Stoksova formula

Dat je krivoliniski integral $\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

gdje je c kontura u prostoru.
Stoksova formula glasi:

$$\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

površinski integral prve vrste

$$\int_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

gdje je S površina u prostoru ograničena konturom c a $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ jedinični vektor normale na površinu S.

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma$$

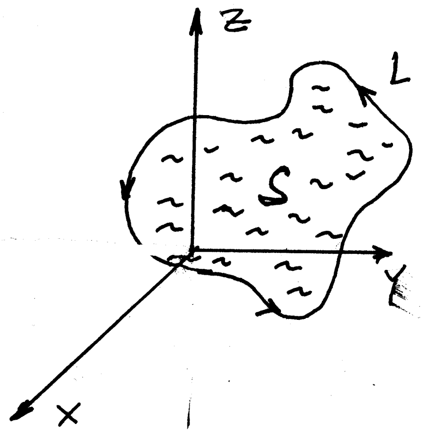
Vidimo da Stoksova formula povezuje krivoliniski integral druge vrste sa površinskim integralom prve i druge vrste.

Ranije smo spomenuli Greenovu formulu koja povezuje krivoliniski integral druge vrste sa dvostrukim integralom. Formula Gauss-Ostrogradski povezuje površinski integral druge vrste sa trostrukim integralom.

Integral $I=\int_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$

uzet po nekoj zatvorenoj konturi L, pretvoriti pomoću formule Stoksa u površinski integral, nad površinom koju zatvara spomenuta kontura.

Rj.



$$\int_L P\,dx+Q\,dy+R\,dz=\iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$R(x,y,z)=x^2+y^2$

$P(x,y,z)=y^2+z^2$

$Q(x,y,z)=x^2+z^2$

$\frac{\partial R}{\partial y}=2y \qquad \frac{\partial Q}{\partial z}=2z$

$\frac{\partial R}{\partial x}=2x \qquad \frac{\partial P}{\partial z}=2z$

$\frac{\partial Q}{\partial x}=2x \qquad \frac{\partial P}{\partial y}=2y$

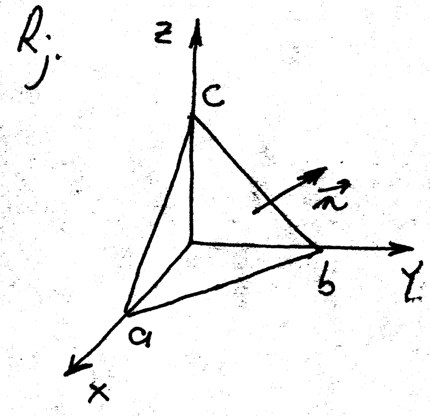
$$I=\iint_S (2y-2z)\,dy\,dz-(2x-2z)\,dx\,dz+(2x-2y)\,dx\,dy=$$

$$=2\iint_S (y-z)\,dy\,dz+(z-x)\,dx\,dz+(x-y)\,dx\,dy$$

(#) Izračunati krivoliniski integral $-\int\limits_c y^2dx+z^2dy+x^2dz$

pri čemu je c kontura ΔABC gdje su tačke $A(a,0,0)$, $B(0,b,0)$ i $C(0,0,c)$, $a,b,c>0$.

Rj.



$$-\int\limits_c y^2dx+z^2dy+x^2dz \xlongequal[\text{formula}]{\text{Stoksova}} -\iint\limits_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$P=y^2$, $Q=z^2$, $R=x^2$

$\frac{\partial Q}{\partial x}=0$, $\frac{\partial P}{\partial y}=2y$, $\frac{\partial R}{\partial y}=0$, $\frac{\partial Q}{\partial z}=2z$

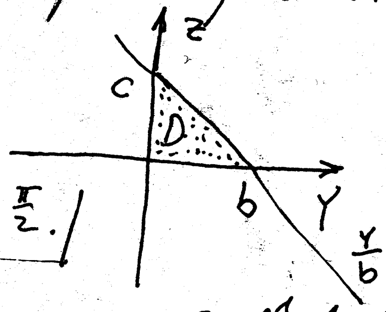
$\frac{\partial R}{\partial x}=2x$ $\quad \frac{\partial P}{\partial z}=0$

$$\begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = -2z\,dydz-2x\,dzdx-2y\,dxdy$$

$$-\int\limits_c y^2dx+z^2dy+x^2dz = 2\iint\limits_S (z\,dydz+x\,dzdx+y\,dxdy)$$

S oblast ograničena ΔABC

Izračunajmo $\iint\limits_S z\,dydz$. Površinu S proicirajmo na yOz ravan:



$\frac{y}{b}+\frac{z}{c}=1$

$cy+bz=bc$

$bz=bc-cy$

$z=c-\frac{c}{b}y=\frac{c}{b}(b-y)$

Ugao koji zaklapa vektor normale $\vec{n}$ na površ S je između $0 : \frac{\pi}{2}$.

$D: \begin{cases} 0\le y\le b \\ 0\le z\le \frac{c}{b}(b-y) \end{cases}$

$\vec{n}=(\frac{1}{a},\frac{1}{b},\frac{1}{c})$

$\cos\alpha>0$

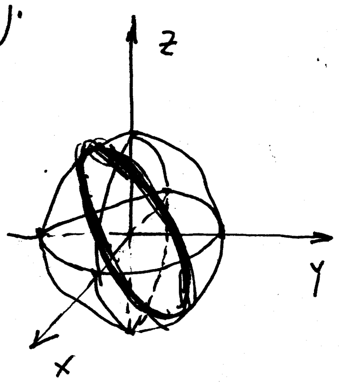
$$\iint\limits_S z\,dydz = +\iint\limits_D z\,dydz = \int\limits_0^b \Big[\int\limits_0^{\frac{c}{b}(b-y)} z\,dz\Big]dy = \int\limits_0^b \frac{1}{2}z^2\Big|_0^{\frac{c}{b}(b-y)}dy = \frac{1}{2}\Big(\frac{c}{b}\Big)^2\int\limits_0^b (b-y)^2dy$$

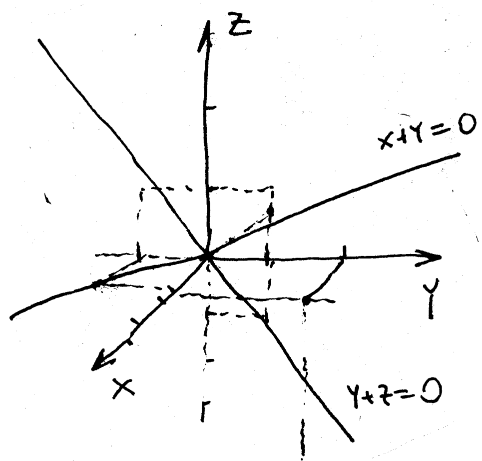
$$= \left| \begin{matrix} b-y=t & -dy=dt \\ y=0 \Rightarrow t=b & dy=-dt \\ y=b \Rightarrow t=0 & \end{matrix} \right| = \frac{1}{2}\frac{c^2}{b^2}\int\limits_0^b t^2dt = \frac{1}{2}\cdot\frac{c^2}{b^2}\frac{t^3}{3}\Big|_0^b = \frac{1}{2}\cdot\frac{bc^2}{3}$$

Analogno izračunamo $\iint\limits_S x\,dzdx=\frac{1}{2}\frac{a^2c}{3}$ i $\iint\limits_S y\,dxdy=\frac{1}{2}\cdot\frac{ab^2}{3} \Rightarrow I=\frac{ab^2+bc^2+ca^2}{3}$

(#) Izračunati krivoliniski integral $\int y\,dx + z\,dy + x\,dz$ ako je c krug dobijen presjekom c sfere $x^2+y^2+z^2=a^2$ i ravni $x+y+z=0$.

Rj.





$$\int_c y\,dx + z\,dy + x\,dz \overset{\text{Stoksova formula}}{=\!=} \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$$\frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial P}{\partial y} = 1 \qquad P = y,\ Q = z,\ R = x$$

S je površina ograničena krugom

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma$$

$$\frac{\partial R}{\partial y} = 0 \qquad \frac{\partial Q}{\partial z} = 1 \qquad \frac{\partial R}{\partial x} = 1 \qquad \frac{\partial P}{\partial z} = 0$$

$$\int_c y\,dx + z\,dy + x\,dz = \iint_S (-\cos\alpha - \cos\beta - \cos\gamma)\,dS$$

gdje je $\vec{n}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$ vektor (jedinični) normale na površinu S

$x+y+z=0$

$\vec{n} = (1,1,1)$ vektor normale na ravan $x+y+z=0$ (a time i na našu površinu S)

$|\vec{n}| = \sqrt{1+1+1} = \sqrt{3}$ $\qquad \vec{n}_0 = \left(\underbrace{\frac{1}{\sqrt{3}}}_{\cos\alpha}, \underbrace{\frac{1}{\sqrt{3}}}_{\cos\beta}, \underbrace{\frac{1}{\sqrt{3}}}_{\cos\gamma}\right)$

$$\iint_S (-\cos\alpha - \cos\beta - \cos\gamma)\,dS = \iint_S \left(-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) dS = -\frac{3}{\sqrt{3}} \iint_S dS$$

$\iint_S dS$ je površina oblasti S (S je krug poluprečnika a $P_{kruga} = a^2\pi$)

$$\int_c y\,dx + z\,dy + x\,dz = -\frac{3}{\sqrt{3}} a^2\pi = -\sqrt{3}\,a^2\pi$$

Formula Gauss-Ostrogradski

Ova formula daje vezu između površinskog integrala druge vrste i trostrukog integrala.

$$\iint P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy =$$

$$= \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S (S je zatvorena površina).

(1.) Izračunati $\iint\limits_{S} xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx$ gdje je S bilo koja zatvorena površ.

Rj.

$$\iint\limits_{S} yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iint\limits_{S} P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy \overset{\text{Formula Gauss-Ostr.}}{=} \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S.

$$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial Q}{\partial y} = 0, \quad \frac{\partial R}{\partial z} = 0$$

$$\iint\limits_{S} yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iiint\limits_{\Omega} 0\,dx\,dy\,dz =$$

$$\Omega : \begin{cases} a \le x \le b \\ c \le y \le d \\ e \le z \le f \end{cases} \qquad = \int_a^b dx \int_c^d dy \int_e^f 0\,dz = 0$$

(#) Površinski integral po zatvorenoj površini pretvoriti uz pomoć formule Ostrogradskog u trostruki integral po zapremini tijela, koje je ograničeno spomenutom površinom $\iint\limits_S \sqrt{x^2+y^2+z^2}\,[\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)]\,dS$

gdje je $\vec{n}$ vanjska normala na površinu S.

Rj. $\cos(\vec{n},x)$ je kosinus ugla između normale i x-ose.
$\cos(\vec{n},y)$ i $\cos(\vec{n},z)$ je kosinus ugla između normale na površinu S i y-ose i z-ose redom.

Uvedimo oznake $\cos(\vec{n},x)=\cos\alpha$, $\cos(\vec{n},y)=\cos\beta$ i $\cos(\vec{n},z)=\cos\gamma$.

Prema formuli Stoksa znamo da je
$$dy\,dz = dS\cos\alpha$$
$$dz\,dx = dS\cos\beta$$
$$dx\,dy = dS\cos\gamma$$

$$I=\iint\limits_S \sqrt{x^2+y^2+z^2}\,(\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)]\,dS =$$
$$= \iint\limits_S \sqrt{x^2+y^2+z^2}\,(dy\,dz+dz\,dx+dx\,dy)$$

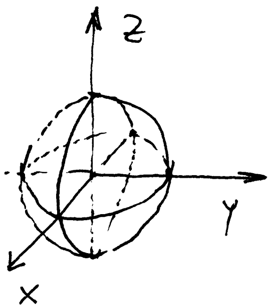
$$\iint\limits_S P\,dy\,dz+Q\,dx\,dz+R\,dx\,dy=\iiint\limits_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$\frac{\partial P}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}$$

$$I=\iiint\limits_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$$

(#) Izračunati $\iint\limits_S x^3 dydz + y^3 dzdx + z^3 dxdy$ gdje je S - vanjski dio sfere $x^2+y^2+z^2=R^2$.

Rj:



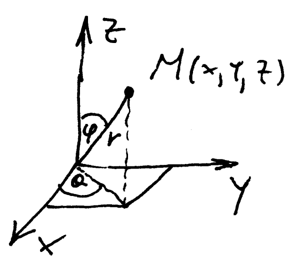
$$I=\iint\limits_S x^3 dydz + y^3 dzdx + z^3 dxdy \overset{\text{formula Gauss-Ostrogradski}}{=}$$

$$=\iiint\limits_\Omega \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right) dxdydz$$

$P=x^3$, $\frac{\partial P}{\partial x}=3x^2$, $Q=y^3$, $\frac{\partial Q}{\partial y}=3y^2$, $R=z^3$, $\frac{\partial R}{\partial z}=3z^2$

$$I=\iiint\limits_\Omega (3x^2+3y^2+3z^2)\,dxdydz \qquad \Omega: x^2+y^2+z^2\le R^2$$

Uvedimo sferne koordinate:



$x = r\sin\varphi\cos\alpha$

$y = r\sin\varphi\sin\alpha$

$z = r\cos\varphi$

$$\Omega': \begin{cases} 0\le r\le R \\ 0\le \alpha \le 2\pi \\ 0\le\varphi\le\pi \\ dxdydz = r^2\sin\varphi\, d\varphi d\alpha dr \end{cases}$$

$x^2+y^2+z^2=r^2$

$$I=3\iiint\limits_{\Omega'} r^2 r^2 \sin\varphi\, d\varphi d\alpha dr = 3\int_0^R r^4 dr \int_0^{2\pi} d\alpha \int_0^\pi \sin\varphi\, d\varphi =$$

$$=3\cdot\frac{1}{5}r^5\Big|_0^R\cdot\alpha\Big|_0^{2\pi}\cdot(-\cos\varphi)\Big|_0^\pi = \frac{3}{5}\cdot R^5\cdot 2\pi\cdot 2 = \frac{12}{5}R^5\pi$$

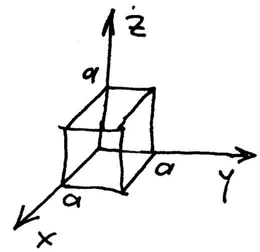
(#) Izračunati $\iint_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$ gdje je S - vanjska strana kocke $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$.

Rj.

formula Gauss-Ostr.

$$\iint_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$$\frac{\partial P}{\partial x} = 2x, \quad \frac{\partial Q}{\partial y} = 2y, \quad \frac{\partial R}{\partial z} = 2z$$



$$\Omega : \begin{cases} 0 \le x \le a \\ 0 \le y \le a \\ 0 \le z \le a \end{cases}$$

Prema tome:

$$\iint_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy =$$

$$= \iiint_\Omega (2x+2y+2z)\,dx\,dy\,dz = 2\int_0^a dx \int_0^a dy \int_0^a (x+y+z)\,dz =$$

$$= 2\int_0^a dx \int_0^a \left(xz\Big|_0^a + yz\Big|_0^a + \frac{1}{2}z^2\Big|_0^a \right) dy = 2\int_0^a dx \int_0^a \left(ax + ay + \frac{1}{2}a^2\right) dy =$$

$$2a\int_0^a \left(xy\Big|_0^a + \frac{1}{2}y^2\Big|_0^a + \frac{1}{2}ay\Big|_0^a \right) dx = 2a\int_0^a \left(ax + \frac{1}{2}a^2 + \frac{1}{2}a^2\right) dx = 2a^2\int_0^a (x+a)\,dx =$$

$$= 2a^2\left(\frac{1}{2}a^2 + a^2\right) = 3a^4$$