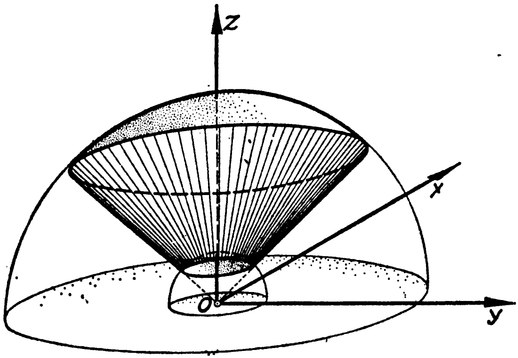
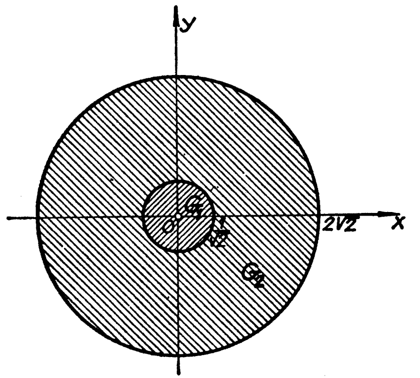
Izračunati zapreminu tela koje ograničavaju površi

$$z=\sqrt{1-x^2-y^2}, \quad z=\sqrt{16-x^2-y^2} \quad \text{i} \quad \sqrt{x^2+y^2}=z.$$

Rešenje Primenom trojnog integrala biće

$$V=\iiint_{\Phi} dx\,dy\,dz,$$ gde je Φ deo prostora čiju zapreminu tražimo.





Prelaskom na cilindrične koordinate $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$ biće: Jakobijanova determinanta $J=\rho$, jednačine površi koje ograničavaju prostor Φ

$$z_1=\sqrt{1-x^2-y^2} \Rightarrow z_1=\sqrt{1-\rho^2}$$

$$z_2=\sqrt{16-x^2-y^2} \Rightarrow z_2=\sqrt{16-\rho^2}$$

$$z_3=\sqrt{x^2+y^2} \Rightarrow z_3=\rho\,;$$

presek površi z_1 i z_3 je kružnica poluprečnika $r=\frac{1}{\sqrt{2}}$, a z_2 i z_3 kružnica poluprečnika $r=2\sqrt{2}$, pa je

$$V=\int_0^{2\pi} d\varphi \int_0^{1/\sqrt{2}} \rho\, d\rho \int_{z_1}^{z_2} dz+\int_0^{2\pi} d\varphi \int_{1/\sqrt{2}}^{2\sqrt{2}} \rho\, d\rho \int_{z_3}^{z_2} dz=$$

(*) $$=\int_0^{2\pi} d\varphi \int_0^{1/\sqrt{2}} \left(\sqrt{16-\rho^2}-\sqrt{1-\rho^2}\right) \rho\, d\rho+\int_0^{2\pi} d\varphi \int_{1/\sqrt{2}}^{2\sqrt{2}} \left(\sqrt{16-\rho^2}-\rho\right) \rho\, d\rho=$$

$$=2\pi\left[\int_0^{2\sqrt{2}} \sqrt{16-\rho^2}\cdot\rho\, d\rho-\int_0^{1/\sqrt{2}} \sqrt{1-\rho^2}\cdot\rho\, d\rho-\int_{1/\sqrt{2}}^{2\sqrt{2}} \rho^2\, d\rho\right]=$$

$$=21\pi(2-\sqrt{2}),$$

Primenom dvojnog integrala tražena zapremina jednaka je

$$V=\iint_{G_1}(z_2-z_1)\, dx\, dy+\iint_{G_2}(z_2-z_3)\, dx\, dy \quad \text{(Vidi (*)!).}$$

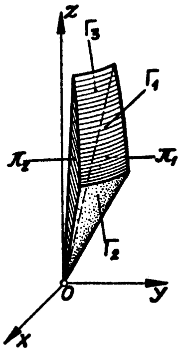
\# Date su površi

$$\Gamma_1:\ z=2-3\,(x^2+y^2)$$

$$\Gamma_2:\ z^2=x^2+y^2$$

$$\Gamma_3:\ z^2=3\,(x^2+y^2)$$

i ravni π_1: $y=x$, π_2: $y=\dfrac{x}{\sqrt{3}}$



Izračunati zapreminu tela ograničenog delovima površi $\Gamma_1, \Gamma_2, \Gamma_3$ i datim ravnima u prvom oktantu.

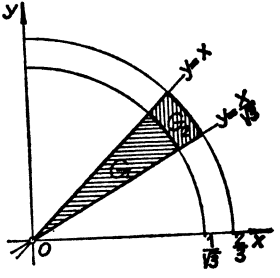
Rešenje. Primenom trojnog integrala imamo

$$V=\iiint\limits_{\Phi} dx\,dy\,dz=\iint\limits_{G_1} dx\,dy\int_{z_2}^{z_3} dz+\iint\limits_{G_2} dx\,dy\int_{z_2}^{z_1} dz=$$

$$=(\sqrt{3}-1)\iint\limits_{G_1}\sqrt{x^2+y^2}\,dx\,dy+\iint\limits_{G_2}\left[2-3\,(x^2+y^2)-\sqrt{x^2+y^2}\right]dx\,dy=$$

$$=(\sqrt{3}-1)\int_{\pi/6}^{\pi/4} d\varphi\int_0^{1/\sqrt{3}}\rho^2\,d\rho+$$

$$+\int_{\pi/6}^{\pi/4} d\varphi\int_{1/\sqrt{3}}^{2/3}(2-3\rho^2-\rho)\,\rho\,d\rho=\frac{19\pi}{3888}.$$



Telo T ograničeno je površima

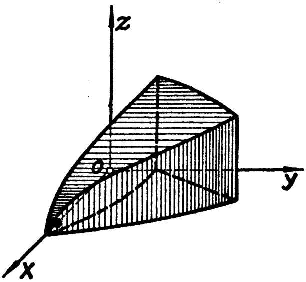
$$z^2+x^2=4 \qquad (0\leqslant x\leqslant 2;\quad z\geqslant 0);$$

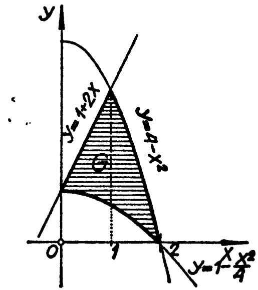
$$y=4-x^2;\quad y=1-\frac{x^2}{4};$$

$$y=1+2x;\quad z=0.$$

Skicirati sliku tela i zatim, primenom dvojnog integrala, izračunati njegovu zapreminu.

Rešenje.





$$V=\iint_G z\,dxdy=\int_0^1\sqrt{4-x^2}\,dx\int_{1-\frac{x^2}{4}}^{1+2x}dy+\int_1^2\sqrt{4-x^2}\,dx\int_{1-\frac{x^2}{4}}^{4-x^2}dy$$

$$V=\frac{64+19\pi-45\sqrt{3}}{12}.$$

Izračunati zapreminu tela koje ograničava površ

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{x}{h} \qquad (h>0).$$

Rešenje.

$$V=\iiint\limits_{\Phi} dx\,dy\,dz =$$

$$=2abc\int_0^{\pi/2} d\varphi \int_0^{\pi} \sin\psi\, d\psi \int_0^{\sqrt[3]{\frac{a\cos\varphi\,\sin\psi}{h}}} \rho^2\, d\rho =$$

$$=\frac{2abc}{3}\int_0^{\pi/2} d\varphi \int_0^{\pi} \sin\psi\, \frac{a\cos\varphi\,\sin\psi}{h}\, d\psi =$$

$$=\frac{2a^2bc}{3h}\int_0^{\pi/2} \cos\varphi\, d\varphi \int_0 \sin^2\psi\, d\psi =$$

$$=\frac{a^2bc\pi}{3h}.$$

$$\frac{x}{a}=\rho\cos\varphi\sin\psi$$

$$\frac{y}{b}=\rho\sin\varphi\sin\psi$$

$$\frac{z}{c}=\rho\cos\psi$$

$$J=abc\,\rho^2\sin\psi$$

Jednačina površi:

$$\rho^3=\frac{a\cos\varphi\,\sin\psi}{h}.$$