\# Površi

$$\Gamma_1:\ x^2+y^2=4 \qquad (x\geqslant 0)$$

$$\Gamma_2:\ x^2-y^2=1 \qquad (x\geqslant 1)$$

$$\Gamma_3:\ z=4-x^2 \qquad (z\geqslant 0)$$

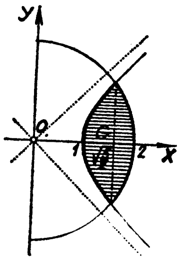
i koordinatna ravan $z=0$ ograničavaju telo T. Izračunati zapreminu tela i nacrtati sliku.

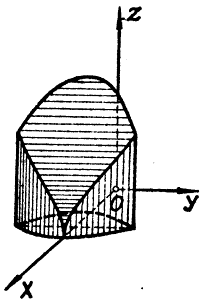
Rešenje.

$$V=\iint_G z\,dx\,dy=\iint_G (4-x^2)\,dx\,dy=$$

$$=2\left[\int_1^{\sqrt{5/2}}(4-x^2)\,dx\int_0^{\sqrt{x^2-1}}dy+\int_{\sqrt{5/2}}^{2}(4-x^2)\,dx\int_0^{\sqrt{4-x^2}}dy=\right.$$

$$=2\left[\frac{3\sqrt{15}}{4}-\frac{15}{8}\ln\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}+3\pi-6\arcsin\frac{\sqrt{5}}{2\sqrt{2}}-\frac{15}{16}\sqrt{15}\right].$$



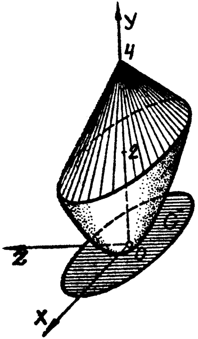


\# Zatvorena površ definisana je jednačinama

$$x^2+4z^2=2y, \quad x^2+4z^2-(y-4)^2=0 \qquad (0\leqslant y\leqslant 4).$$

Izračunati zapreminu tela koje te površi ograničavaju.

Rešenje.



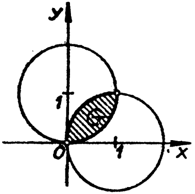
$$V=\iint\limits_G (y_2-y_1)\,dx\,dz \qquad G:\ \frac{x^2}{4}+z^2\leqslant 1.$$

$$=\iint\limits_G \left(4-\sqrt{x^2+4z^2}-\frac{x^2+4z^2}{2}\right)dx\,dz \qquad \begin{aligned} x&=2\rho\cos\varphi\\ z&=\rho\sin\varphi\\ J&=2\rho.\end{aligned}$$

$$=\int\limits_0^{2\pi} d\varphi \int\limits_0^1 (4-2\rho-2\rho^2)\,2\rho\,d\rho=\frac{10\pi}{3}.$$

\# Izračunati zapreminu tela ograničenog cilindrima

$$x^2+y^2=2x, \quad x^2+y^2=2y \quad \text{i ravnima} \quad z=x+2y, \quad z=0.$$



Rešenje.

$$V=\iint_G z\,dx\,dy=\iint_G (x+2y)\,dx\,dy=$$

$$=\int_0^{\pi/4}(\cos\varphi+2\sin\varphi)\,d\varphi\int_0^{2\sin\varphi}\rho^2\,d\rho+\int_{\pi/4}^{\pi/2}(\cos\varphi+2\sin\varphi)\,d\varphi\int_0^{2\cos\varphi}\rho^2\,d\rho=$$

$$=\frac{3\pi}{4}-\frac{3}{2}.$$

Naći zapreminu tela ograničenog cilindrom $xy=1$ i ravni $x+y=\frac{5}{2}$ između koordinatne ravni $z=0$ i površi $z=\frac{\sqrt{2}}{2}\ln\frac{x}{y}$.

Rešenje. Funkcija $z=\frac{\sqrt{2}}{2}\ln\frac{x}{y}$ u oblasti $xy\geqslant 1$, $x+y\leqslant\frac{5}{2}$ menja znak te je tražena zapremina

$$V=\frac{\sqrt{2}}{2}\iint_G\left|\ln\frac{x}{y}\right|dx\,dy=$$

$$=\frac{\sqrt{2}}{2}\left(-\iint_{G_1}\ln\frac{x}{y}\,dx\,dy+\iint_{G_2}\ln\frac{x}{y}\,dx\,dy\right)=$$

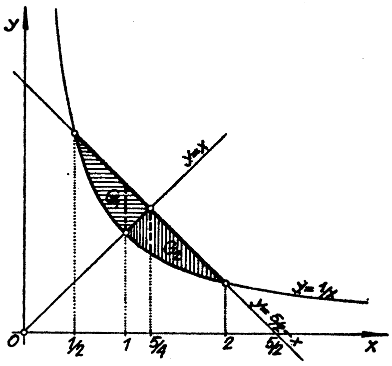
$$=\frac{\sqrt{2}}{2}\left[-\int_{\frac{1}{2}}^{1}dx\int_{1/x}^{5/2-x}(\ln x-\ln y)\,dy-\right.$$

$$-\int_{1}^{5/4}dx\int_{x}^{5/2-x}(\ln x-\ln y)\,dy+$$

$$+\int_{1}^{5/4}dx\int_{1/x}^{x}(\ln x-\ln y)\,dy+$$

$$\left.+\int_{5/4}^{2}dx\int_{1/x}^{5/2-x}(\ln x-\ln y)\,dy\right]=$$

$$=5\sqrt{2}\ln 2-\frac{25\sqrt{2}}{16}\ln 5-\sqrt{2}\ln^2 2.$$



\# Izračunati zapreminu tela ograničenog površima

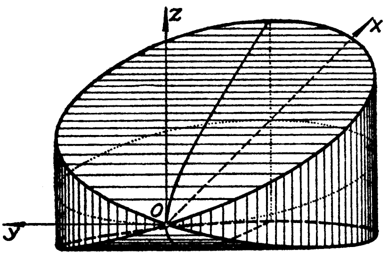
$$\left(\frac{x-a}{a}\right)^2+\left(\frac{y}{b}\right)^2=1, \quad z^2=x.$$

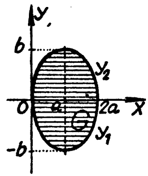
Rešenje. Zbog simetrije u odnosu na ravan xOy možemo pisati

$$V=2\iint_G z\,dx\,dy=2\iint_G \sqrt{x}\,dx\,dy=$$

$$=2\int_0^{2a}\sqrt{x}\,dx\int_{y_1}^{y_2}dy=4\int_0^{2a}\sqrt{x}\,dx\int_0^{y_2}dy=$$

$$=\frac{4b}{a}\int_0^{2a}x\sqrt{2a-x}\,dx=\frac{64\,ab\sqrt{2a}}{15}.$$

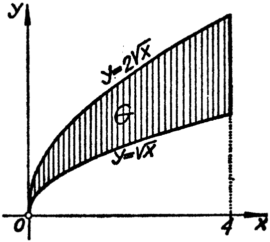




\# Izračunati zapreminu tela ograničenog površima

$$y^2 = x, \quad y^2 = 4x, \quad z = 0, \quad x + z = 4, \quad y > 0.$$

Rešenje.



$$V = \iint_G z \, dx\, dy = \iint_G (4-x)\, dx\, dy =$$

$$= \int_0^4 (4-x)\, dx \int_{\sqrt{x}}^{2\sqrt{x}} dy = \frac{128}{15}.$$