\sin\varphi\sin\theta$
$z = r\cos\varphi$

$$\Rightarrow \Omega': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le \pi/2 \end{cases}$$

$dx\, dy\, dz = r^2 \sin\varphi\, d\varphi\, dr\, d\theta$

$$\Phi = \iiint_{\Omega'} (r\sin\varphi\cos\theta + \ldots)\, r^2 \sin\varphi \ldots = \ldots = \frac{3\pi}{16}$$

(#) Izračunati tok (fluks) vektora $\vec{v} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ kroz sferu $x^2+y^2+z^2=R^2$.

Rj. $\vec{v} = (v_x, v_y, v_z) = (x^3, y^3, z^3)$

Tok vektorskog polja (kroz površ S) je površinski integral

$$\phi = \iint_S v_x\,dy\,dz + v_y\,dx\,dz + v_z\,dx\,dy$$

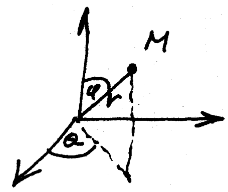
Kako je data zatvorena površina S to možemo upotrebiti formulu Gauss-Ostrogradski

$$\iint_S v_x\,dy\,dz + v_y\,dx\,dz + v_z\,dx\,dy = \iiint_\Omega \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\,dy\,dz$$

$\frac{\partial v_x}{\partial x} = 3x^2$, $\frac{\partial v_y}{\partial y} = 3y^2$, $\frac{\partial v_z}{\partial z} = 3z^2$, Ω oblast ograničena sferom $x^2+y^2+z^2=R^2$

$$\phi = \iiint_\Omega 3(x^2+y^2+z^2)\,dx\,dy\,dz \overset{(\Delta)}{=}$$

uvedimo sferne koordinate



$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi$

$x^2+y^2+z^2 = r^2[\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi] = r^2$

$$\Omega': \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\overset{(\Delta)}{=} 3\iiint_{\Omega'} r^2 r^2 \sin\varphi\,dr\,d\varphi\,d\theta =$$

$$= 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,d\varphi \int_0^R r^4\,dr = 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,\frac{1}{5}r^5\Big|_0^R d\varphi = 3\frac{R^5}{5}\int_0^{2\pi}(-\cos\varphi)\Big|_0^{\pi} d\theta$$

$$= \frac{6R^5}{5}\pi\,\theta\Big|_0^{2\pi} = \frac{12R^5}{5}\pi$$ traženi tok vektora kroz sferu

Izračunati cirkulaciju polja $\vec{v} = x\vec{i} + y\vec{j} + (x+y-1)\vec{k}$ duž odsječka prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

Rj. Cirkulacija vektorskog polja $\vec{v}$ ($\vec{v}=(v_x, v_y, v_z)$) duž krive c je integral

$$C = \int_c v_x\,dx + v_y\,dy + v_z\,dz .$$

U našem slučaju $\vec{v} = (x,\ y,\ x+y-1)$, dok je c dio prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

Imamo krivolinijski integral druge vrste

$$C = \int_c x\,dx + y\,dy + (x+y-1)\,dz$$

$A(1,1,1)$
$B(2,3,4)$

Kako glasi jednačina prave kroz dvije tačke u prostoru?

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3} \qquad (=t)$$

Napišimo pravu u parametarskom obliku:

$x = t+1$
$y = 2t+1$
$z = 3t+1$

Dio prave između tačke $A(1,1,1)$ i $B(2,3,4)$ je za $t \in [0,1]$.

$dx = dt, \quad dy = 2\,dt, \quad dz = 3\,dt$

$$C = \int_0^1 (t+1)\,dt + (2t+1)\,2\,dt + (3t+1)\,3\,dt = \int_0^1 (t+1+4t+2+9t+3)\,dt$$

$$= \int_0^1 (14t+6)\,dt = 14\cdot\frac{1}{2}t^2\Big|_0^1 + 6t\Big|_0^1 = 7+6 = 13$$

vrijednost cirkulacije polja