Krivoliniski integral druge vrste (po koordinatama)

Ako je c data kriva u ravni opisana jednačinom $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

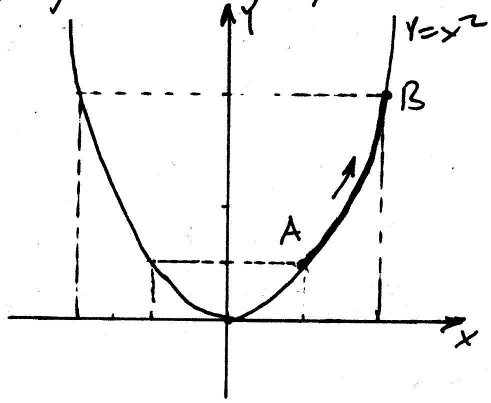
Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

Analogne formule vrijede za krivoliniski integral druge vrste uzete po prostornoj krivoj. Krivoliniski integral druge vrste OVISI O SMJERU PUTA INTEGRACIJE (bitna je orjentacija i u kom smjeru ide luk).

Izračunati krivoliniski integral $\int_c (x^2-2xy)\,dx+(2xy+y^2)\,dy$ gdje je c luk parabole $y=x^2$ od tačke $A(1,1)$ do $B(2,4)$.

Rj.



$y=x^2$

$\frac{\partial y}{\partial x}=2x$

$1\le x\le 2$

Ako je data kriva $y=\eta(x)$, $a\le x\le b$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=$$

$$=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

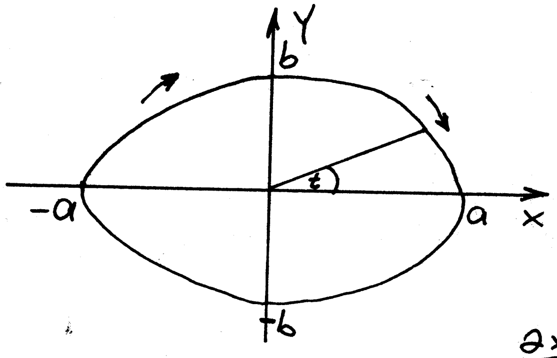
$$\int_c (x^2-2xy)\,dx+(2xy+y^2)\,dy=\int_1^2\left(x^2-2x^3+(2x^3+x^4)\,2x\right)dx=\int_1^2(2x^5+4x^4-2x^3+x^2)\,dx$$

$$=2\cdot\frac{1}{6}x^6\Big|_1^2+4\cdot\frac{1}{5}x^5\Big|_1^2-2\cdot\frac{1}{4}x^4\Big|_1^2+\frac{1}{3}x^3\Big|_1^2=\frac{1}{3}\cdot 63+\frac{4}{5}\cdot 31-\frac{1}{2}\cdot 15+\frac{1}{3}\cdot 7=40+\frac{19}{30}$$

(#) Izračunati krivolinijski integral $\int\limits_c y^2dx + x^2dy$

gdje je c gornja polovina elipse $x=a\cos t$, $y=b\sin t$ $(a>0, b>0)$, koja se prelazi u smislu pomjeranja kazaljke na satu.

Rj.



Ako je kriva c zadana parametarski $x=\varphi(t)$, $y=\psi(t)$ gdje $\alpha\le t\le\beta$ imamo

$$\int\limits_c P(x,y)dx+Q(x,y)dy = \int\limits_\alpha^\beta \left[P(\varphi(t),\psi(t))\varphi'(t)+Q(\varphi(t),\psi(t))\psi'(t)\right]dt$$

$$\frac{\partial x}{\partial t} = -a\sin t \qquad \frac{\partial y}{\partial t} = b\cos t$$

$$\int\limits_c y^2dx+x^2dy = \int\limits_\pi^0 \left[b^2\sin^2 t\cdot(-a\sin t)+a^2\cos^2 t\cdot b\cos t\right]dt =$$

$$= -ab^2\int\limits_\pi^0 \sin^3 t\,dt + a^2b\int\limits_\pi^0 \cos^3 t\,dt \overset{(*)}{=} \frac{4}{3}ab^2$$

$$\int\limits_\pi^0 \sin^3 t\,dt = \int\limits_\pi^0 \sin t(1-\cos^2 t)\,dt = \left|\begin{matrix}\cos t = u & t=\pi \Rightarrow u=-1\\ -\sin t\,dt = du & t=0 \Rightarrow u=1\\ \sin t\,dt=-du & \end{matrix}\right| = -\int\limits_{-1}^{1}(1-u^2)\,du =$$

$$= -\left(u\Big|_{-1}^{1} - \frac{1}{3}u^3\Big|_{-1}^{1}\right) = -\left(2-\frac{1}{3}\cdot 2\right) = -\left(\frac{6-2}{3}\right) = -\frac{4}{3}$$

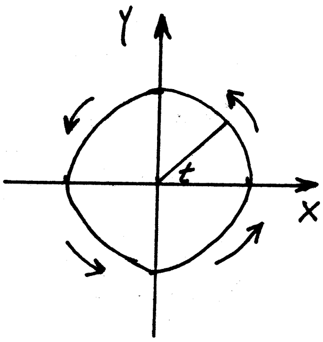
$\ldots(*)$

$$\int\limits_\pi^0 \cos^3 t\,dt = \int\limits_\pi^0 \cos t(1-\sin^2 t)\,dt = \left|\begin{matrix}\sin t=u\\ \cos t\,dt = du\\ t=\pi \Rightarrow u=0\\ t=0 \Rightarrow u=0\end{matrix}\right| = \int\limits_0^0 (1-u^2)\,du = 0$$

Izračunati krivoliniski integral $\int\limits_C \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2}$

gdje je c krug $x^2+y^2=a^2$ koji se prelazi u smislu suprotnom pomjeranju kazaljke na satu.

Rj.



Krug $x^2+y^2=a^2$ napisan parametarski

$$x=a\cos t$$
$$y=a\sin t$$
$$0\le t\le 2\pi$$

Ako je c kriva zadana parametarski $x=\mu(t)$, $y=\eta(t)$, $a\le t\le b$

$$\int\limits_C P(x,y)\,dx+Q(x,y)\,dy = \int_a^b [P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)]\,dt$$

$$\frac{\partial x}{\partial t}=-a\sin t$$

$$\frac{\partial y}{\partial t}=a\cos t$$

$$\int\limits_C \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2} = \int\limits_C \frac{x+y}{x^2+y^2}\,dx-\frac{x-y}{x^2+y^2}\,dy = \int_0^{2\pi}\left[\frac{a\cos t+a\sin t}{a^2}\cdot(-a\sin t)-\right.$$

$$\left.-\frac{a\cos t-a\sin t}{a^2}\cdot a\cos t\right]dt = \int_0^{2\pi}[(\cos t+\sin t)\cdot(-\sin t)-(\cos t-\sin t)\cdot\cos t]\,dt$$

$$=\int_0^{2\pi}(\underline{\underline{-\sin t\cos t}}-\sin^2 t-\cos^2 t\,\underline{\underline{+\sin t\cos t}})\,dt=\int_0^{2\pi}(-1)\,dt=-2\pi$$

(#) Izračunati krivolinijski integral $\int_C x^3 dx + 3zy^2 dy - x^2 y\, dz$ gdje je c dio prave od tačke $A(3,2,1)$ do tačke $O(0,0,0)$.

Rj.

jednačina prave kroz dvije tačke $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

$p(O,A)$: $\dfrac{x}{3} = \dfrac{y}{2} = \dfrac{z}{1} \quad (=t)$

$$\begin{cases} x = 3t \\ y = 2t \\ z = t \end{cases} \qquad \begin{aligned} dx &= 3\,dt \\ dy &= 2\,dt \\ dz &= dt \end{aligned}$$

Trebaju nam još granice za t

$A(3,2,1)$ $\quad \begin{aligned} x&=3t \\ y&=2t \\ z&=t \end{aligned} \quad \Rightarrow \quad t=1$

$O(0,0,0)$ $\quad \begin{aligned} x&=3t \\ y&=2t \\ z&=t \end{aligned} \quad \Rightarrow \quad t=0$

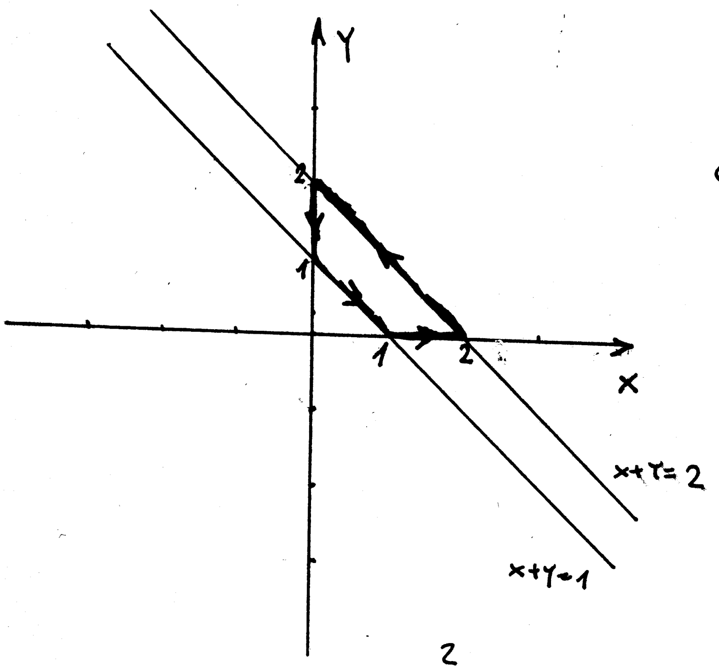
$$\int_C x^2 dx + 3zy^2 dy - x^2 y\, dz =$$

$$= \int_1^0 \left[(3t)^3\cdot 3 + 3\cdot t\cdot (2t)^2\cdot 2 - (3t)^2\cdot 2t\cdot 1\right] dt$$

$$= \int_1^0 (81t^3 + 24t^3 - 18t^3)\, dt = -\int_0^1 87t^3\, dt = -87\cdot\frac{1}{4}t^4\Big|_0^1 = -\frac{87}{4}$$

(#) Izračunati krivoliniski integral $I = \int_C (x^2+y^2)\,dx + x^2y\,dy$ gdje je c kontura trapeza koga obrazuju prave $x=0,\ y=0,\ x+y=1,\ x+y=2$.

Rj.



Ako je $c:\ y=\eta(x),\ a\le x\le b$

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(x,\eta(x)) + Q(x,\eta(x))\cdot\eta'(x)\,]\,dx$$

U našem slučaju postoje 4 krive

$C_1:\ y=0,\ 1\le x\le 2$

$C_1:\ y=-x+2,\ 2\ge x\ge 0$

$C_3:\ x=0,\ 2\ge y\ge 1$

$C_4:\ y=-x+1,\ 0\le x\le 1$

$$I = I_1+I_2+I_3+I_4, \quad I_1 = \int_1^2 (x^2 + x^2\cdot 0)\,dx = \int_1^2 x^2\,dx = \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}(8-1) = \frac{7}{3}$$

$$I_2 = \int_2^0 \left(x^2+(-x+2)^2 + x^2(-x+2)\cdot(-1)\right)dx = \int_2^0 (\underline{\underline{x^2}}+\underline{\underline{x^2}}-4x+4\ \underline{+x^3}\ \underline{\underline{-2x^2}})\,dx =$$

$$= \int_2^0 (x^3-4x+4)\,dx = \frac{1}{4}x^4\Big|_2^0 - 4\cdot\frac{1}{2}x^2\Big|_2^0 + 4x\Big|_2^0 = -4+8-8 = -4$$

$$I_3 = \int_2^1 (y^2\cdot 0 + 0\cdot y)\,dy = 0$$

$$I_4 = \int_0^1 \left(x^2+(-x+1)^2 + x^2(-x+1)\cdot(-1)\right)dx = \int_0^1 (x^2+\underline{x^2}-2x+1\underline{\underline{+x^3}}\underline{-x^2})\,dx =$$

$$= \int_0^1 (x^3+x^2-2x+1)\,dx = \frac{1}{4}x^4\Big|_0^1 + \frac{1}{3}x^3\Big|_0^1 - 2\cdot\frac{1}{2}x^2\Big|_0^1 + x\Big|_0^1 = \frac{1}{4}+\frac{1}{3}-1+1 = \frac{7}{12}$$

$$I = I_1+I_2+I_3+I_4 = \frac{7}{3}+(-4)+\frac{7}{12} = \frac{28-48+7}{12} = -\frac{13}{12}$$ vrijednost krivoliniskog integrala

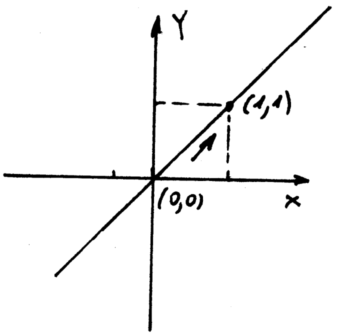
II način: Greenova formula ...

Izračunati krivoliniski integral $\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$ ako prelazimo po liniji

a) $y = x$ b) $y = x^2$ c) $y = x^3$ d) $y^2 = x$.

Rj.

a)



$\boxed{\text{Ako je data kriva } y = \eta(x),\ \alpha \le x \le \beta \quad \int_c P(x,y)\,dx + Q(x,y)\,dy = \int_\alpha^\beta [P(x,\eta(x)) + Q(x,\eta(x))\cdot \eta'(x)]\,dx}$

$y = x$

$y' = 1$

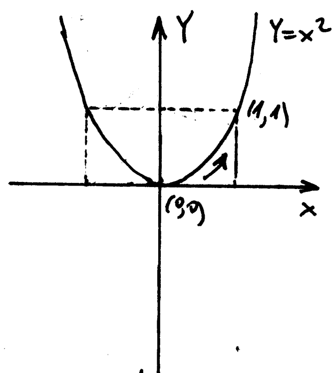
$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy = \int_0^1 (2x^2 + x^2\cdot 1)\,dx = 3\int_0^1 x^2\,dx = 3\cdot\frac{1}{3}x^3\Big|_0^1 = 3\cdot\frac{1}{3} = 1$$

b)

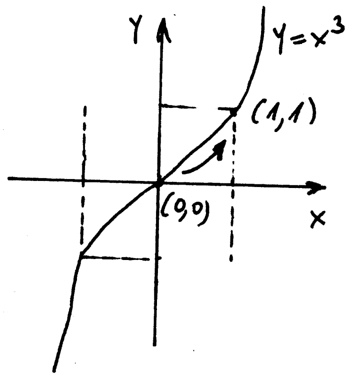
$y = x^2$

$y' = 2x$

$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy = \int_0^1 (2x\cdot x^2 + x^2\cdot 2x)\,dx = \int_0^1 4x^3\,dx = 4\cdot\frac{1}{4}x^4\Big|_0^1 = 4\cdot\frac{1}{4} = 1$$



c)



$y = x^3$

$y' = 3x^2$

$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy = \int_0^1 (2x\cdot x^3 + x^2\cdot 3x^2)\,dx = \int_0^1 5x^4\,dx = 5\cdot\frac{1}{5}x^5\Big|_0^1 = 5\cdot\frac{1}{5} = 1$$

d)

$x = y^2$

$x'_y = 2y$

$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy = \int_0^1 \left(2\cdot y^2\cdot y\cdot 2y + (y^2)^2\right)dy = \int_0^1 (4y^4 + y^4)\,dy = \int_0^1 5y^4\,dy = 5\cdot\frac{1}{5}y^5\Big|_0^1 = 1$$