\# Izračunati integral po površi prve vrste

$$I=\iint_{\Gamma}\frac{d\sigma}{\sqrt{1+z}}$$

gde je Γ polusfera $x^2+y^2+z^2=a^2$, $z\geqslant 0$.

Rešenje. Kako je na Γ: $z=\sqrt{a^2-x^2-y^2}$, $p=-\dfrac{x}{z}$, $q=-\dfrac{y}{z}$,

$$1+p^2+q^2=\frac{a^2}{z^2}\Rightarrow d\sigma=\frac{2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy,$$ to je dati integral jednak dvojnom integralu

$$I=\iint_{G}\frac{a\,dx\,dy}{\sqrt{a^2-x^2-y^2}\cdot\sqrt{1+(a^2-x^2-y^2)^{1/2}}},\qquad G:\ x^2+y^2\leqslant a^2$$

ili, prelaskom na polarne koordinate

$$I=a\int_0^{2\pi}d\varphi\int_0^a\frac{\rho\,d\rho}{\sqrt{a^2-\rho^2}\cdot\sqrt{1+(a^2-\rho^2)^{1/2}}}=$$

$$=2a\pi\int_0^a\frac{t\ dt}{\sqrt{t^2(1+t)}}=\qquad\begin{cases}a^2-\rho^2=t^2\\ -\rho\,d\rho=t\,dt\end{cases}$$

$$=2a\pi\int_0^a\frac{dt}{\sqrt{1+t}}=4a\pi(\sqrt{1+a}-1).$$

Izračunati integral po površi

$$\iint\limits_{\Gamma} \frac{x\cos\alpha + y\cos\beta + z\cos\gamma}{\sqrt{x^2+y^2+z^2}}\, d\sigma .$$

gde su α, β, γ uglovi između spoljne normale površi i koordinatnih osa, a Γ površ $x^2+y^2+z^2=a^2$.

Rešenje. Dati integral se može napisati u obliku

$$I = \iint\limits_{\Gamma} \frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{\sqrt{x^2+y^2+z^2}}$$

ili

$$I = I_1 + I_2 + I_3,$$

gde je

$$I_1 = \iint\limits_{\Gamma} \frac{x\,dy\,dz}{\sqrt{x^2+y^2+z^2}}, \quad I_2 = \iint\limits_{\Gamma} \frac{y\,dz\,dx}{\sqrt{x^2+y^2+z^2}} \quad \text{i} \quad I_3 = \iint\limits_{\Gamma} \frac{z\,dx\,dy}{\sqrt{x^2+y^2+z^2}} .$$

Zbog potpune simetrije sva tri integrala imaju istu vrednost, pa je dovoljno izračunati samo jedan od njih, napr.

$$I_3 = \iint\limits_{\Gamma} = \iint\limits_{\Gamma_1+} + \iint\limits_{\Gamma_2-},$$

gde je

$$\Gamma_1: \quad z_1 = \sqrt{a^2-x^2-y^2}, \quad \Gamma_2: \quad z_2 = -\sqrt{a^2-x^2-y^2},$$

pa je

$$I_3 = \iint\limits_{G} \frac{z_1\,dx\,dy}{\sqrt{x^2+y^2+z_1^2}} - \iint\limits_{G} \frac{z_2\,dx\,dy}{\sqrt{x^2+y^2+z_2^2}} =$$

$$= \frac{2}{a} \iint\limits_{G} \sqrt{a^2-x^2-y^2}\, dx\,dy; \qquad G:\ x^2+y^2 \leqslant a^2$$

$$I_3 = \frac{4a^2\pi}{3} .$$ Stoga je

$$I = 4a^2\pi .$$

\# Izračunati integral

$$\iint\limits_{G} 2\,dxdy + y\,dzdx - x^2 z\,dydz$$

gde je Γ spoljna strana dela elipsoida $4x^2+y^2+4z^2=1$ koji pripada prvom oktantu.

Rešenje.

$$I = I_1 + I_2 + I_3,$$

gde je

$$I_1 = \iint\limits_{\Gamma} 2\,dx\,dy, \qquad I_2 = \iint\limits_{\Gamma} y\,dz\,dx, \qquad I_3 = \iint\limits_{\Gamma} -x^2 z\,dy\,dz.$$

$$I_1 = \iint\limits_{\Gamma+} 2\,dx\,dy = 2\iint\limits_{G} dx\,dy = 2\,m(G) = 2\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot 1\cdot\pi = \frac{\pi}{4}; \quad G: \frac{x^2}{\frac{1}{4}} + y^2 \leqslant 1.$$

$$I_2 = \iint\limits_{\Gamma+} y\,dz\,dx = \iint\limits_{G} \sqrt{1-4x^2-4z^2}\,dx\,dz = \qquad G:\ x^2+y^2 \leqslant \frac{1}{4}$$

$$= \int_0^{\pi/2} d\varphi \int_0^{1/2} \sqrt{1-4\rho^2}\,\rho\,d\rho = \frac{\pi}{24}.$$

$$I_3 = -\iint\limits_{\Gamma+} x^2 z\,dy\,dz =$$

$$= -\frac{1}{4}\iint\limits_{G} z(1-y^2-4z^2)\,dy\,dz =$$

$$= -\frac{1}{4}\int_0^{\pi/2} d\varphi \int_0^1 \frac{1}{2}\rho\sin\varphi\,(1-\rho^2)\cdot\frac{1}{2}\rho\,d\rho =$$

$$= -\frac{1}{120}.$$

$G:\ y^2+4z^2 \leqslant 1$

$y = \rho\cos\varphi$

$z = \frac{1}{2}\rho\sin\varphi$

$J = \frac{1}{2}\rho$

Konačno je

$$I = \frac{7\pi}{24} - \frac{1}{120}.$$

Izračunati integral

$$\iint_{\Gamma} x^2\,dydz + y^2\,dzdx + z^2\,dxdy$$

gde je Γ spoljna strana dela površi hiperboloida

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad 0 \leqslant z \leqslant c.$$

Rešenje. Zadatak se može rešiti na dva načina: direktno, izračunavanjem datog integrala po površi; ili, tako, što bi se površ integracije dopunila elipsama u ravni $z=0$ i $z=c$, pa za izračunavanje integrala po zatvorenoj površi primenila formula Ostrogradskog. Rešimo zadatak na oda načina.

I način. $I = I_1 + I_2 + I_3$,

$$I_1 = \iint_{\Gamma} x^2 dy\,dz = \iint_{\Gamma_1 +} + \iint_{\Gamma_2 -} \qquad \Gamma_1; \quad x_1 = a\sqrt{1 + \frac{z^2}{c^2} - \frac{y^2}{b^2}}$$

$$= + \iint_{G} x_1^2\,dy\,dz - \iint_{G} x_2^2\,dy\,dz = 0. \qquad \Gamma_2: \quad x_2 = -a\sqrt{1 + \frac{z^2}{c^2} - \frac{y^2}{b^2}}$$

Slično je

$$I_2 = \iint_{\Gamma} y^2\,dz\,dx = 0.$$

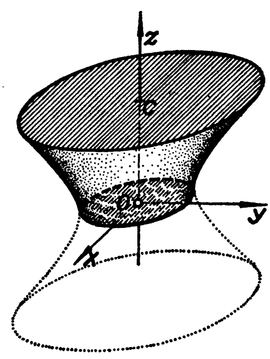
Ostaje da se izračuna

$$I_3 = \iint_{\Gamma^-} z^2\,dx\,dy =$$

$$= -c^2 \iint_{G} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dx\,dy =$$

$$= -c^2 \int_0^{2\pi} d\varphi \int_1^{\sqrt{2}} (\rho^2 - 1)\; ab\,\rho\,d\rho =$$

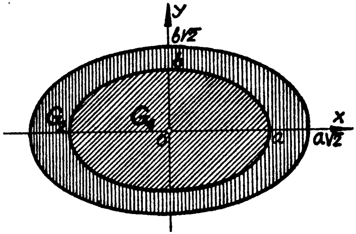
$$= -2\,abc^2\pi \cdot \frac{1}{4} = -\frac{1}{2}\,abc^2\pi.$$



II način. Neka je $\Gamma_3=\Gamma_1\cup\Gamma_2\cup\Gamma$

gde je Γ_1 elipsa u ravni $z=0$ $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\leqslant 1$, a Γ_2 elipsa u ravni $z=c$ $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\leqslant 2$.

Tada je



$$\oiint\limits_{\Gamma_3}=\iint\limits_{\Gamma_1}+\iint\limits_{\Gamma_2}+\iint\limits_{\Gamma}$$

$$\oiint\limits_{\Gamma_3}=\iiint\limits_{\Phi}2\,(x+y+z)\,dx\,dy\,dz$$

$$=2\left[\iint\limits_{G_1}\left[(x+y)\,z+\frac{z^2}{2}\right]\Bigg|_0^c dx\,dy+\iint\limits_{G_2}\left[(x+y)\,z+\frac{z^2}{2}\right]\Bigg|_{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1}}^{c} dx\,dy\right]=$$

$$=2\iint\limits_{G_1\cup G_2}\left[c\,(x+y)+\frac{c^2}{2}\right]dx\,dy-$$

$$-2\iint\limits_{G_2}\left[c\,(x+y)\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1}+\frac{c^2}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]dx\,dy=$$

$$\frac{x}{a}=\rho\cos\varphi \qquad \frac{y}{b}=\rho\sin\varphi \qquad J=ab\,\rho$$

$$=2\int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}\left[c\rho\,(a\cos\varphi+b\sin\varphi)+\frac{c^2}{2}\right]ab\,\rho\,d\rho-$$

$$-2ab\int_0^{2\pi}d\varphi\int_1^{\sqrt{2}}\left[\rho\,(a\cos\varphi+b\sin\varphi)\,c\sqrt{\rho^2-1}+\frac{c^2}{2}(\rho^2-1)\right]\rho\,d\rho=$$

$$=2\,abc\int_0^{2\pi}(a\cos\varphi+b\sin\varphi)\,d\varphi\int_0^{\sqrt{2}}\rho^2\,d\rho+$$

$$+abc^2\int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}\rho\,d\rho-2\,abc\int_0^{2\pi}(a\cos\varphi+b\sin\varphi)\,d\varphi\int_1^{\sqrt{2}}\rho^2\sqrt{\rho^2-1}\,d\rho\,-$$

$$-\,abc^2\int_0^{2\pi}d\varphi\int_1^{\sqrt{2}}\rho\,(\rho^2-1)\,d\rho=\frac{3}{2}\,abc^2\,\pi.$$

$$\iint_{\Gamma_1^-}=0,\quad \iint_{\Gamma_2^+}=\iint_G c^2\,dx\,dy=2\,abc^2\,\pi \qquad G:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 2.$$

i $$I=\frac{3}{2}\,abc^2\,\pi-2\,abc^2\,\pi=-\frac{1}{2}\,abc^2\,\pi.$$

Izračunati integral

$$\oiint_\Gamma (x^2\cos\alpha+y^2\cos\beta+z^2\cos\gamma)\,d\sigma,$$

gde je Γ spoljna strana sfere

$$x^2+y^2+z^2=R^2.$$

Rezultat. $I=0$.

Izračunati integral

$$\oiint_\Gamma \frac{dy\,dz}{x}+\frac{dz\,dx}{y}+\frac{dx\,dy}{z}.$$

gde je Γ spoljna strana elipsoida

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Rezultat. $4\pi\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)$

Formula Ostrogradskog se ne može primeniti.