Površinski integral prve vrste

Trebamo izračunati integral $\iint\limits_S f(x,y,z)\,dS$ gde je S - površ u prostoru.

I način:

Ako je D projekcija površi $S: z=z(x,y)$ na xOy ravan tada

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

II način:

L je projekcija površi $S: y=y(x,z)$ na xOz ravan

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_L f(x,y(x,z),z)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz$$

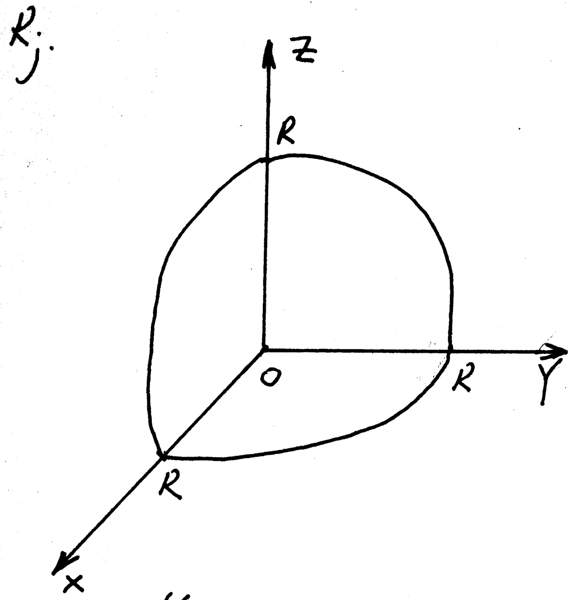
III način:

Neka je C projekcija površi $S: x=x(y,z)$ na yOz ravan

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_C f(x(y,z),y,z)\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$$

Izračunati integral $I=\iint\limits_S x\,dS$ gdje je S dio sfere $x^2+y^2+z^2=R^2$ u prvom oktantu.

Rj.:



$$x^2+y^2+z^2=R^2$$

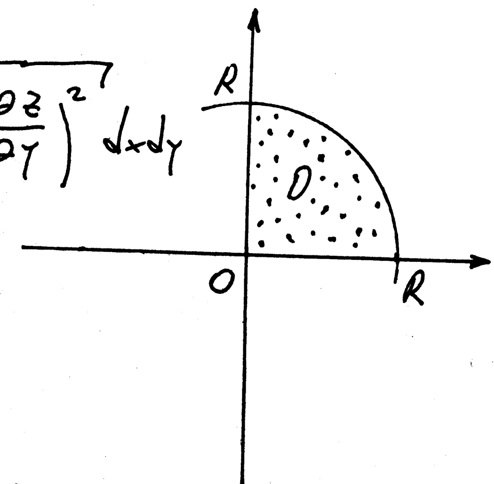
$$z^2=R^2-x^2-y^2$$

$$z=\pm\sqrt{R^2-x^2-y^2}$$

$$z\geq 0 \qquad z=\sqrt{R^2-x^2-y^2}$$

Projekcija površi na xOy ravan

$$\iint\limits_S f(x,y,z)\,dS=\iint\limits_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$



S' projekcija površi S na xOy ravan

$$\frac{\partial z}{\partial x}=\frac{-2x}{2\sqrt{R^2-x^2-y^2}}=\frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{R^2-x^2-y^2}} \qquad \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}}=\frac{R}{\sqrt{R^2-x^2-y^2}}$$

$$I=\iint\limits_S x\,dS=\iint\limits_D x\cdot\frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

uvedimo polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$ u našem slučaju

$$D':\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ 0\leq\varphi\leq\frac{\pi}{2} \\ 0\leq r\leq R\end{cases} \qquad dx\,dy=r\,dr\,d\varphi \qquad x^2+y^2=r^2$$

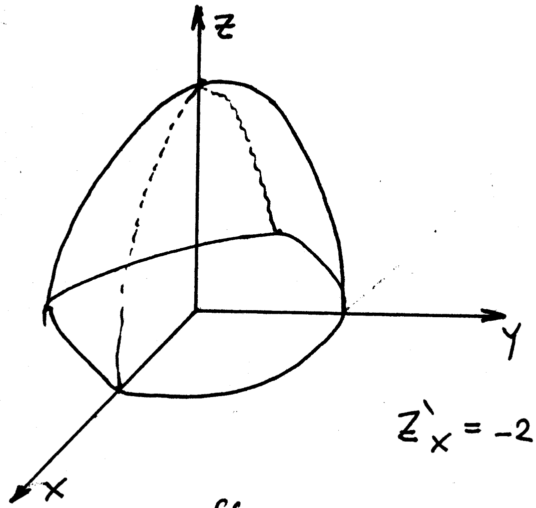
$$\iint\limits_D \frac{xR}{\sqrt{R^2-x^2-y^2}}\,dx\,dy=\iint\limits_{D'}\frac{r\cos\varphi\,R\cdot r}{\sqrt{R^2-r^2}}\,dr\,d\varphi$$

$$=R\int_0^{\pi/2}\cos\varphi\left[\int_0^R\frac{r^2}{\sqrt{R^2-r^2}}\,dr\right]d\varphi=\left|\begin{array}{l} r=R\sin t \\ r=0\Rightarrow t=0 \\ r=R\Rightarrow t=\frac{\pi}{2} \\ dr=R\cos t\,dt\end{array}\right|=R\int_0^{\pi/2}\cos\varphi\left[\int_0^{\frac{\pi}{2}}\frac{1}{R}\,\frac{R^2\sin^2 t}{\sqrt{1-\sin^2 t}}\,R\cos t\,dt\right]d\varphi$$

$$=R^3\int_0^{\pi/2}\cos\varphi\left[\int_0^{\pi/2}\sin^2 t\,dt\right]d\varphi=R^3\int_0^{\pi/2}\cos\varphi\left[\frac{1}{2}\int_0^{\pi/2}(1-\cos 2t)\,dt\right]=R^3\sin\varphi\Big|_0^{\pi/2}\cdot\frac{1}{2}\left(t\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}\sin 2t\Big|_0^{\pi/2}\right)=\frac{R^3\pi}{4}$$

Izračunati $\iint_S U(x,y,z)\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy ravni i $U(x,y,z)$ je jednako
a) 1
b) x^2+y^2
c) $3z$.

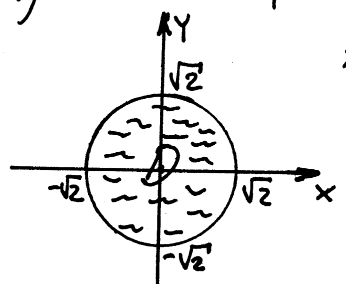
Rj. $\iint_S U(x,y,z)\,dS = \iint_D U(x,y,z)\sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$ gdje je oblast D projekcija površi S na xOy ravan



$z=2-(x^2+y^2)$

Projekcija na xOy ravan

$x^2+y^2=2$



$z_x' = -2x$

$z_y' = -2y$

$$\iint_S U(x,y,z)\,dS = \iint_D U(x,y,z)\sqrt{1+4x^2+4y^2}\,dx\,dy$$

a) $U(x,y,z)=1$ $\qquad I=\iint_D \sqrt{1+4x^2+4y^2}\,dx\,dy$

Da izračunamo ovo transformišimo na polarne koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$

$$D': \begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ 0\le\varphi\le 2\pi \\ 0\le r\le\sqrt{2} \end{cases} \qquad dx\,dy = r\,d\varphi\,dr$$

$$I=\iint_{D'}\sqrt{1+4r^2}\cdot r\,dr\,d\varphi =$$

$$=\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\sqrt{1+4r^2}\cdot r\,dr\right]d\varphi = \left|\begin{matrix} 1+4r^2=t^2 & r=0 \Rightarrow t=1 \\ 8r\,dr=2t\,dt & r=\sqrt{2}\Rightarrow t=3 \\ r\,dr=\frac{1}{4}t\,dt & \therefore \end{matrix}\right| =$$

$$=\int_0^{2\pi}\left[\int_1^3 t\cdot\frac{1}{4}t\,dt\right]d\varphi = \frac{1}{4}\int_0^{2\pi}\frac{1}{3}t^3\Big|_1^3\,d\varphi = \frac{1}{12}\cdot\varphi\Big|_0^{2\pi}\cdot 26 = \frac{13}{6}\cdot 2\pi = \frac{13\pi}{3}$$

b) vježbu $\quad I=\iint_D (x^2+y^2)\sqrt{1+4x^2+4y^2} = \iint_{D'} r^3\sqrt{1+4r^2}\,dr\,d\varphi = \frac{149}{30}$

c) vjež. $\quad I=\frac{111\pi}{10}$

$$= 3\int_0^{2\pi}\left[\int_1^3 \frac{1}{16}(t^2-1)t\cdot t\,dt\right]d\varphi = \frac{3}{16}\int_0^{2\pi}\left[\int_1^3 (t^4-t^2)\,dt\right]d\varphi = \frac{3}{16}\cdot\varphi\Big|_0^{2\pi}\cdot\left(\frac{1}{5}t^5\Big|_1^3 - \frac{1}{3}t^3\Big|_1^3\right)$$

$$= \frac{3}{8}\pi\cdot\left(\frac{242}{5} - \frac{26}{3}\right) = \frac{1}{8}\pi\left(\frac{726}{5} - 26\right) = \frac{1}{8}\pi\,\frac{726-130}{5} = \frac{596\pi}{40} = \frac{149\pi}{10}$$

$$\iint_S 3z\,dS = 26\pi - \frac{149\pi}{10} = \frac{260-149}{10}\pi = \frac{111\pi}{10}$$ traženo rješenje

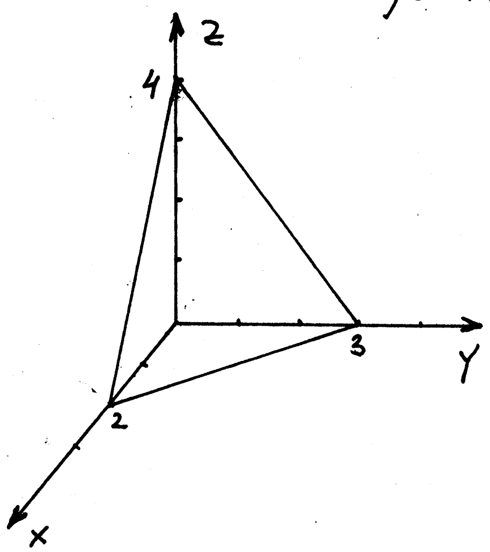
(#) Izračunati površinski integral $\iint_S \left(z+2x+\frac{4}{3}y\right)dS$ gdje je S dio ravni $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ u prvom oktantu.

Rj.

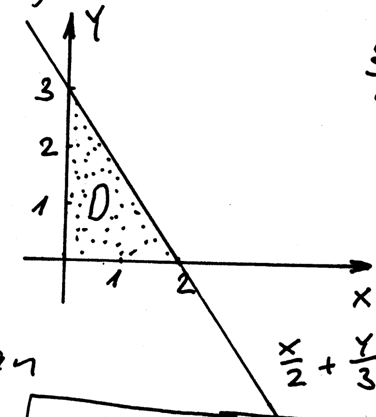
$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ segmentni oblik jednačine ravni

$\frac{z}{4} = 1 - \frac{x}{2} - \frac{y}{3} \quad / \cdot 4$

$z = 4 - 2x - \frac{4}{3}y$



Projekcija na xOy ravan izgleda



$\frac{x}{2}+\frac{y}{3}=1$

$\frac{\partial z}{\partial x} = -2$

$\frac{\partial z}{\partial y} = -\frac{4}{3}$

S' projekcija površi S na xOy ravan

$$I = \iint_S f(x,y,z)\,dS = \iint_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+4+\frac{16}{9}} = \sqrt{\frac{61}{9}} = \frac{\sqrt{61}}{9}$$

$$\iint_S \left(z+2x+\frac{4}{3}y\right)dS = \iint_D \left(4-\underline{\underline{2x}}-\underline{\frac{4}{3}y}+\underline{\underline{2x}}+\underline{\frac{4}{3}y}\right)\cdot\frac{\sqrt{61}}{3}\,dx\,dy = \frac{4\sqrt{61}}{3}\underbrace{\iint_D dx\,dy}_{\text{površina oblasti } D}$$

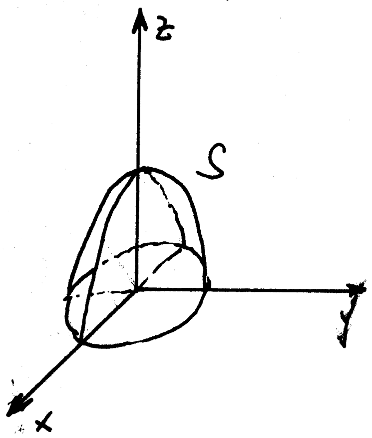
$$= \frac{4\sqrt{61}}{3}\cdot\frac{2\cdot 3}{2} = 4\sqrt{61}.$$

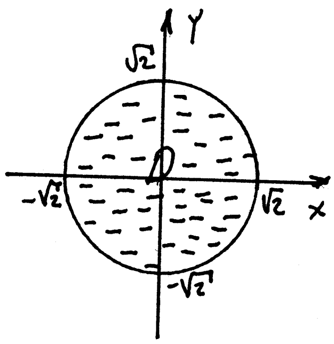
\# Izračunati površinski integral $\iint\limits_S 3z\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy-ravni.

Rj. Neka je D projekcija površi S na xOy ravan. Tada

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

Pronađimo projekciju paraboloida $z=2-(x^2+y^2)$ na xOy ravan.





$z=0 \Rightarrow x^2+y^2=2$ krug sa centrom u tački $(0,0)$ poluprečnika $\sqrt{2}$

$$\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial y} = -2y$$

$$I=\iint\limits_S 3z\,dS = 3\iint\limits_D \left[2-(x^2+y^2)\right]\sqrt{1+4x^2+4y^2}\,dx\,dy$$

Da bi smo riješili ovaj dvostruki integral potrebno je uvesti smjenu promjenjivih.

Uvedimo polarne koordinate $x = r\cos\varphi$, $y = r\sin\varphi$

$$D':\begin{cases} 0\le\varphi\le 2\pi \\ 0\le r\le\sqrt{2} \\ dx\,dy = r\,dr\,d\varphi \end{cases}$$ ove granice čitamo sa slike

$$x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$$

$$1+4x^2+4y^2 = 1+4(x^2+y^2) = 1+4r^2$$

$$I = 3\iint\limits_{D'}(2-r^2)\sqrt{1+4r^2}\cdot r\,dr\,d\varphi = 3\iint\limits_{D'}2r\sqrt{1+4r^2}\,dr\,d\varphi - 3\iint\limits_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi$$

$$6\iint\limits_{D'}r\sqrt{1+4r^2}\,dr\,d\varphi = 6\int_0^{2\pi}\left[\int_0^{\sqrt{2}} r\sqrt{1+4r^2}\,dr\right]d\varphi = \left|\begin{array}{ll} 1+4r^2=t^2 & r=0\Rightarrow t=1 \\ 8r\,dr = 2t\,dt & r=\sqrt{2}\Rightarrow t=3 \\ r\,dr=\frac{1}{4}t\,dt & \end{array}\right| =$$

$$= 6\int_0^{2\pi}\left[\int_1^3 \frac{1}{4}t^2\,dt\right]d\varphi = 6\cdot\frac{1}{4}\,\varphi\Big|_0^{2\pi}\cdot\frac{t^3}{3}\Big|_1^3 = \frac{3}{2}\cdot\frac{1}{3}\cdot 2\pi\cdot 26 = 26\pi$$

$$3\iint\limits_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi = 3\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\underbrace{r^3}_{r^2\cdot r}\sqrt{1+4r^2}\,dr\right]d\varphi = \left|\begin{array}{ll} 1+4r^2=t^2 & r\,dr=\frac{1}{4}t\,dt \\ 4r^2=t^2-1 & r=0\Rightarrow t=1 \\ r^2=\frac{1}{4}(t^2-1) & r=\sqrt{2}\Rightarrow t=3 \\ 8r\,dr=2t\,dt & \end{array}\right| =$$