Krivoliniski integral prve vrste (po luku)

Ako je c kriva data u ravni opisana jednačinom $y=\eta(x)$ gdje je $a \le x \le b$ tada

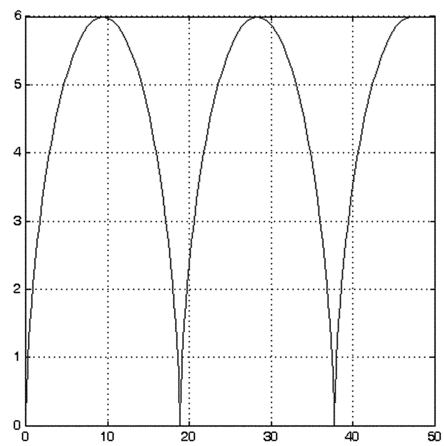
$$\int_c f(x,y)\,ds = \int_a^b f(x,\eta(x))\underbrace{\sqrt{1+(\eta'(x))^2}\,dx}_{ds}$$

Ako je c kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1 \le t \le t_2$ tada

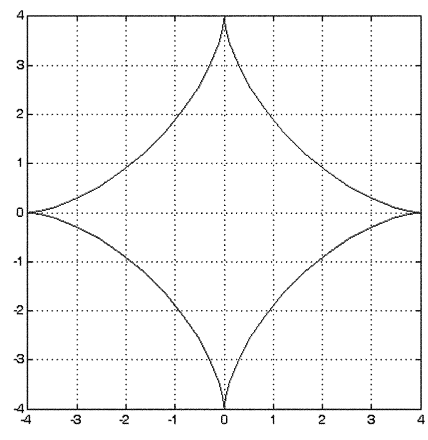
$$\boxed{\mu'(t)=\frac{\partial \mu}{\partial t}}$$

$$\int_c f(x,y)\,ds = \int_{t_1}^{t_2} f(\mu(t),\eta(t))\underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt}_{ds}$$

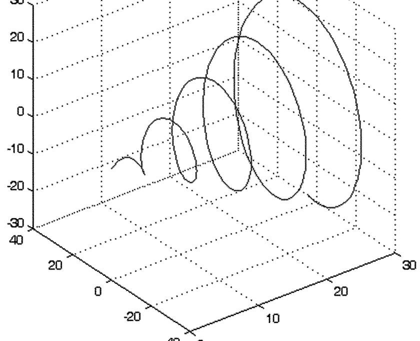
Krivoliniski integrali prve vrste f-ja triju promenjivih $f(x,y,z)$ uzeti po prostornoj krivoj se računaju analogno.
Krivoliniski integral prve vrste NE OVISI O SMJERU PUTA INTEGRACIJE.



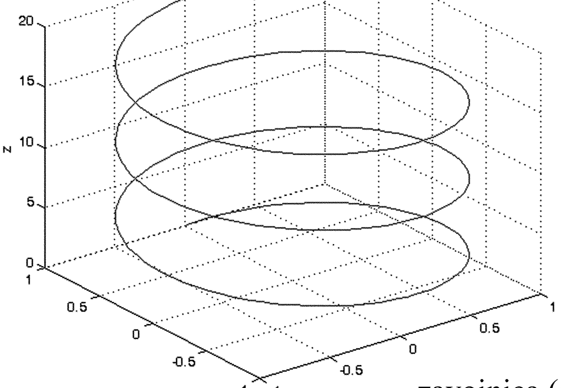
cikloida
$x=3(t-\sin t), y=3(1-\cos t)$, $0 \le t \le 5\pi$.



funkcija $x=4\cos^3 t, y=4\sin^3 t$, $0 \le t \le 2\pi$.



funkcija $x=t, y=t\cos t, z=t\sin t$, $0 \le t \le 30$.

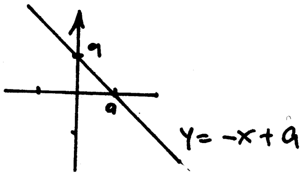


zavojnica (spirala)
$x=\sin t, y=\cos t, z=t$, $0 \le t \le 6\pi$.

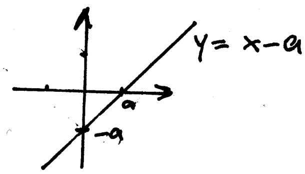
(#) Izračunati integral po krivoj C $\int_C xy\,ds$ gdje je C kvadrat $|x|+|y|=a$, $a>0$.

Rj: Kako nacrtati kvadrat $|x|+|y|=a$?

1° $x>0$, $y>0$: $x+y=a$



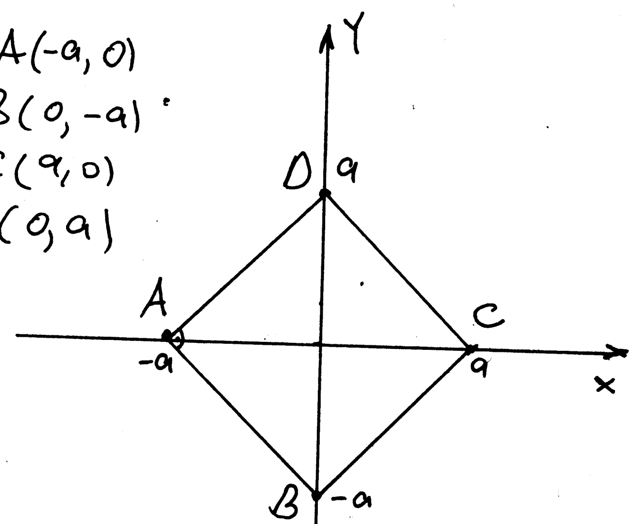
2° $x>0$, $y<0$: $x-y=a$



3° $x<0$, $y>0$: $-x+y=a$

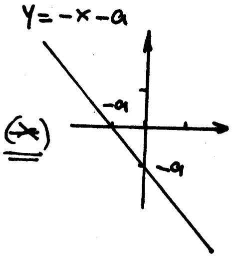
4° $x<0$, $y<0$: $-x-y=a$

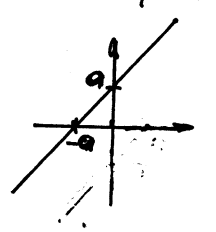
$A(-a,0)$
$B(0,-a)$
$C(a,0)$
$D(0,a)$



Kriva po kojoj se integrali mora biti glatka, ako ima čošak razbije se na dijelove.

$$\int_C xy\,ds = \int_{AB} xy\,ds + \int_{BC} xy\,ds + \int_{CD} xy\,ds + \int_{DA} xy\,ds \overset{(*)}{=}$$





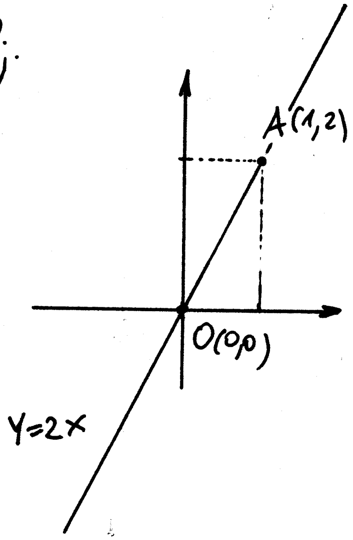
$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx, \quad \text{gdje je } C: y=\varphi(x) \text{ kriva}, \ x\in[a,b]}$$

$$\overset{(*)}{=} \int_{-a}^{0} x\cdot(-x-a)\sqrt{1+(-1)^2}\,dx + \int_0^a x(x-a)\sqrt{1+1^2}\,dx + \int_0^a x(-x+a)\sqrt{1+(-1)^2}\,dx$$

$$+ \int_{-a}^{0} x(x+a)\sqrt{1+1^2}\,dx = \sqrt{2}\left(\int_{-a}^{0}(-x^2-ax+x^2+ax)\,dx + \int_0^a (x^2-ax-x^2+ax)\,dx\right) = 0$$

(#) Izračunati integral $\int\limits_c \frac{ds}{\sqrt{x^2+y^2+4}}$ gdje je c duž koja spaja tačke O(0,0) i tačku A(1,2).

Rj.



c: $y=2x$. $y'=2$

$$\int\limits_c f(x,y)\,ds = \int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx$$

gdje je $y=\varphi(x)$. kriva $x\in[a,b]$

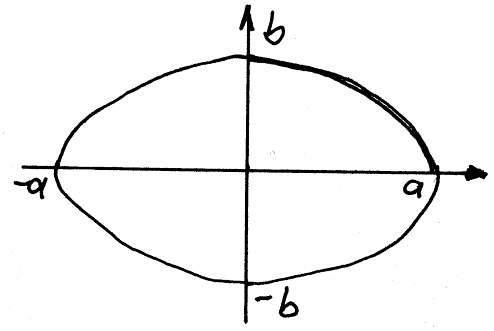
$$\int\limits_c \frac{1}{\sqrt{x^2+y^2+4}}\,dS = \int_0^1 \frac{\sqrt{1+2^2}}{\sqrt{x^2+(2x)^2+4}}\,dx = \sqrt{5}\int_0^1 \frac{dx}{\sqrt{5x^2+4}} =$$

$$= \sqrt{5}\int_0^1 \frac{dx}{\sqrt{4(\frac{5}{4}x^2+1}} = \frac{\sqrt{5}}{2}\int_0^1 \frac{d(\frac{\sqrt{5}}{2}x)}{\sqrt{(\frac{\sqrt{5}}{2}x)^2+1}}\cdot\frac{2}{\sqrt{5}} = \ln\left|\frac{\sqrt{5}x}{2}+\sqrt{\left(\frac{\sqrt{5}x}{2}\right)^2+1}\right|\Bigg|_0^1$$

$$= \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{5}{4}+1}\right| - \ln 1 = \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{9}{4}}\right| = \ln\frac{\sqrt{5}+3}{2}$$

Izračunati $\int_c xy\,ds$ gdje je c četvrtina elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ koja leži u prvom kvadrantu.

Rj. I način:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2-\frac{b^2}{a^2}x^2$$

$$y=\pm\sqrt{b^2\left(1-\frac{1}{a^2}x^2\right)}$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$c:\begin{cases} y=\frac{b}{a}\sqrt{a^2-x^2} \\ 0\le x\le a \end{cases}$$

$$y'=\frac{b}{a}\cdot\frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}$$

$$\int_c xy\,ds=\int_0^a x\frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\left(\frac{b}{a}\cdot\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx$$

$$=\dots$$

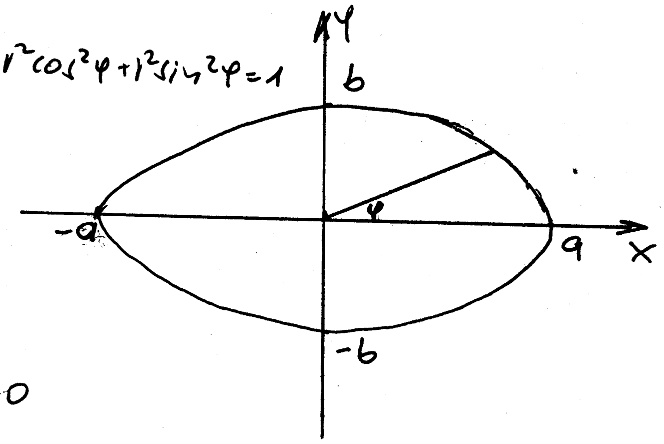
II način

Uvodimo poopštene polarne koordinate

$$x=ar\cos\varphi \qquad x^2=a^2r^2\cos^2\varphi \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2\cos^2\varphi+r^2\sin^2\varphi=1$$
$$y=br\sin\varphi \qquad y^2=b^2r^2\sin^2\varphi$$

$$\Downarrow$$

za $\varphi=0$ imamo $x=a,\ y=0$
za $\varphi=\frac{\pi}{2}$ imamo $x=0,\ y=b$
$\Big\}\Rightarrow r=1$



Sad elipsu možemo napisati u parametarskom obliku tj. imamo

$$c:\begin{cases} x=a\cos\varphi \\ y=b\sin\varphi \\ 0\le\varphi\le\frac{\pi}{2}\end{cases} \qquad \frac{\partial x}{\partial\varphi}=-a\sin\varphi \qquad \frac{\partial y}{\partial\varphi}=b\cos\varphi$$

$$\boxed{\int_c f(x,y)\,ds=\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\psi}{\partial t}\right)^2}\,dt \quad \text{gdje je} \quad \begin{matrix} x=\varphi(t) \\ y=\psi(t) \\ \alpha\le t\le\beta\end{matrix}}$$

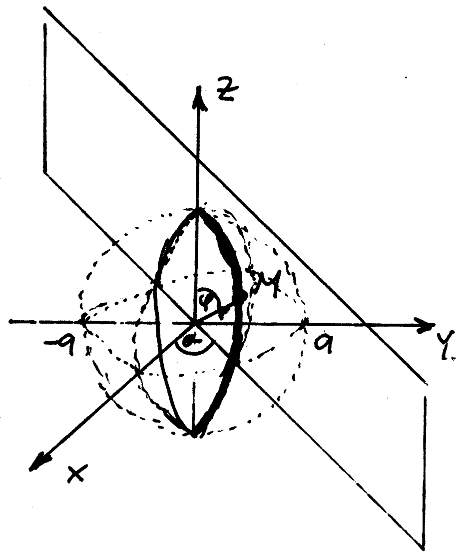
$$\int_c xy\,ds=\int_0^{\pi/2}(a\cos\varphi)(b\sin\varphi)\sqrt{a^2\underbrace{\sin^2\varphi}_{1-\cos^2\varphi}+b^2\cos^2\varphi}\,d\varphi=ab\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\sqrt{a^2+(b^2-a^2)\cos^2\varphi}\,d\varphi$$

$$=\left|\begin{matrix} a^2+(b^2-a^2)\cos^2\varphi=u & \varphi=0\Rightarrow u=b^2 \\ (b^2-a^2)\,2\cos\varphi(-\sin\varphi)\,d\varphi=du & \varphi=\frac{\pi}{2}\Rightarrow u=a^2\end{matrix}\right|=ab\cdot\frac{-1}{2(b^2-a^2)}\int_{b^2}^{a^2}\sqrt{u}\,du$$

$$=\frac{-ab}{2(b^2-a^2)}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{b^2}^{a^2}=\frac{-ab}{(b-a)(b+a)}\cdot\frac{1}{3}\cdot\frac{(a^3-b^3)}{(a-b)(a^2+ab+b^2)}=\frac{ab}{3(a+b)}(a^2+ab+b^2)$$

Izračunati $\int\limits_c \sqrt{2y^2+z^2}\,ds$ gdje je c krug dobijen presjekom sfere $x^2+y^2+z^2=a^2$ i ravni $x=y$.

Rj.



Kako ćemo opisati datu sferu parametarski? (sferne koordinate)

$$x = r\sin\varphi\cos\theta,\quad y = r\sin\varphi\sin\theta,\quad z = r\cos\varphi, \qquad r=a,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi$$

Kako da parametarski opišemo krug dobijen presjekom sfere i ravni?

Za pravu $x=y$ znamo da je ugao između ove prave i x-ose 45°. Prema tome $\theta=45^\circ$, $(r=a)$:

$$c:\begin{cases} x = \frac{\sqrt{2}}{2}a\sin\varphi \\ y = \frac{\sqrt{2}}{2}a\sin\varphi \\ z = a\cos\varphi \\ 0\le\varphi\le 2\pi \end{cases}$$

θ i r su fiksirani

$$2y^2+z^2 = 2\cdot\frac{2}{4}a^2\sin^2\varphi + a^2\cos^2\varphi = a^2$$

Ako je kriva c opisana sa $x=\mu(t)$, $y=\eta(t)$, $z=\xi(t)$, $\alpha<t<\beta$ onda je

$$\int\limits_c f(x,y,z)\,ds = \int_\alpha^\beta f(\mu(t),\eta(t),\xi(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\xi'(t))^2}\,dt$$

$$\frac{\partial x}{\partial\varphi} = \frac{\sqrt{2}}{2}a\cos\varphi$$

$$\frac{\partial y}{\partial\varphi} = \frac{\sqrt{2}}{2}a\cos\varphi$$

$$\frac{\partial z}{\partial\varphi} = -a\sin\varphi$$

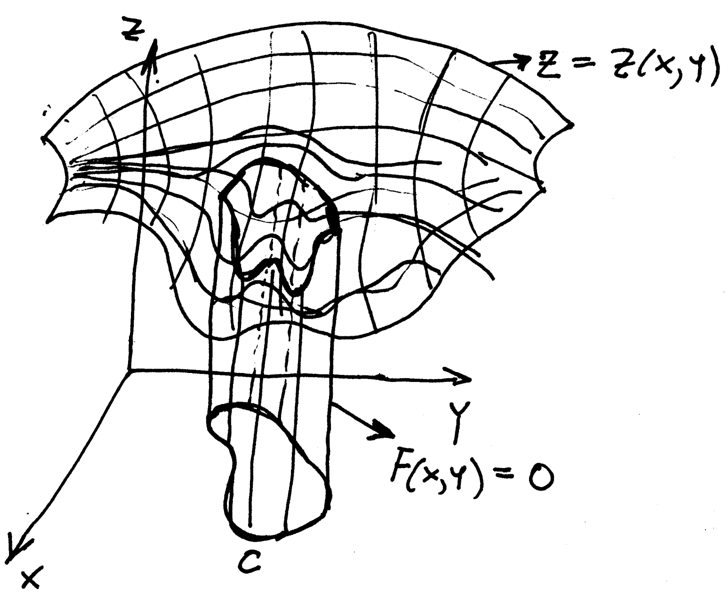
$$\int\limits_c \sqrt{2y^2+z^2}\,ds = \int_0^{2\pi}\sqrt{a^2}\cdot\sqrt{\tfrac{2}{4}a^2\cos^2\varphi+\tfrac{2}{4}a^2\cos^2\varphi+a^2\sin^2\varphi}\,dt$$

$$= \int_0^{2\pi} a\cdot\sqrt{a^2(\cos^2\varphi+\sin^2\varphi)}\,dt = a^2\int_0^{2\pi}dt = 2a^2\pi.$$

Računanje površine cilindrične površi

Ako je S dio cilindrične površine $F(x,y)=0$ između xOy ravni i neke površine $z=z(x,y)$ tada se površina $P(S)$ površi S računa po formuli

$$P(S)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad C:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}$$



$P(S)$ – površina dijela cilindrične površi

\# Izračunati površinu eliptičkog valjka $9x^2+5y^2=45$ koji se nalazi između površi $z=0$ i $z=y$.

Rj.

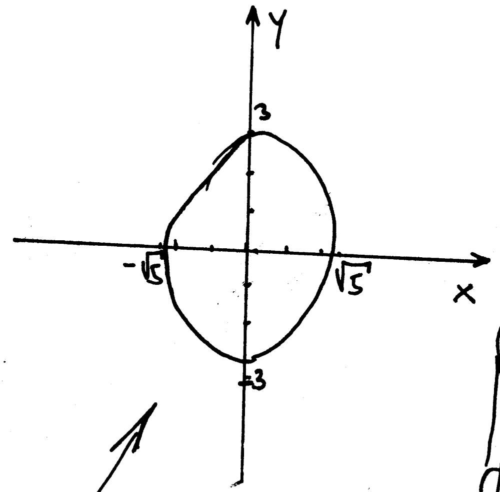
$$P(s)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad c:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}.$$

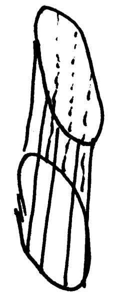
Skicirajmo valjak $9x^2+5y^2=45.\ /:45$ u xOy ravni on izgleda

$\frac{x^2}{5}+\frac{y^2}{9}=1$

elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\sqrt{5}\approx 2{,}23$



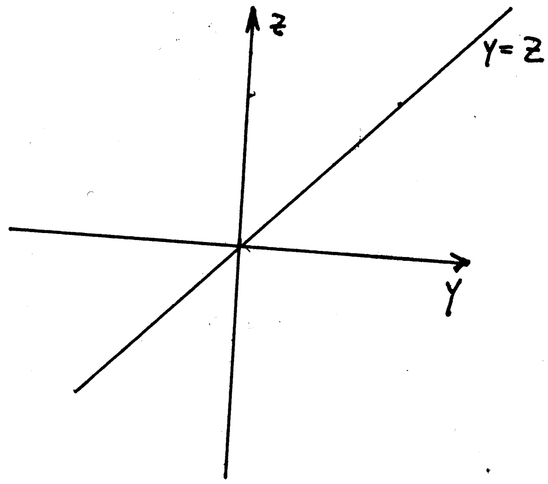


$z(x,y)=y$

$c:\begin{cases} 9x^2+5y^2=45 \\ z=0 \end{cases}$ c je elipsa

$z=0$ je xOy ravan

$z=y$ u yOz ravni



$z=y$ je ravan koja sadrži x-osu a u yOz ravni sadrži $y=z$ pravu

Svedimo elipsu $\frac{x^2}{5}+\frac{y^2}{9}=1$ na parametarski oblik $\quad x=a\cos t,\ y=b\sin t$

U našem slučaju $\quad x=\sqrt{5}\cos t,\quad y=3\sin t,\quad 0\le t\le 2\pi$

$$c:\begin{cases} x=\mu(t) \\ y=\eta(t) \\ t_1\le t\le t_2 \end{cases} \qquad \int_C f(x,y)\,ds=\int_{t_1}^{t_2} f(\mu(t),\eta(t))\underbrace{\sqrt{\mu'(t)^2+\eta'(t)^2}\,dt}_{ds}$$

površina figure koja se nalazi u I i II oktantu

$dS=\sqrt{5\sin^2 t+9\cos^2 t}\,dt$ Kako se ravni $z=0$ i $z=y$ sijeku u x-osi, to će parametar t uzimati vrijednosti od 0 do π

$$P(s)=\int_C y\,dS=\int_0^{\pi} 3\sin t\sqrt{5\sin^2 t+9\cos^2 t}\,dt=3\int_0^{\pi}\sin t\sqrt{5(1-\cos^2 t)+9\cos^2 t}\,dt=$$

$$=3\int_0^{\pi}\sin t\sqrt{5+4\cos^2 t}\,dt=\left|\begin{matrix} 2\cos t=u & t=0 \Rightarrow u=2 \\ -2\sin t\,dt=du & t=\pi \Rightarrow u=-2 \\ \sin t\,dt=-\frac{1}{2}u & \end{matrix}\right|=3\int_2^{-2}\left(-\frac{1}{2}\right)\underbrace{\sqrt{5+u^2}}_{\text{parna f-ja}}\,du=$$

$$=3\cdot\frac{1}{2}\cdot 2\int_0^2\sqrt{5+u^2}\,du \overset{\text{umjesto } u \text{ stavimo } x}{=} 3\int_0^2\frac{5+x^2}{\sqrt{5+x^2}}\,dx=3\int_0^2\frac{5}{\sqrt{5+x^2}}\,dx+3\int_0^2\frac{x^2}{\sqrt{5+x^2}}=\left|\begin{matrix} u=x & dv=\frac{x}{\sqrt{5+x^2}} \\ \text{ZAVRŠITI} & \\ \text{SAMI} \ldots & \end{matrix}\right|=\frac{15\ln 5}{4}+9$$

Izračunati površinu dijela valjka $x^2+y^2=1$ koji se nalazi između površi $z=0$ i $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

Rj: $P(S)=\int_C z(x,y)\,dS$ gdje je $C:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}$

U ovom slučaju je $z(x,y)=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

$C:\begin{cases} x^2+y^2=1 \\ z=0 \end{cases}$ tj. $C: x^2+y^2=1$

Parametrizirajmo kružnicu: $x=r\cos\varphi$, $y=r\sin\varphi$, $0\le\varphi\le 2\pi$. U našem slučaju: $x=\cos\varphi$, $y=\sin\varphi$, $0\le\varphi\le 2\pi$

$C:\begin{cases} x=\mu(t) \\ y=\eta(t) \\ t_1\le t\le t_2 \end{cases}$ $\quad \int_C f(x,y)\,dS=\int_{t_1}^{t_2} f(\mu(t),\eta(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt$

$(\cos\varphi)'=-\sin\varphi$

$(\sin\varphi)'=\cos\varphi$

$dS=\sqrt{\sin^2\varphi+\cos^2\varphi}\,d\varphi=d\varphi$

$\sqrt{x^2+y^2}=1$

$\sqrt{1-x^2}=\cos\varphi$ | $\sqrt{1-y^2}=\sin\varphi$

Definiciono područje f-je $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$ je $\{(x,y)\,|\,-1\le x\le 1 \ ; \ -1\le y\le 1\}$

zbog simetričnosti četiri dijela

$$P(S)=\int_C\left(\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}\right)dS=4\int_0^{\frac{\pi}{2}}(1+\sin\varphi+\cos\varphi)\,d\varphi=$$

$$=4\left[\varphi\Big|_0^{\frac{\pi}{2}}-\cos\varphi\Big|_0^{\frac{\pi}{2}}+\sin\varphi\Big|_0^{\frac{\pi}{2}}\right]=4\left(\frac{\pi}{2}+1+1\right)=2\pi+8$$

Izračunati površinu cilindra $x^2+y^2=R^2$ između ravni $z=0$ i površi $z=R+\frac{x^2}{R}$.