Diferenciranje pod znakom integrala

Neka je dat integral koji zavisi od parametra α:

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\, dx$$

Ako su $f(x,\alpha)$, $f'(x,\alpha)$ neprekidne f-je, ako postoje $b'(\alpha)$ i $a'(\alpha)$ tada

$$I'(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f'_{\alpha}(x,\alpha)\, dx + b'(\alpha)\, f(b(\alpha),\alpha) - a'(\alpha)\cdot f(a(\alpha),\alpha)$$

Ako granice a i b ne zavise od α tada

$$I'(\alpha) = \int_a^b f'_{\alpha}(x,\alpha)\, dx$$

(#) Polazeći od integrala $\int_0^b \frac{dx}{1+\alpha x}$ izračunati

$$\int_0^b \frac{x\,dx}{(1+\alpha x)^2} \quad ; \quad \int_0^b \frac{x^2\,dx}{(1+\alpha x)^3} .$$

Rj.

$$I(\alpha)=\int_0^b \frac{dx}{1+\alpha x} = \left| \begin{matrix} 1+\alpha x = t & x=0 \Rightarrow t=1 \\ \alpha dx = dt & x=b \Rightarrow t=1+\alpha b \\ dx = \frac{1}{\alpha} dt & \end{matrix} \right| =$$

$$= \frac{1}{\alpha}\int_1^{1+\alpha b} \frac{dt}{t} = \frac{1}{\alpha} \ln|t| \Big|_1^{1+\alpha b} = \frac{1}{\alpha} \ln|1+\alpha b|$$

$$f'_\alpha(x,\alpha) = \left(\frac{1}{1+\alpha x}\right)'_\alpha = (-1)(1+\alpha x)^{-2}\cdot x = \frac{-x}{(1+\alpha x)^2}$$

$$I'(\alpha) = \int_a^b f'_\alpha(x,\alpha)\,dx \quad \Rightarrow \quad I'(\alpha) = \int_0^b \frac{-x}{(1+\alpha x)^2}\,dx \quad \Rightarrow$$

$$\Rightarrow \quad \int_0^b \frac{x\,dx}{(1+\alpha x)^2} = -I'(\alpha)$$

Kako je $I(\alpha) = \frac{1}{\alpha}\ln|1+\alpha b|$ to je $I'_\alpha = -\frac{1}{\alpha^2}\ln|1+\alpha b| + \frac{1}{\alpha}\cdot\frac{1}{1+\alpha b}\cdot b$

Prema tome

$$\int_0^b \frac{x\,dx}{(1+\alpha x)^2} = \frac{1}{\alpha^2}\ln|1+\alpha b| - \frac{b}{\alpha(1+\alpha b)}$$

Slično bi imali $I''(\alpha) = \int_0^b \left(\frac{-x}{(1+\alpha x)^2}\right)'_\alpha dx = \int_0^b \frac{2x^2}{(1+\alpha x)^3}\,dx \Rightarrow$

$$\Rightarrow \int_0^b \frac{x^2}{(1+\alpha x)^3}\,dx = \frac{1}{2} I''(\alpha), \quad I''_\alpha = (I'_\alpha)' = \frac{2}{\alpha^3}\ln|1+\alpha b| + \left(-\frac{1}{\alpha^2}\right)\frac{b}{1+\alpha b}$$

$$- \frac{b}{\alpha^2(1+\alpha b)} + \frac{1}{\alpha}\cdot\frac{-b}{(1+\alpha b)^2}\cdot b \quad \Rightarrow \quad \int_0^b \frac{x^2}{(1+\alpha x)^3}\,dx = \frac{1}{\alpha^3}\ln|1+\alpha b| - \frac{b}{\alpha^2(1+\alpha b)}$$

$$- \frac{b^2}{\alpha(1+\alpha b)^2}$$ traženo rješenje

(#) Izračunati $I(\alpha)=\int_0^\infty \frac{1-e^{-\alpha x}}{xe^x}dx$ ako je $\alpha>-1$.

Rj.

$$I'(\alpha)=\int_a^b f'_\alpha(\alpha,x)\,dx$$

$$f(\alpha,x)=\frac{1-e^{-\alpha x}}{x\,e^x}\ ,\quad f'_\alpha=\frac{-e^{-\alpha x}\cdot(-x)}{xe^x}$$

$$I'(\alpha)=\int_0^\infty \frac{x\,e^{-\alpha x}}{xe^x}dx=\int_0^\infty e^{-\alpha x-x}dx=\int_0^\infty e^{-(\alpha+1)x}dx=\left|\begin{array}{l}-(\alpha+1)x=s\\ -(\alpha+1)dx=ds\\ dx=-\dfrac{ds}{\alpha+1}\\ x=0\Rightarrow s=0\\ x=\infty\Rightarrow s=-\infty\end{array}\right|=-\frac{1}{\alpha+1}\int_0^{-\infty}e^s ds=\frac{-1}{\alpha+1}\,e^s\Big|_0^{-\infty}=0-\frac{(-1)}{\alpha+1}=\frac{1}{\alpha+1}$$

$$I'_\alpha=\frac{1}{\alpha+1}\quad\Rightarrow\quad I(\alpha)=\int\frac{1}{\alpha+1}d\alpha=\ln|\alpha+1|+C$$

Kako je $I(0)=\int_0^\infty\frac{1-e^0}{xe^x}dx=0$ to je $I(0)=\ln 1+C=0$

$\Rightarrow\ C=0$

$I(\alpha)=\ln|\alpha+1|$ traženo rješenje

(#) za vježbu

Izračunati $I(\alpha)=\int_0^\infty\frac{1-e^{-\alpha x^2}}{xe^{x^2}}dx$, ako je $\alpha>-1$.

(#) za vježbu

Izračunati $\int_0^{\pi/2}\frac{\mathrm{arctg}(\alpha\,\mathrm{tg}x)}{\mathrm{tg}x}dx$

rješenje: $I(\alpha)=\frac{\pi}{2}\ln|1+\alpha|$.

(#) Izračunati
$$\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}\,dx .$$

Rj:
$$I'(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx, \qquad I(\alpha)=\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}\,dx$$

$$f(\alpha,x)=e^{-x}\frac{\sin\alpha x}{x}, \qquad f'_\alpha=\frac{e^{-x}}{x}\cdot x\cos\alpha x=e^{-x}\cos\alpha x$$

$$I'(\alpha)=\int_0^\infty e^{-x}\cos\alpha x\,dx=\lim_{R\to\infty}\int_0^R e^{-x}\cos\alpha x\,dx=$$

$$=\left|\begin{matrix} u=e^{-x} & dv=\cos\alpha x\,dx \\ du=-e^{-x}dx & v=\frac{1}{\alpha}\sin\alpha x\end{matrix}\right|=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-x}\sin\alpha x\Big|_0^R+\frac{1}{\alpha}\int_0^R e^{-x}\sin\alpha x\,dx\right)$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R+\frac{1}{\alpha}\int_0^R e^{-x}\sin\alpha x\,dx\right)=\left|\begin{matrix} u=e^{-x} & dv=\sin\alpha x\,dx \\ du=-e^{-x}dx & v=-\frac{1}{\alpha}\cos\alpha x\end{matrix}\right|=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R+\frac{1}{\alpha}\left(-\frac{1}{\alpha}\underbrace{e^{-x}\cos\alpha x\Big|_0^R}_{(e^{-R}\cos\alpha R-e^0\cdot 1)}-\frac{1}{\alpha}\int_0^R e^{-x}\cos\alpha x\,dx\right)\right)=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R-\frac{1}{\alpha^2}e^{-R}\cos\alpha R+\frac{1}{\alpha^2}-\frac{1}{\alpha^2}\int_0^R e^{-x}\cos\alpha x\,dx\right)=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\underbrace{\sin\alpha R}_{\text{ovo je između }-1\text{ i }1}-\frac{1}{\alpha^2}e^{-R}\underbrace{\cos\alpha R}_{\text{uzima vrijednosti između }-1\text{ i }1}+\frac{1}{\alpha^2}\right)-\frac{1}{\alpha^2}\underbrace{\lim_{R\to\infty}\int_0^R e^{-x}\cos\alpha x\,dx}_{I'(\alpha)}$$

Sad imamo
$$\left(1+\frac{1}{\alpha^2}\right)\int_0^\infty e^{-x}\cos\alpha x\,dx=\frac{1}{\alpha^2} \Rightarrow \int_0^\infty e^{-x}\cos\alpha x\,dx=\frac{\frac{1}{\alpha^2}}{\frac{\alpha^2+1}{\alpha^2}}=\frac{1}{\alpha^2+1}$$

Kako je $I'(\alpha)=\frac{1}{\alpha^2+1}$ to je $I(\alpha)=\int\frac{1}{\alpha^2+1}\,d\alpha=\operatorname{arctg}\alpha+C$

$I(0)=0=\operatorname{arctg}0+C \Rightarrow C=0$

Prema tome $\displaystyle\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}\,dx=\operatorname{arctg}\alpha$ traženo rješenje