Zadaci za vježbu

1. Izračunati krivoliniski integral $\int\limits_L \frac{ds}{x-y}$ gdje je L-duž prave $y=\frac{1}{2}x-2$ čije su ivice tačke $A(0,-2)$ i $B(4,0)$.

2. Izračunati krivoliniski integral $\int\limits_L xy\,ds$ gdje je L-konture pravougaonika sa tjemenima $A(0,0)$, $B(4,0)$, $C(4,2)$ i $D(0,2)$

3. Izračunati krivoliniski integral $\int\limits_L y\,ds$ gdje je L-luk parabole $y^2=2px$ koji otsjeca parabola $x^2=2py$.

4. Izračunati krivoliniski integral $\int\limits_L (x^2+y^2)^n ds$ gdje je L-krug $x=a\cos t$, $y=a\sin t$.

5. Izračunati krivoliniski integral $\int\limits_L \sqrt{2y}\,ds$ gdje je L-prvi luk cikloide $x=a(t-\sin t)$, $y=a(1-\cos t)$.

6. Izračunati krivoliniski integral $\int\limits_L x\,dy$ gdje je L-duž prave $\frac{x}{a}+\frac{y}{b}=1$ iz tačke njenog presjeka sa apcisom do tačke njenog presjeka sa ordinatom.

7. Izračunati krivoliniski integral $\int\limits_L (x^2+y^2)\,dy$ gdje je L-kontura četverougla sa tjemenima (navedenih po poretku prelaza) u tačkama $A(0,0)$, $B(2,0)$, $C(4,4)$ i $D(0,4)$.

8. Izračunati krivoliniski integral
$$\int\limits_{(0,0)}^{(1,1)} xy\,dx+(y-x)\,dy$$
ako prelazimo po liniji
a) $y=x$ b) $y=x^2$ c) $y^2=x$ d) $y=x^3$.

9. Izračunati krivoliniski integral $\int\limits_L y\,dx+x\,dy$ gdje je L-četvrtina kruga $x=r\cos t$, $y=r\sin t$ od $t_1=0$ do $t_2=\frac{\pi}{2}$.

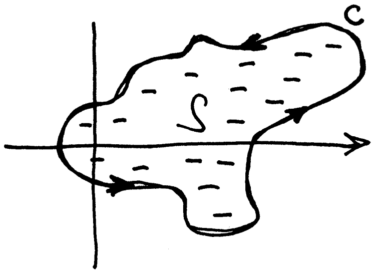
Rješenja:

1. $\sqrt{5}\ln 2$ 2. 24 3. $\frac{p^2}{3}(5\sqrt{5}-1)$ 4. $2\pi a^{2n+3}$ 5. $4\pi a\sqrt{a}$

6. $\frac{ab}{2}$ 7. $37\frac{1}{3}$ 8. a) $\frac{1}{3}$ b) $1/12$ c) $\frac{17}{30}$ d) $-1/20$ 9. 0

Greenova formula za ravan

Ako je c po djelovima glatka granica područja S, a f-je $P(x,y)$ i $Q(x,y)$ neprekidne zajedno sa svojim parcijalnim izvodima prvog reda u zatvorenom području $S+C$, onda vrijedi Greenova formula



$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

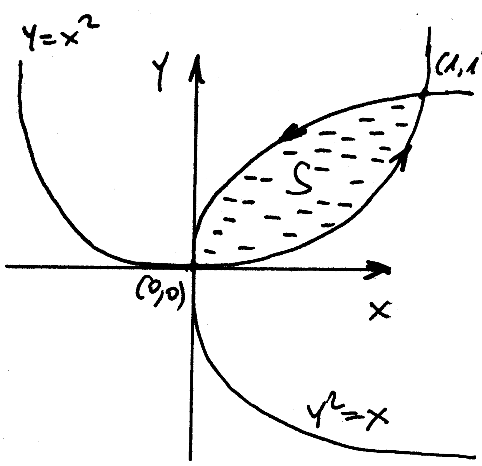
c - zatvorena kontura

S - oblast ograničena konturom

\# Izračunati integral $\displaystyle\int_C (2xy - x^2)\,dx + (x+y^2)\,dx$

gdje je c kontura površine ograničene sa $y=x^2$ i $y^2=x$.

Rj.



$P(x,y) = 2xy - x^2 \qquad \dfrac{\partial P}{\partial y} = 2x$

$Q(x,y) = x + y^2 \qquad \dfrac{\partial Q}{\partial x} = 1.$

$$\int_C P\,dx - Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

formula Greena

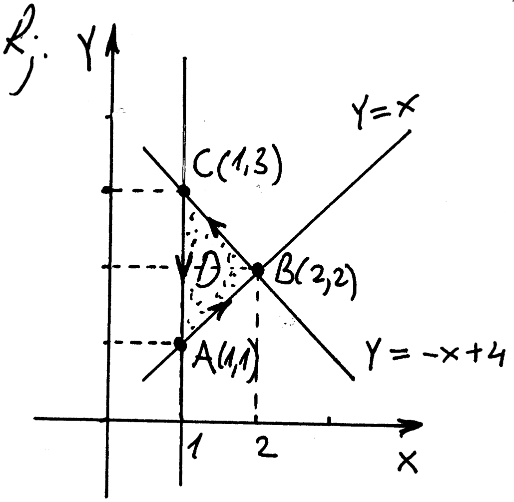
$$\int_C (2xy - x^2)\,dx + (x+y^2)\,dx = \iint_S (1-2x)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (1-2x)\,dy\right] dx =$$

$$= \int_0^1 \left(y\Big|_{x^2}^{\sqrt{x}} - 2xy\Big|_{x^2}^{\sqrt{x}} \right) dx = \int_0^1 \left(\sqrt{x} - x^2 - 2x(\sqrt{x} - x^2)\right) dx =$$

$$= \int_0^1 \left(2x^3 - x^2 - 2x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) = 2\cdot\frac{1}{4}x^4\Big|_0^1 - \frac{1}{3}x^3\Big|_0^1 - 2\cdot\frac{2}{5}x^{\frac{5}{2}}\Big|_0^1 + \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{1}{30}$$

Izračunati $\int\limits_C 2(x^2+y^2)dx+(x+y)^2dy$ gdje je c kontura trougla ΔABC pozitivno orjentisana ($A(1,1)$, $B(2,2)$, $C(1,3)$).

Rj.



$$P(x,y)=2(x^2+y^2)=2x^2+2y^2$$

$$Q(x,y)=(x+y)^2=x^2+2xy+y^2$$

$$\int\limits_C P(x,y)\,dx+Q(x,y)\,dy=\iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

formula Grina

$$y-y_1=\frac{x_2-x_1}{y_2-y_1}(x-x_1)$$

$$y-2=\frac{-1}{1}(x-2)$$

$$y-2=-x+2 \Rightarrow y=-x+4$$

$$\frac{\partial P}{\partial y}=4y \qquad \frac{\partial Q}{\partial x}=2x+2y \qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=2x+2y-4y=2x-2y$$

$$D:\begin{cases}1\le x\le 2\\ x\le y\le 4-x\end{cases} \qquad \int\limits_C 2(x^2+y^2)dx+(x+y)^2dy=\iint\limits_D(2x-2y)\,dx\,dy=$$

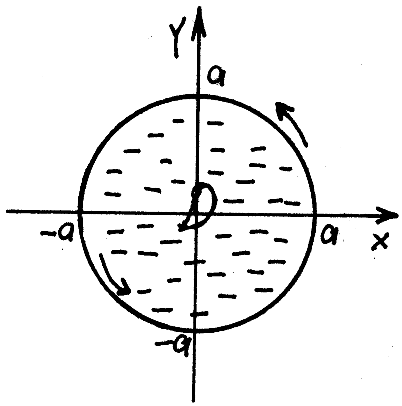
$$=\int_1^2\left[\int_x^{4-x}(2x-2y)\,dy\right]dx=\int_1^2\left(\underbrace{2xy\Big|_x^{4-x}}_{4-x-x}-\underbrace{2\cdot\frac{1}{2}y^2\Big|_x^{4-x}}_{(4-x)^2-x^2}\right)dx=$$

$$=\int_1^2\Big(2x(4-2x)-(16-8x)\Big)dx=\int_1^2(8x-4x^2-16+8x)\,dx=\int_1^2(-4x^2+16x-16)\,dx$$

$$=-4\cdot\frac{1}{3}x^3\Big|_1^2+16\cdot\frac{1}{2}x^2\Big|_1^2-16x\Big|_1^2=-\frac{4}{3}\cdot 7+8\cdot 3-16=8-\frac{28}{3}=-\frac{4}{3}$$

Izračunati $\int\limits_C xy^2\,dy - x^2y\,dx$ gdje je c krug $x^2+y^2=a^2$. Integraciju izvesti u pozitivnom smjeru.

Rj.



$P(x,y) = -x^2y \qquad \frac{\partial P}{\partial y} = -x^2$

$Q(x,y) = xy^2 \qquad \frac{\partial Q}{\partial x} = y^2 \qquad D:\ x^2+y^2 \le a^2$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + x^2 = x^2 + y^2$$

$$\int\limits_C P(x,y)\,dx + Q(x,y)\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Formula Greena

polarne koordinate $\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array} \Rightarrow D':\ \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases} \quad dx\,dy = r\,dr\,d\varphi$

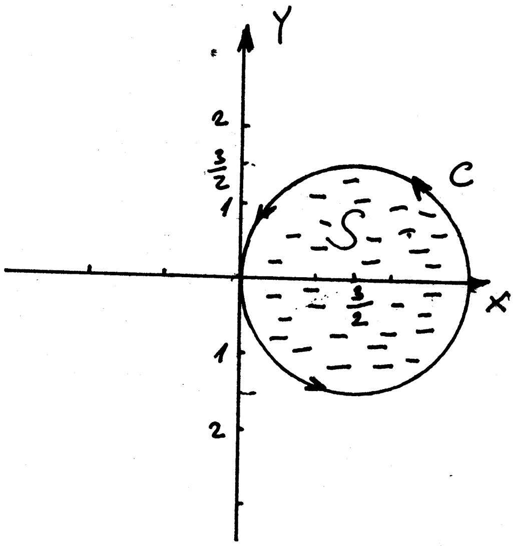
$$\int\limits_C xy^2\,dy - x^2y\,dx = \iint\limits_D (x^2+y^2)\,dx\,dy = \iint\limits_{D'} (r^2\cos^2\varphi + r^2\sin^2\varphi)\cdot r\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\left[\int_0^a r^3\,dr\right] d\varphi = \int_0^{2\pi} \frac{1}{4} r^4\Big|_0^a\,d\varphi = \frac{a^4}{4}\cdot\varphi\Big|_0^{2\pi} = \frac{\pi a^4}{2}$$

Izračunati krivoliniski integral

$$I=\int_C (xy+x+y)\,dx+(xy+x-y)\,dy \quad \text{ako je } c: x^2+y^2=3x.$$

Rj.



$$x^2+y^2=3x$$
$$x^2-3x+y^2=0$$
$$x^2-2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{9}{4}+y^2=0$$
$$\left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$$

C: Krug sa centrom u tački $\left(\frac{3}{2},0\right)$ poluprečnika $r=\frac{3}{2}$

I način: Greenova formula za ravan

$$\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

c - zatvorena kontura

S - oblast ograničena konturom

$P=xy+x+y$, $\frac{\partial P}{\partial y}=x+1$, $Q=xy+x-y$, $\frac{\partial Q}{\partial x}=y+1$.

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y+1-(x+1)=y-x$$

Kako je c krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate

$$x=\frac{3}{2}+r\cos\varphi$$
$$y=r\sin\varphi$$
$$dx\,dy=r\,dr\,d\varphi$$

$$D:\begin{cases}0\le r\le\frac{3}{2}\\ 0\le\varphi\le 2\pi\end{cases}$$

$$I=\int_C(xy+x+y)\,dx+(xy+x-y)\,dy=\iint_S(y-x)\,dx\,dy=\iint_D\left(r\sin\varphi-\left(\frac{3}{2}+r\cos\varphi\right)\right)\cdot r\,dr\,d\varphi$$

$$=\int_0^{3/2}\left[\int_0^{2\pi}\left(r^2\sin\varphi-\frac{3}{2}r-r^2\cos\varphi\right)d\varphi\right]dr=\int_0^{3/2}\left(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0}-\frac{3r}{2}\varphi\Big|_0^{2\pi}\underbrace{-r\sin\varphi\Big|_0^{2\pi}}_{=0}\right)dr$$

$$=\int_0^{3/2}-3\pi r\,dr=-3\pi\frac{r^2}{2}\Big|_0^{3/2}=-\frac{3}{2}\pi\cdot\frac{9}{4}=-\frac{27}{8}\pi$$

II način: Klasičan način

c kriva u ravni opisana jednačinom $y=\eta(x)$, $a\leq x\leq b$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\leq t\leq t_2$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

U našem slučaju c je kružnica. Parametrizirajmo kružnicu

$x=\frac{3}{2}+r\cos\varphi$

$y=r\sin\varphi$

U našem slučaju $r=\frac{3}{2}$ a umjesto promjenjive φ stavimo promjenjivu t

$\frac{\partial x}{\partial t}=-\frac{3}{2}\sin t$

$\frac{\partial y}{\partial t}=\frac{3}{2}\cos t$

$x=\frac{3}{2}+\frac{3}{2}\cos t$

$y=\frac{3}{2}\sin t$

gdje $0\leq t\leq 2\pi$

$$I=\int_c (xy+x+y)\,dx+(xy+x-y)\,dy=\int_0^{2\pi}\Big[\Big(\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)\left(\tfrac{3}{2}\sin t\right)+\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)+$$
$$+\left(\tfrac{3}{2}\sin t\right)\Big)\left(-\tfrac{3}{2}\sin t\right)+\Big(\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)\left(\tfrac{3}{2}\sin t\right)+\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)-$$
$$-\left(\tfrac{3}{2}\sin t\right)\Big)\tfrac{3}{2}\cos t\Big]dt=\ldots$$

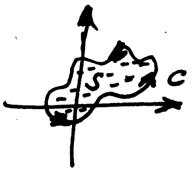
na klasičan način ovo je komplikovano ali se može izračunati

$$I=-\frac{27}{8}\pi$$

(#) Pomoću Greenove formule izračunati integral

$$I=\int\limits_C (xy+x+y)\,dx+(xy+x-y)\,dy,$$ ako je c kontura kružnice $x^2+y^2=ax$ pređena u pozitivnom smislu.

Rj. Greenova formula $$\int\limits_C P\,dx+Q\,dy=\iint\limits_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy .$$



$P(x,y)=xy+x+y$ $\qquad \dfrac{\partial P}{\partial y}=x+1,\quad \dfrac{\partial Q}{\partial x}=y+1$

$Q(x,y)=xy+x-y$

$x^2+y^2=ax$

$x^2-ax+y^2=0$

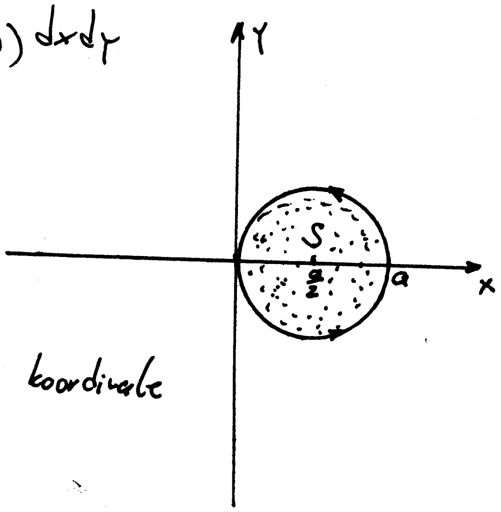
$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

krug sa centrom u $\left(\frac{a}{2},0\right)$ poluprečnika $\frac{a}{2}$

$$I=\iint\limits_S\left(y+1-(x+1)\right)dx\,dy$$

$$I=\iint\limits_S (y-x)\,dx\,dy$$



uvedimo polarne koordinate

$x=\frac{a}{2}+r\sin\varphi$

$y=r\cos\varphi$

$dx\,dy=r\,dr\,d\varphi$

$$S \xrightarrow{\text{transformiš}} S':\begin{cases}0\le r\le \frac{a}{2}\\ 0\le\varphi\le 2\pi\end{cases}$$

$$I=\iint\limits_{S'}\left(r\cos\varphi-\frac{a}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint\limits_{S'}\left(r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right)dr\,d\varphi=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\frac{a}{2}}\left[r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right]dr=\int_0^{2\pi}\left[\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}(\cos\varphi-\sin\varphi)-\frac{a}{2}\cdot\frac{1}{2}r^2\Big|_0^{\frac{a}{2}}\right]d\varphi$$

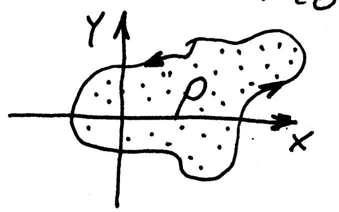
$$=\frac{a^3}{24}\int_0^{2\pi}(\cos\varphi-\sin\varphi)\,d\varphi-\frac{a^3}{16}\int_0^{2\pi}d\varphi=\frac{a^3}{24}\Big(\underbrace{\sin\varphi\Big|_0^{2\pi}}_{=0}+\underbrace{\cos\varphi\Big|_0^{2\pi}}_{1-1}\Big)-\frac{a^3}{16}2\pi=$$

$$=-\frac{a^3\pi}{8}$$ traženo rješenje

Računanje površine ravne figure

Površinu figure ograničenu zatvorenom linijom c računamo po formuli:

$$P = \frac{1}{2}\int_c x\,dy - y\,dx .$$



Podrazumjeva se da po liniji c prelazimo u pozitivnom smjeru

(#) Pokazati da se površina ograničena jednostavnom zatvorenom krivom (konturom) C računa po formuli

$$\frac{1}{2}\int_c x\,dy - y\,dx$$

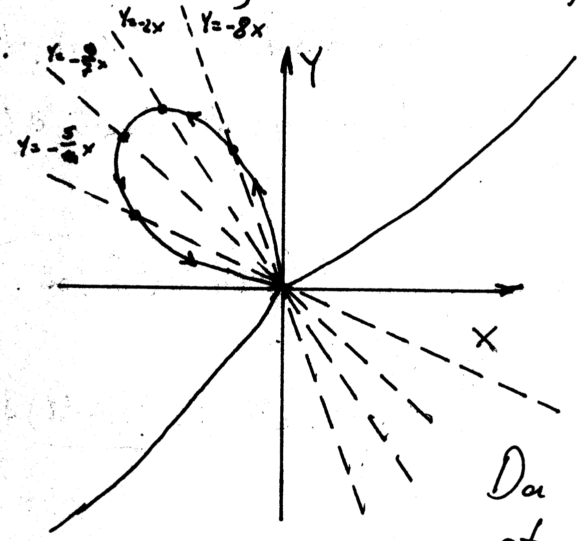
Rj. U formuli Greena stavimo $P(x,y) = -y$, $Q(x,y) = x$. Tada

$$\int_c x\,dy - y\,dx = \iint_S \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)dx\,dy = 2\iint_S dx\,dy = 2\cdot P$$

gdje je P tražena površina. Prema tome $P = \frac{1}{2}\int_c x\,dy - y\,dx$

(#) Uz pomoć krivolinijskog integrala izračunati površinu Dekartovog lista dobijen petljom $x^3+y^3-3axy=0$.

Rj.



$$P=\frac{1}{2}\int_C x\,dy-y\,dx$$

Da bismo upotrebili ovu formulu potrebno je parametrizovati krivu.

Da bismo parametrizovali datu petlju, stavimo $y=tx$. Tada iz jednačine krive dobijamo:

$$x^3+y^3-3axy=0 \qquad y=tx$$

$$x^3+t^3x^3-3atx^2=0 \quad /:x^2$$

$$x(1+t^3)=3at \qquad y=\frac{3at^2}{1+t^3}$$

$$x=\frac{3at}{1+t^3}$$

(Pokušajte sa slike shvatiti zašto smo stavili $y=tx$!!!)

$$dx=3a\,d\left(\frac{t}{1+t^3}\right)=3a\frac{1+t^3-t\cdot 3t^2}{(1+t^3)^2}dt=3a\frac{1-2t^3}{(1+t^3)^2}dt$$

$$dy=3a\,d\left(\frac{t^2}{1+t^3}\right)=3a\frac{2t(1+t^3)-t^2\cdot 3t^2}{(1+t^3)^2}=3at\frac{2+2t^3-3t^3}{(1+t^3)^2}=3at\frac{2-t^3}{(1+t^3)^2}$$

$$x\,dy=3at\cdot\frac{1}{1+t^3}\cdot 3at\cdot\frac{2-t^3}{(1+t^3)^2}dt=(3at)^2\frac{2-t^3}{(1+t^3)^3}dt$$

$$y\,dx=3at\,\frac{t}{1+t^3}\cdot 3a\frac{1-2t^3}{(1+t^3)^2}dt=(3at)^2\frac{1-2t^3}{(1+t^3)^3}dt$$

$$P=\frac{1}{2}\int_C x\,dy-y\,dx=\frac{1}{2}\int_{-\infty}^{0}9a^2t^2\frac{\overbrace{2-t^3-1+2t^3}^{1+t^3}}{(1+t^3)^3}dt=\frac{9a^2}{2}\int_{-\infty}^{0}\frac{t^2}{(1+t^3)^2}dt=$$

$$=\left|\begin{array}{l}1+t^3=u\\3t^2dt=du\\t^2dt=\frac{1}{3}du\end{array}\right|=\frac{3a^2}{2}\int_{-\infty}^{0}\frac{du}{u^2}=\frac{3a^2}{2}\cdot\frac{u^{-1}}{-1}=-\frac{3a^2}{2}\cdot\frac{1}{1+t^3}\Bigg|_{-\infty}^{0}$$

$$=-\frac{3a^2}{2}(1-0)=-\frac{3a^2}{2}$$

Površina je uvijek pozitivna $\quad P=\frac{3a^2}{2}$

Izračunati površinu figure koja je ograničena krivom $x=a\cos^3 t$, $y=a\sin^3 t$, $0\le t\le 2\pi$.

Rj.

$$P=\frac{1}{2}\int_C x\,dy-y\,dx,\qquad C:\begin{cases}x=a\cos^3 t\\ y=a\sin^3 t\\ 0\le t\le 2\pi\end{cases}\qquad \begin{aligned}dx&=3a\cos^2 t\cdot(-\sin t)\,dt\\ dy&=3a\sin^2 t\cos t\,dt\end{aligned}$$

$$P=\frac{1}{2}\int_C x\,dy-y\,dx=\frac{1}{2}\int_0^{2\pi}(a\cos^3 t\cdot 3a\sin^2 t\cos t-a\sin^3 t\cdot 3a\cos^2 t\cdot(-\sin t))\,dt$$

$$=\frac{1}{2}\cdot 3a^2\int_0^{2\pi}(\sin^2 t\cos^4 t+\sin^4 t\cos^2 t)\,dt=\frac{3}{2}a^2\int_0^{2\pi}\sin^2 t\cos^2 t\underbrace{(\cos^2 t+\sin^2 t)}_{1}\,dt$$

$$=\frac{3}{2}a^2\int_0^{2\pi}\frac{1}{4}\underbrace{(2\sin t\cos t)}_{\sin 2t}{}^2\,dt=\frac{3}{8}a^2\int_0^{2\pi}\sin^2 2t\,dt\overset{(*)}{=}\frac{3}{8}a^2\int_0^{2\pi}\frac{1}{2}(1-\cos 4t)\,dt$$

$$=\frac{3}{16}a^2\left(t\Big|_0^{2\pi}-\frac{1}{4}\sin 4t\Big|_0^{2\pi}\right)$$

$$=\frac{3}{16}a^2(2\pi-0)=\frac{3}{8}a^2\pi$$

...(*)

$$1=\sin^2 2t+\cos^2 2t$$
$$\cos 4t=\cos^2 2t-\sin^2 2t$$
$$\Rightarrow 1-\cos 4t=2\sin^2 2t$$