Vektorska teorija polja

Skalarno polje je f-ja $u = f(T) = f(x, y, z)$ u oblasti prostora ili na površi (na primjer, temperatura u svakoj tački prostora, nadmorska visina tačke i dr.) Skalarno polje se predstavlja nivoskim površima tj. površima s jednačinom $u = c \cdot f(T) = c \cdot f(x,y,z)$ (gdje je c-konstanta) i u ima neprekidne parcijalne izvode koji se ne anuliraju istovremeno).

Na primjer $u = x^2 + y^2 + z^2$ je skalarno polje.

Ranije smo spomenuli da je gradijent f-je $u = f(x,y,z)$, date u nekoj oblasti prostora, vektor čije su projekcije na ose Dekartovog koordinatnog sistema $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ i $\frac{\partial u}{\partial z}$. Označava se simbolom

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

Izvod u pravcu gradijenta u datoj tački dostiže najveću vrijednost jednaku $|\operatorname{grad} u| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}$ tj. pravac gradijenta je pravac najbržeg rasta f-je.

Vektorsko polje je oblast prostora u čijoj je svakoj tački definisan vektor.

$$\vec{v} = (V_x, V_y, V_z) = V_x \vec{i} + V_y \vec{j} + V_z \vec{k} \quad \text{gdje su } V_x, V_y, V_z \text{ skalarna polja.}$$

Na primjer $\vec{v} = (y^2 + z^2)\vec{i} + x^2 \vec{j} + xyz^2 \vec{k}$ je vektorsko polje.

Nabla operator (∇ operator ili Hamiltonov operator) je diferencijalni operator oblika $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ gdje su $\vec{i}, \vec{j}, \vec{k}$ jedinični ortogonalni vektori.

Ako je $u = f(x, y, z)$ skalarna f-ja biće

$$\nabla \cdot f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \operatorname{grad} f$$

Ako je $\vec{v} = (V_x, V_y, V_z)$ vektorska f-ja onda je $\nabla \vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Važne osobine vektorskog polja su divergencija i rotor vektorskog polja

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$ (skalarni proizvod ∇ i $\vec{v}$)

$$\operatorname{rot} \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$ (vektorski proizvod ∇ i $\vec{v}$)

Ako je $\operatorname{div}\vec{v} = 0$ tada kažemo da je $\vec{v}$ solenoidno polje.
Ako je $\operatorname{rot}\vec{v} = \vec{0}$ tada kažemo da je $\vec{v}$ potencijalno polje.
F-ju u za koju vrijedi da je $\vec{v} = \operatorname{grad} u$ zovemo potencijalom polja $\vec{v}$.

$$\boxed{\text{relacija } u(x,y,z) = C \text{ gdje je } C \text{ konstanta, predstavlja površ koju zovemo ekviskalarna površ (nivo površ) skalarnog polja}}$$

(#) Nađi veličinu i pravac gradijenta skalarnog polja: a) $u = x^2+y^2+z^2$ u tački $T(2,-2,1)$

b) $u = xyz$ u tački $T(1,2,3)$.

Rj. a)

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\operatorname{grad} u = (2x, 2y, 2z) \Rightarrow \operatorname{grad} u(T) = (4,-4,2)$$

$$|\operatorname{grad} u| = \sqrt{16+16+4} = 6$$ veličina gradijenta

$$\frac{\operatorname{grad} u(T)}{|\operatorname{grad} u(T)|} = \left(\frac{4}{6}, -\frac{4}{6}, \frac{2}{6}\right) = \left(\underbrace{\frac{2}{3}}_{\cos\alpha}, \underbrace{-\frac{2}{3}}_{\cos\beta}, \underbrace{\frac{1}{3}}_{\cos\gamma}\right)$$ jedinični vektor pravca gradijenta

$\alpha = \arccos\frac{2}{3}$

$\beta = \arccos\left(-\frac{2}{3}\right)$

$\gamma = \arccos\frac{1}{3}$

b) $|\operatorname{grad} u(T)| = 7$ $\qquad \dfrac{\operatorname{grad} u(T)}{|\operatorname{grad} u(T)|} = \left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$

(#) Dato je skalarno polje $u=x^3+y^3+z^3-3xyz$. U kojim tačkama je a) $\operatorname{grad} u=\vec{0}$

b) $\vec{k}\cdot \operatorname{grad} u = 0$.

Rj.

a) $\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$

$\operatorname{grad} u = (3x^2-3yz,\ 3y^2-3xz,\ 3z^2-3xy)$

$$\operatorname{grad} u=\vec{0} \Rightarrow \begin{array}{l} 3x^2-3yz=0 \\ 3y^2-3xz=0 \\ 3z^2-3xy=0 \end{array} \qquad \begin{array}{ll} x^2-yz=0 & (I) \\ y^2-xz=0 & (II) \\ z^2-xy=0 & (III) \end{array}$$

Trivijalno rješenje sistema je $x=0$, $y=0$, $z=0$.
Ako pomnožimo (I) sa x ($x\neq 0$), (II) sa y ($y\neq 0$) i (III) sa z ($z\neq 0$) dobijamo

$$\begin{array}{l} x^3-xyz=0 \\ y^3-xyz=0 \\ z^3-xyz=0 \end{array} \qquad \begin{array}{l} xyz=x^3 \\ xyz=y^3 \\ xyz=z^3 \end{array} \qquad \begin{array}{c} x^3=y^3=z^3 \\ x=y=z \end{array}$$

Ako ovu zadnju jednakost napišemo u obliku $\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-0}{1}$ (prava u prostoru) vidimo da je $\operatorname{grad} u=\vec{0}$ za sve tačke ove prave.

b) $\operatorname{grad} u = (3x^2-3yz)\vec{i}+(3y^2-3xz)\vec{j}+(3z^2-3xy)\vec{k}$

$\vec{k}\cdot\operatorname{grad} u = 3z^2-3xy=0$

$\vec{k}\cdot\operatorname{grad} u=0$ je za sve tačke krive $z^2-xy=0$

(#) Odrediti ugao kojeg zatvaraju gradijenti polja $z=\sqrt{x^2+y^2}$ i $u=x-3y+\sqrt{3xy}$ u tački $A(3,4)$.

Rj.

Gradijent f-je $z=f(x,y)$ se računa po formuli

$$\operatorname{grad} z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right).$$

$$\operatorname{grad} z=\left(\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x,\ \frac{1}{2\sqrt{x^2+y^2}}\cdot 2y\right)=\left(\frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$u=x-3y+\sqrt{3xy},\quad \frac{\partial u}{\partial x}=1+\frac{1}{2\sqrt{3xy}}\cdot 3y=1+\frac{3y}{2\sqrt{3xy}}$$

$$\frac{\partial u}{\partial y}=-3+\frac{3x}{2\sqrt{3xy}}$$

$$\operatorname{grad} u=\left(1+\frac{3y}{2\sqrt{3xy}},\ -3+\frac{3x}{2\sqrt{3xy}}\right).$$

$$A(3,4),\quad \operatorname{grad} z(A)=\left(\frac{3}{\sqrt{9+16}},\ \frac{4}{\sqrt{9+16}}\right)=\left(\frac{3}{5},\frac{4}{5}\right)=\frac{3}{5}\vec{i}+\frac{4}{5}\vec{j},$$

$$\operatorname{grad} u(A)=\left(1+\frac{12}{2\sqrt{36}},\ -3+\frac{9}{2\sqrt{36}}\right)=\left(1+1,\ -3+\frac{\cancel{9}^{\,3}}{\cancel{12}_{\,4}}\right)=$$

$$=\left(2,-\frac{9}{4}\right)=2\vec{i}-\frac{9}{4}\vec{j}$$

$$\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cdot\cos\varphi\quad\Rightarrow\quad \cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}.$$

U našem slučaju $\vec{a}=\left(\frac{3}{5},\frac{4}{5}\right)$, $\vec{b}=\left(2,-\frac{9}{4}\right)$:.

$$\vec{a}\cdot\vec{b}=\frac{3}{5}\cdot 2+\frac{4}{5}\cdot\left(-\frac{9}{4}\right)=\frac{6\cdot 4}{5\cdot 4}-\frac{36}{20}=\frac{24-36}{20}=\frac{-12}{20}=\frac{-3}{5}$$

$$|\vec{a}|=\sqrt{\frac{9}{25}+\frac{16}{25}}=1,\quad |\vec{b}|=\sqrt{4+\frac{81}{16}}=\sqrt{\frac{64+81}{16}}=\frac{\sqrt{145}}{4}$$

$$\cos\varphi=\frac{-\frac{3}{5}}{1\cdot\frac{\sqrt{145}}{4}}=\frac{-12}{\sqrt{145}}\quad\Rightarrow\quad \varphi=\arccos\left(\frac{-12}{\sqrt{145}}\right)$$ ugao kojeg zatvaraju gradijenti polja

(#) Odrediti divergenciju i rotor vektorskih polja

a) $\vec{v} = (y^2+z^2)\vec{i} + (z^2+x^2)\vec{j} + (x^2+y^2)\vec{k}$

b) $\vec{v} = x^2yz\,\vec{i} + xy^2z\,\vec{j} + xyz^2\,\vec{k}$

Rj. a) $\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$, $(\vec{v} = (V_x, V_y, V_z))$

$V_x = y^2 + z^2$ $\quad \frac{\partial V_x}{\partial x} = 0$

$V_y = z^2 + x^2$ $\quad \frac{\partial V_y}{\partial y} = 0$ $\quad \frac{\partial V_z}{\partial z} = 0$ $\quad \operatorname{div}\vec{v} = 0+0+0 = 0$

$V_z = x^2 + y^2$

divergencija vektorskog polja

Kako je $\operatorname{div}\vec{v} = 0$ to je polje $\vec{v}$ solenoidno

$$\operatorname{rot}\vec{v} = \nabla\times\vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(V_x, V_y, V_z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$\frac{\partial V_z}{\partial y} = 2y$ $\quad \frac{\partial V_y}{\partial z} = 2z$

$\frac{\partial V_z}{\partial x} = 2x$ $\quad \frac{\partial V_x}{\partial z} = 2z$ $\quad \operatorname{rot}\vec{v} = (2y-2z)\vec{i} - (2x-2z)\vec{j}$

$\frac{\partial V_y}{\partial x} = 2x$ $\quad \frac{\partial V_x}{\partial y} = 2y$ $\quad + (2x-2y)\vec{k} =$

$= (2y-2z,\ 2z-2x,\ 2x-2y)$

rotor vektorskog polja

Kako je $\operatorname{rot}\vec{v} \neq 0$ to polje nije potencijalno polje.

b) URADITI ZA VJEŽBU

Rj. $\operatorname{div}\vec{v} = 6xyz$

$\operatorname{rot}\vec{v} = (yx^2 - yz^2)\vec{j} + (zy^2 - zx^2)\vec{k} + (xz^2 - xy^2)\vec{i}$.

(#) Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2+z^2)\vec{i} + 2y(x^2+z^2)\vec{j} + 2z(x^2+y^2)\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $\text{rot}\,\vec{v} = \vec{0}$. Rotor vektorskog polja $\text{rot}\,\vec{v}$ se računa . .

$$\text{rot}\,\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$$V_x = 2x(y^2+z^2) \qquad V_y = 2y(x^2+z^2) \qquad V_z = 2z(x^2+y^2)$$

$$\frac{\partial V_x}{\partial y} = 4xy \qquad \frac{\partial V_y}{\partial x} = 4xy \qquad \frac{\partial V_z}{\partial x} = 4xz$$

$$\frac{\partial V_x}{\partial z} = 4xz \qquad \frac{\partial V_y}{\partial z} = 4yz \qquad \frac{\partial V_z}{\partial y} = 4yz$$

$$\text{rot}\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \vec{i}(4yz-4yz) - \vec{j}(4xz-4xz) + \vec{k}(4xy-4xy)$$

$$= (0,0,0) = \vec{0}$$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v} = \text{grad}\,u$.

$$\text{grad}\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x(y^2+z^2) \\ \frac{\partial u}{\partial y} &= 2y(x^2+z^2) \\ \frac{\partial u}{\partial z} &= 2z(x^2+y^2) \end{aligned}\right\} \ldots (1)$$

$$u = u(x,y,z)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$u = \int 2x(y^2+z^2)\,dx + \varphi(y,z)$$

$$u = x^2(y^2+z^2) + \varphi(y,z)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= 2yx^2 + \varphi'_y \\ \frac{\partial u}{\partial z} &= 2x^2z + \varphi'_z \end{aligned}\right\} \ldots (2)$$

$$(1) \text{ i } (2) \Rightarrow \quad \varphi'_y = 2yz^2, \quad \varphi'_z = 2zy^2 \ldots (*)$$

Odredimo f-ju φ: $\varphi = \int 2yz^2\,dy + \psi(z)$

$$\varphi = y^2z^2 + \psi(z)$$

$$\varphi' = 2y^2z + \psi' \ldots (**)$$

$$(*) \text{ i } (**) \Rightarrow \psi'_z = 0 \Rightarrow \psi(z) = C$$

$$\Rightarrow \varphi = y^2z^2 + C \Rightarrow u = x^2y^2 + x^2z^2 + y^2z^2 + C$$

Potencijal vektorskog polja je $u = x^2y^2 + x^2z^2 + y^2z^2 + C$

(#) Odrediti konstante a, b i c tako da vektorsko polje $\vec{v} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ bude potencijalno i naći njegov potencijal.

Rj. Ako je $\operatorname{rot}\vec{v} = \vec{0}$ tada je vektorsko polje $\vec{v}$ potencijalno.

$$\operatorname{rot}\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \qquad \begin{aligned} V_x &= x+2y+az \\ V_y &= bx-3y-z \\ V_z &= 4x+cy+2z \end{aligned}$$

$$\frac{\partial V_z}{\partial y} = c \quad \frac{\partial V_y}{\partial z} = -1 \qquad \frac{\partial V_y}{\partial x} = b$$

$$\frac{\partial V_z}{\partial x} = 4 \quad \frac{\partial V_x}{\partial z} = a \qquad \frac{\partial V_x}{\partial y} = 2$$

$$\operatorname{rot}\vec{v} = (c+1)\vec{i} - (4-a)\vec{j} + (b-2)\vec{k} = (c+1,\ a-4,\ b-2)$$

Za vrijednosti $a=4$, $b=2$ i $c=-1$ vektorsko polje $\vec{v}$ je potencijalno polje.

$\vec{v} = (x+2y+4z,\ 2x-3y-z,\ 4x-y+2z)$

Potencijal polja $\vec{v}$ je f-ja u koja zavisi od 3 promjenjive $u = u(x,y,z)$ i za koju vrijedi $\vec{v} = \operatorname{grad} u$.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Nađimo f-ju u.

$$\frac{\partial u}{\partial x} = x+2y+4z$$

$$\frac{\partial u}{\partial y} = 2x-3y-z \qquad \frac{\partial u}{\partial z} = 4x-y+2z \quad \ldots(*)$$

$$u = \int (x+2y+4z)\,dx + \varphi(y,z)$$

$$u = \frac{1}{2}x^2 + (2y+4z)x + \varphi(y,z) \Rightarrow \frac{\partial u}{\partial y} = 2x + \varphi'_y, \quad \frac{\partial u}{\partial z} = 4x + \varphi'_z \stackrel{(*)}{\Longrightarrow}$$

$$\stackrel{(*)}{\Longrightarrow} \varphi'_y = -3y-z \quad ; \quad \varphi'_z = -y+2z \quad \ldots(**)$$

Odredimo f-ju φ.

$$\varphi = \int(-3y-z)\,dy + \psi(z) = -\frac{3}{2}y^2 - yz + \psi(z)$$

$$\varphi'_z = -y + \psi'_z \stackrel{(**)}{\Longrightarrow} \psi'_z = 2z \Rightarrow \psi(z) = \int 2z\,dz = z^2 + C$$

$$\varphi(y,z) = -\frac{3}{2}y^2 - yz + z^2 + C \Rightarrow u = \frac{1}{2}x^2 + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2 + C$$