Primjena trostrukog integrala

a) Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi

$$V = \iiint_{\Omega} dx\,dy\,dz$$

b) Težište $T(x_T, y_T, z_T)$ trodimenzionalnog homogenog tijela ograničenog oblašću Ω tražimo po formulama

$$x_T = \frac{1}{V}\iiint_{\Omega} x\,dx\,dy\,dz, \qquad y_T = \frac{1}{V}\iiint_{\Omega} y\,dx\,dy\,dz$$

$$z_T = \frac{1}{V}\iiint_{\Omega} z\,dx\,dy\,dz$$

Homogeno tijelo je tijelo kojem je masa jednako raspoređena u svim njegovim dijelovima

Izračunati zapreminu tijela koje je određeno oblašću $\Omega: |x+y+z|+|x-y+z|+|x+y-z|=1$.

Rj. $V=\iiint\limits_{\Omega} dx\,dy\,dz$

uvedimo smjenu
$$u=x+y+z$$
$$v=x-y+z$$
$$w=x+y-z$$

$dx\,dy\,dz = J\,du\,dv\,dw$

↑ Jakobijan

$$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$J^{-1}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \frac{I_v+II_v}{II_v+III_v}$$

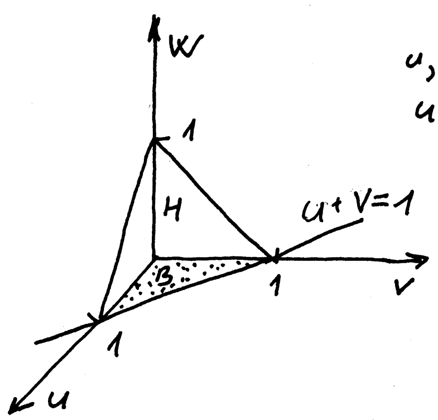
$$=\begin{vmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (-1)\cdot\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(-4)=4 \Rightarrow$$

$$\Rightarrow J=\frac{1}{4}$$

pa je
$$dx\,dy\,dz=\frac{1}{4}\,du\,dv\,dw$$

$\Omega': |u|+|v|+|w|=1$

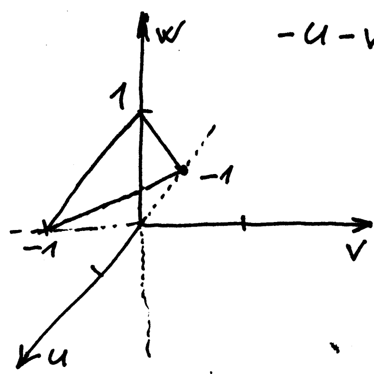
$$V=\iiint\limits_{\Omega'} \frac{1}{4}\,du\,dv\,dw$$



$u,v,w>0$
$u+v+w=1$

pored ovoga imamo još 7 slučajeva

npr. $u,v<0 \quad w>0$
$$-u-v+w=1$$



Vidimo da je dovoljno oblast integrirati u 1. oktantu jer imamo simetričnu oblast po svim oktantima.

$$V=8\cdot\frac{1}{4}\iiint\limits_{\Omega''} du\,dv\,dw =$$

$$=2\int_0^1 du\int_0^{1-u} dv\int_0^{1-u-v} dw = 2\int_0^1 du\int_0^{1-u} w\Big|_0^{1-u-v} dv =$$

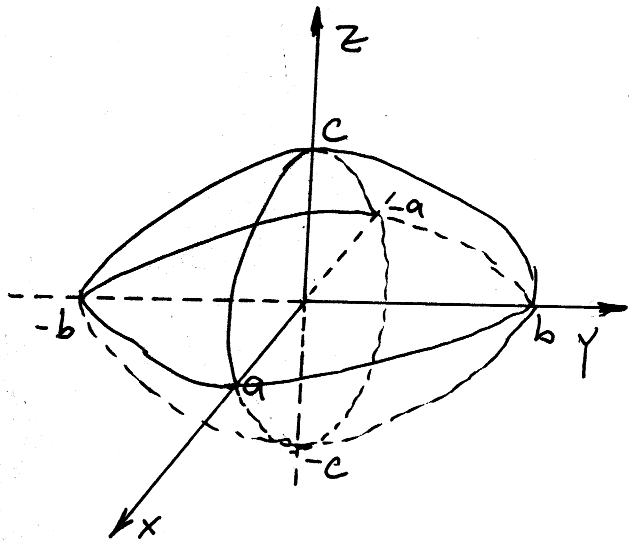
$u+v=1$
$v=1-u$
$u+v+w=1$
$w=1-u-v$

$$=2\int_0^1 du\int_0^{1-u}(1-u-v)\,dv = 2\int_0^1\left(v\Big|_0^{1-u}-uv\Big|_0^{1-u}-\frac{1}{2}v^2\Big|_0^{1-u}\right)du = \ldots = 2\cdot\frac{1}{6}=\frac{1}{3}$$

Na drugi način: $V_1=\frac{B\cdot H}{3}=\frac{\frac{1\cdot 1}{2}\cdot H}{3}=\frac{\frac{1}{2}\cdot 1}{3}=\frac{1}{6}$, $\quad V=2\cdot\frac{1}{6}=\frac{1}{3}$ zapremina tijela

Izračunati zapreminu elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Rj.



$$V=\iiint\limits_S dx\,dy\,dz$$

$$S:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

smjena: uopštene sferne koordinate

$x = a r \sin\varphi\cos\theta \qquad 0\le r\le 1$

$y = b r \sin\varphi\sin\theta \qquad 0\le\varphi\le\pi$

$z = c r\cos\varphi \qquad 0\le\theta\le 2\pi$

$$dx\,dy\,dz = J\,dr\,d\varphi\,d\theta$$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta}\end{vmatrix}=\begin{vmatrix} a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\varphi & -cr\sin\varphi & 0\end{vmatrix}$$

$$=abc\begin{vmatrix}\text{ista determinanta}\\ \text{kao kod standardnih}\\ \text{sfernih koordinata}\end{vmatrix}=abc\,r^2\sin\varphi$$

$$V=\int_0^{\pi}d\varphi\int_0^{1}dr\int_0^{2\pi}abc\,r^2\sin\varphi\,d\theta=\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2\,dr\int_0^{2\pi}abc\,d\theta=$$

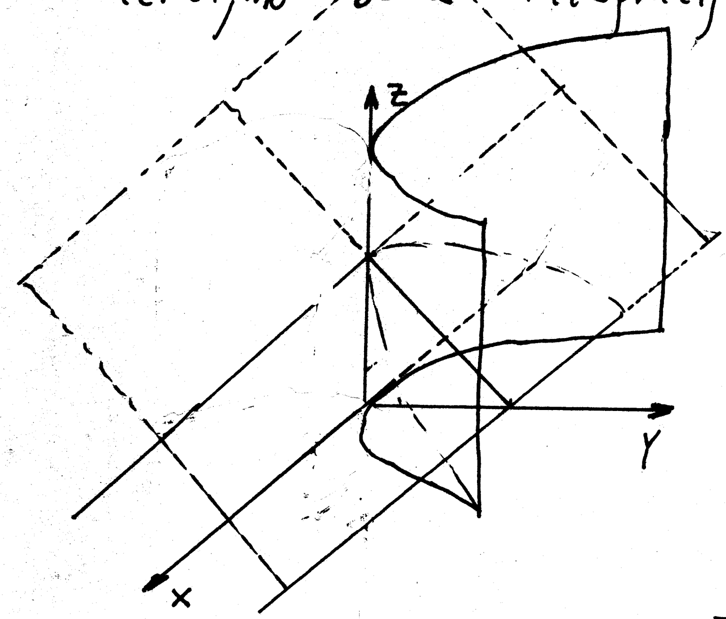
$$=abc\,\theta\Big|_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2\,dr=2\pi abc\int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^1 d\varphi=$$

$$=\frac{2}{3}\pi abc\int_0^{\pi}\sin\varphi\,d\varphi=\frac{2}{3}\pi abc\left(-\cos\varphi\Big|_0^{\pi}\right)=\frac{2}{3}\pi abc(1+1)=\frac{4}{3}\pi abc$$

q.e.d.

(#) Izračunati zapreminu tijela koje je ograničeno cilindrom $y=2x^2$ i ravnima $y+z=8$, $z=0$.

Rj. Nacrtajmo oblast integracije $\quad \Omega: \begin{cases} y=2x^2 \\ y+z=8 \\ z=0 \end{cases}$

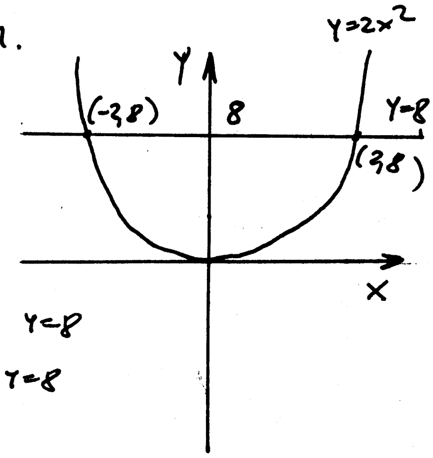


Ravan $y+z=8$ siječe cilindar

Napravimo projekciju oblasti Ω na xOy ravan.

Nađimo presjek krive $y=2x^2$ i prave $y=8$.

$$\begin{array}{l} y=2x^2 \\ \underline{y=8} \\ x^2=4 \\ x_1=-2,\ x_2=2 \end{array} \qquad \begin{array}{l} x_1=-2 \Rightarrow y=8 \\ x_2=2 \Rightarrow y=8 \end{array}$$



$$\Omega: \begin{cases} -2 \le x \le 2 \\ 2x^2 \le y \le 8 \\ 0 \le z \le 8-y \end{cases} \qquad V=\iiint_{\Omega} dx\,dy\,dz$$

$$V=\iiint_{\Omega} dx\,dy\,dz = \int_{-2}^{2} dx \int_{2x^2}^{8} dy \int_{0}^{8-y} dz = \int_{-2}^{2} dx \int_{2x^2}^{8} z\Big|_0^{8-y} dy = \int_{-2}^{2} dx \int_{2x^2}^{8} (8-y)\,dy =$$

$$= \int_{-2}^{2} \left(8y\Big|_{2x^2}^{8} - \frac{1}{2}y^2\Big|_{2x^2}^{8}\right) dx = \int_{-2}^{2} \left[8(8-2x^2) - \frac{1}{2}(8^2-4x^4)\right] dx =$$

$$= \int_{-2}^{2} (64-16x^2-32+2x^4)\,dx = \int_{-2}^{2} (2x^4-16x^2+32)\,dx =$$

$$= 2\cdot\frac{1}{5}x^5\Big|_{-2}^{2} - 16\cdot\frac{1}{3}x^3\Big|_{-2}^{2} + 32x\Big|_{-2}^{2} = \frac{2}{5}\cdot 64 - \frac{16}{3}\cdot 16 + 32\cdot 4 =$$

$$= \frac{384-1280+1920}{15} = \frac{1024}{15}$$

Izračunati zapreminu tijela ograničenog ravninom XOY, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

Rj. Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi $V=\iiint\limits_{\Omega} dx\,dy\,dz$. Pokušajmo skicirati tijelo čiju zapreminu tražimo.

valjak $x^2+y^2=2ax$

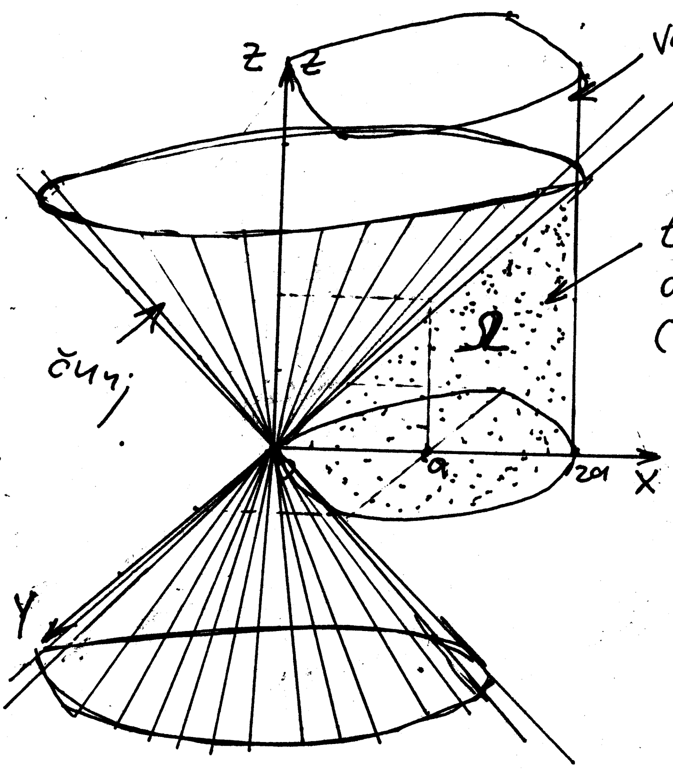
$$x^2-2ax+y^2=0$$

$$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$$

$$(x-a)^2+y^2=a^2$$

valjak u presjeku sa XOY ravni je krug sa centrom u tački $(a,0)$ poluprečnika a

čunj $x^2+y^2=z^2$ u presjeku sa XOY ravni je tačka, a u presjeku sa YOZ ili sa XOZ su po dvije prave



Oblast Ω je najlakše projicirati na XOY ravan.

Uvodimo cilindrične koordinate

$$x=a+r\cos\varphi$$
$$y=r\sin\varphi$$
$$z=z$$

$$dx\,dy\,dz=r\,dr\,d\varphi\,dz$$

$$\Omega': \begin{cases} 0\le r\le a \\ 0\le\varphi\le 2\pi \end{cases}$$

$$x^2+y^2=z^2$$

$$z=\pm\sqrt{x^2+y^2} \quad \text{čunj}$$

$$x^2+y^2=(a+r\cos\varphi)^2+(r\sin\varphi)^2=$$

$$=a^2+2ar\cos\varphi+r^2\cos^2\varphi+r^2\sin^2\varphi=$$

$$=a^2+2ar\cos\varphi+r^2$$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=\int_0^a dr\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2+2ar\cos\varphi+r^2}} r\,dz = \ldots$$ ao je teško izračunati

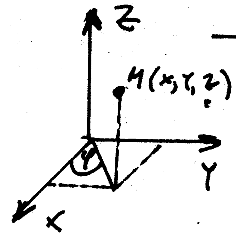
Pokušajmo uvesti drugačije smjene.

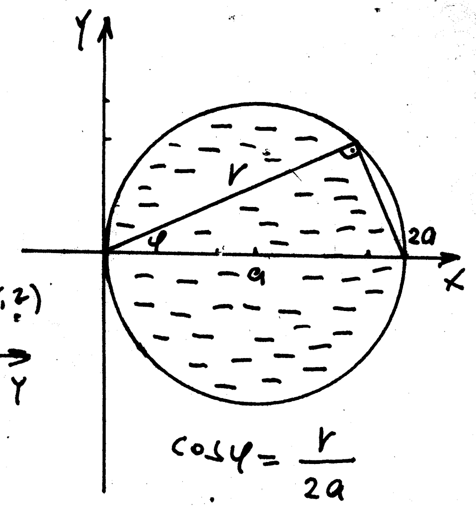
$x = r\cos\varphi$ $\qquad dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$y = r\sin\varphi$

$z = z$ $\qquad x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$

$$\Omega'': \begin{cases} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2a\cos\varphi \\ 0 \le z \le \sqrt{r^2} \end{cases}$$





$\cos\varphi = \frac{r}{2a}$

$r = 2a\cos\varphi$

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iiint\limits_{\Omega''} r\,dr\,d\varphi\,dz =$$

$$= \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} dr \int_0^{r} r\,dz = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} \left(r z\Big|_0^r\right) dr = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} r^3\Big|_0^{2a\cos\varphi}\right) d\varphi = \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\varphi\,d\varphi = \frac{8}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi$$

$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\cos^2\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi(1-\sin^2\varphi)\,d\varphi = \left| \begin{array}{l} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi = -\frac{\pi}{2} \Rightarrow t = -1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array} \right|$$

$$= \int_{-1}^{1} (1-t^2)\,dt = t\Big|_{-1}^{1} - \frac{1}{3}t^3\Big|_{-1}^{1} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$V = \dfrac{32}{9} a^3$ tražena zapremina

II način: $V = \iint\limits_{S} f(x,y)\,dx\,dy$ $\quad$ uvedimo smjene $\quad x = r\cos\varphi$

$y = r\sin\varphi$

$0 \le \varphi \le 2a\cos\varphi$

$-\frac{\pi}{2} \le r \le \frac{\pi}{2}$

$$V = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr$$

ZAVRŠITI ZA VJEŽBU

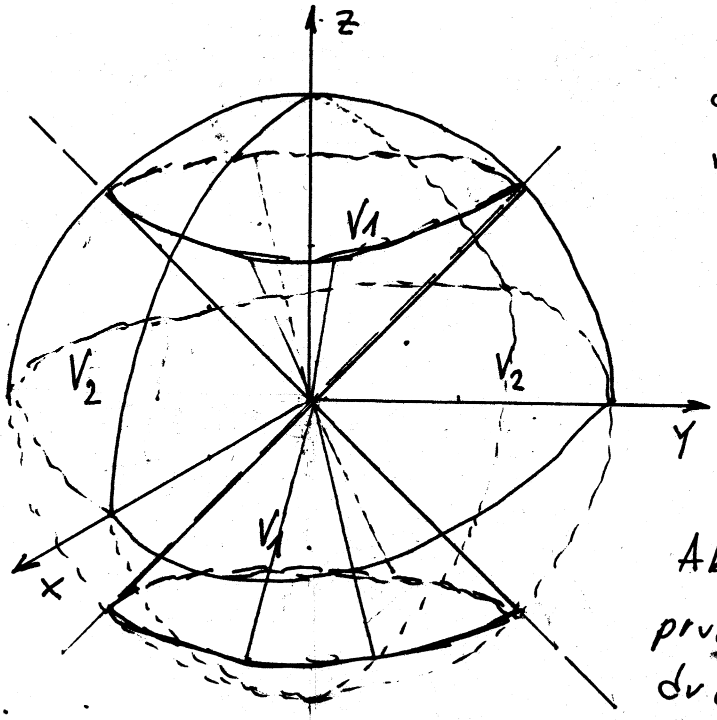
Izračunati zapreminu tijela koje je ograničeno površima $z^2 = x^2 + y^2$, $x^2+y^2+z^2=4$.

Rj.

$x^2+y^2+z^2=4$ je kugla sa centrom u $(0,0,0)$ poluprečnika $r=2$

$z^2 = x^2+y^2$ je konus

Skicirajmo ove dvije figure u prostoru.



Presjek konusa i kugle daje dva tijela za koje možemo računati zapreminu: prvo tijelo je određeno u presjeku unutrašnjosti konusa i kugle, a drugo tijelo je određeno dijelom lopte van konusa.

Ako sa V_1 označimo zapreminu prvog, a sa V_2 zapreminu drugog tijela, imamo da je

$$V = V_1 + V_2 = \frac{4}{3} r^3 \pi \quad (\text{zapremina kugle})$$

Kako je $r=2 \Rightarrow V = \frac{4}{3}\cdot 8\pi = \frac{32\pi}{3}$.

$V = \iiint\limits_{\Omega} dx\,dy\,dz$ – zapremina tijela ograničenog sa oblašću Ω

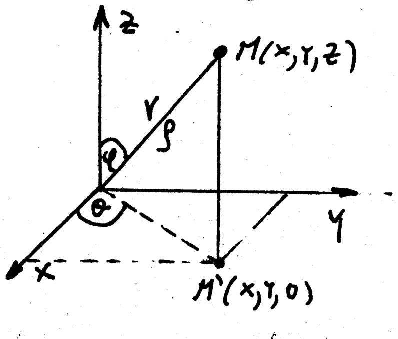
Uvedimo sferne koordinate

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$dx\,dy\,dz = \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$



$z^2 = x^2+y^2$

$\rho^2\cos^2\varphi = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta =$

$= \rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta) = \rho^2\sin^2\varphi$

$\Rightarrow \cos^2\varphi = \sin^2\varphi \quad /:\sin^2\varphi$

$\operatorname{tg}^2\varphi = 1 \Rightarrow \operatorname{tg}\varphi = \pm 1$

$x^2+y^2+z^2=4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = \pm 2$ tj. $\rho = 2$ (udaljenost tačke)

$$\Omega: \begin{cases} z^2 = x^2+y^2 \\ x^2+y^2+z^2=4 \end{cases} \xrightarrow{\text{transformacije}} \Omega': \begin{cases} \operatorname{tg}\varphi = \pm 1 \\ \rho = 2 \end{cases}$$

Odredimo granice za drugo tijelo $\Omega'_{V_2}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \end{cases}$

$$V_2 = \iiint\limits_{\Omega'_{V_2}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi \, d\varphi \int_0^2 \rho^2 \, d\rho =$$

$$= 2\pi \cdot (-\cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \frac{1}{3}\rho^3\Big|_0^2 = 2\pi\left(-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} =$$

$$= 2\pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 2\pi\sqrt{2} \cdot \frac{8}{3} = \frac{16\pi\sqrt{2}}{3}$$ traženo rješenje

Zapreminu V_1 sad možemo odrediti na dva načina

I način:

$$V = V_1 + V_2 = \frac{32\pi}{3} \Rightarrow V_1 = \frac{32\pi}{3} - V_2 = \frac{32\pi}{3} - \frac{16\pi\sqrt{2}}{3}$$

$$V_1 = \frac{16\pi}{3}(2 - \sqrt{2})$$ traženo rješenje

II način:

Ako uzmemo u obzir simetričnost date oblasti Ω' u odnosu na xOy-ravan, možemo računati polovinu zapremine V_1 za $z \ge 0$ i tada bi trebalo odabrati sljedeće granice $\Omega'_{V_1}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{cases}$

$$V_1 = \iiint\limits_{\Omega'_{V_1}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$\frac{1}{2}V_1 = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^2 \rho^2 \, d\rho = 2\pi(-\cos\varphi)\Big|_0^{\pi/4} \cdot \frac{\rho^3}{3}\Big|_0^2 =$$

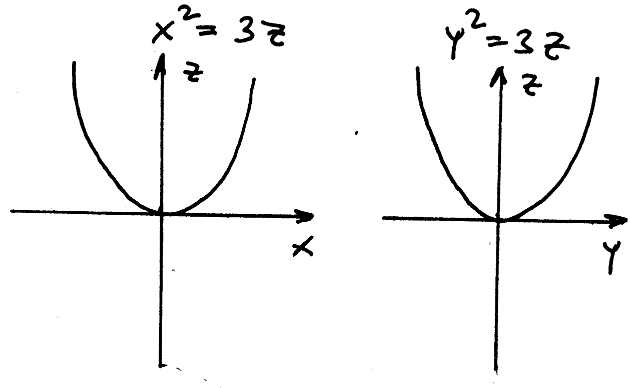
$$= 2\pi\left(1 - \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3}$$

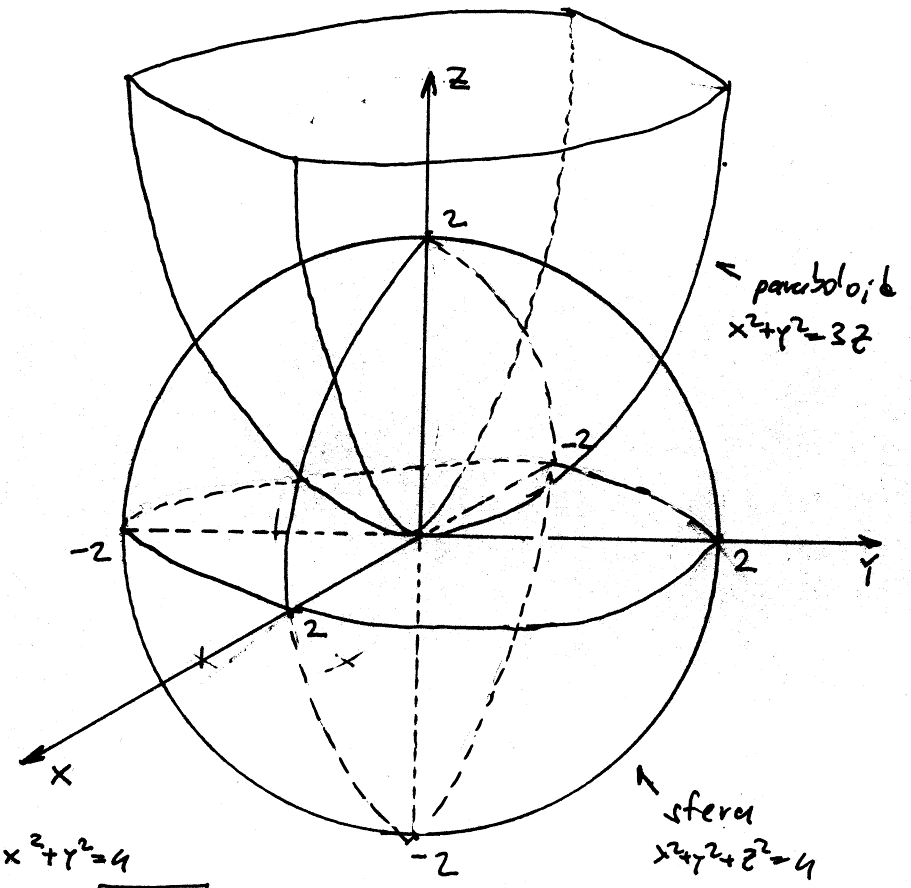
$$\Rightarrow V_1 = 4\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 4\pi \cdot \frac{2-\sqrt{2}}{2} \cdot \frac{8}{3} = \frac{16\pi}{3}(2 - \sqrt{2}).$$

Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u $(0,0,0)$ poluprečnika 2
$x^2+y^2=3z$ je paraboloid

Skicirajmo ova dva tijela





$x^2+y^2=4$
$x=\pm\sqrt{4-y^2}$
$y=\pm\sqrt{4-x^2}$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz$$

Primetimo da je tijelo dobijeno presjekom simetrično na ravni xOz i na yOz.
Prema tome

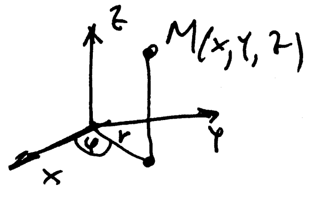
$$V=4\iiint\limits_{\Omega_1} dx\,dy\,dz \quad \text{gdje je}$$

Ω_1 oblast u presjeku dva tijela u prvom oktantu

$$\Omega_1:\begin{cases}0\le x\le 2\\ 0\le y\le\sqrt{4-x^2}\\ 0\le z\le \frac{1}{3}(x^2+y^2)\end{cases}$$

$$V=4\int_0^2 dx\int_0^{\sqrt{4-x^2}} dy\int_0^{\frac{1}{3}(x^2+y^2)} dz=4\int_0^2 dx\int_0^{\sqrt{4-x^2}}\frac{1}{3}(x^2+y^2)\,dy$$

$$=\frac{4}{3}\int_0^2\left(x^2y\Big|_0^{\sqrt{4-x^2}}+\frac{1}{3}y^3\Big|_0^{\sqrt{4-x^2}}\right)dx=\underset{\text{komplikovano}}{\ldots}=\frac{8\pi}{3}$$



II način:
Uvedimo cilindrične koordinate $x=r\cos\varphi$, $y=r\sin\varphi$, $z=z$

$dx\,dy\,dz=r\,dr\,d\varphi\,dz$

Oblast $\Omega_1 \overset{\text{transformise}}{\rightsquigarrow} \Omega_1' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \frac{1}{3} r^2 \end{cases}$

$$V = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 dr \int_0^{\frac{1}{3}r^2} r\,dz = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 r\cdot\frac{1}{3}r^2\,dr = \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4}r^4\Big|_0^2 d\varphi = \frac{1}{3}\cdot 16\cdot\frac{\pi}{2} = \frac{8\pi}{3}$$

$V = \dfrac{8\pi}{3}$ tražena zapremina

Zadaci za vježbu

1. Izračunajte $\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz$ transformiravši ga prethodno na cilindarske koordinate.
2. Izračunati $\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,dz$ transformiravši ga prethodno na sferne koordinate.
3. Izračunajte zapreminu tijela ograničenog ravninom xOY, valjkom $x^2+y^2=ax$ i kuglom $x^2+y^2+z^2=a^2$ (unutrašnjeg s obzirom na valjak.
4. Izračunajte volumen tijela omeđenog paraboloidom $\frac{y^2}{b^2}+\frac{z^2}{c^2}=2\frac{x}{a}$ i ravninom $x=a$.

Rj. 1. $\frac{8}{9}a^2$ 2. $\frac{4}{15}\pi R^5$ 3. $\frac{a^3}{9}(3\pi-4)$ 4. πabc