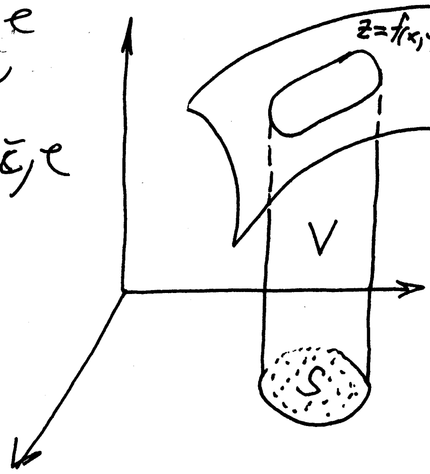
Primjena dvostrukog integrala

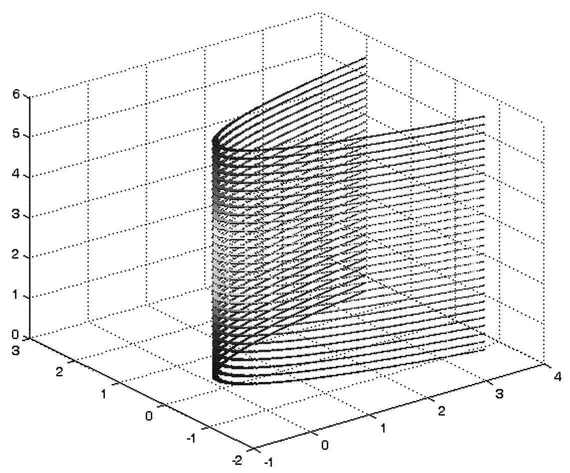
1° Površina zatvorene i ograničene oblasti D

$$P = \iint_D dx\,dy$$

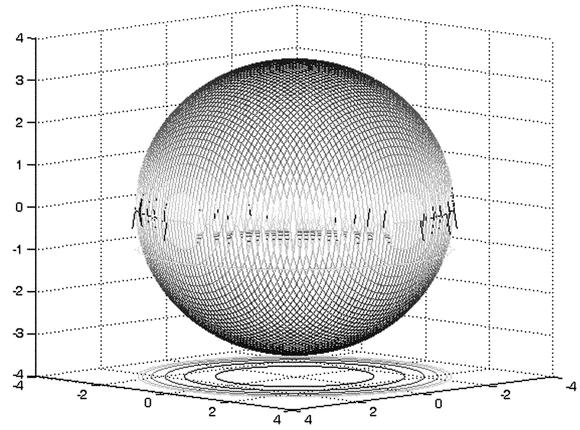
2° Zapremina tijela koje odozgo određuje površ $z = f(x,y)$, odozdo ravan $z=0$ a postranice valjkasta ploha koja na ravni XOY izrezuje omeđeno zatvoreno područje S iznosi



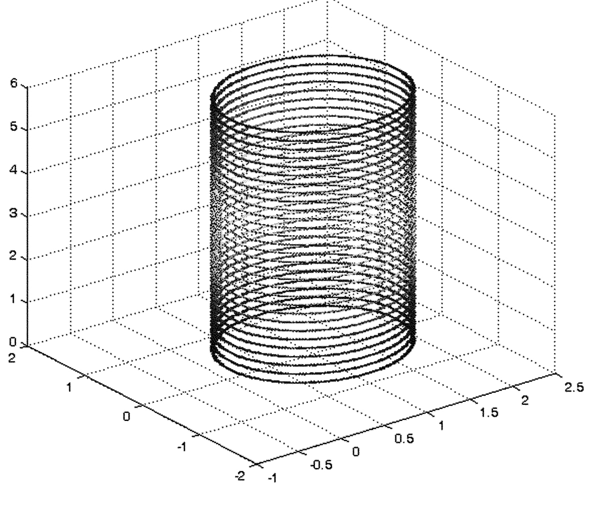
$$V = \iint_S f(x,y)\,dx\,dy$$



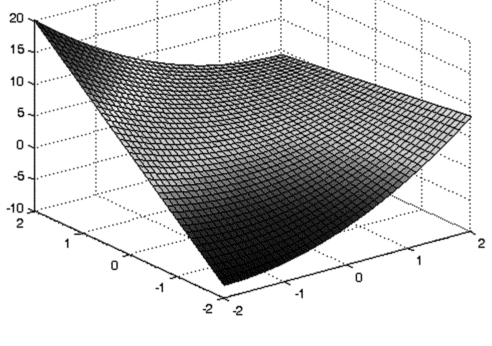
cilindar $x = 2y^2$



kugla $x^2 + y^2 + z^2 = 12$



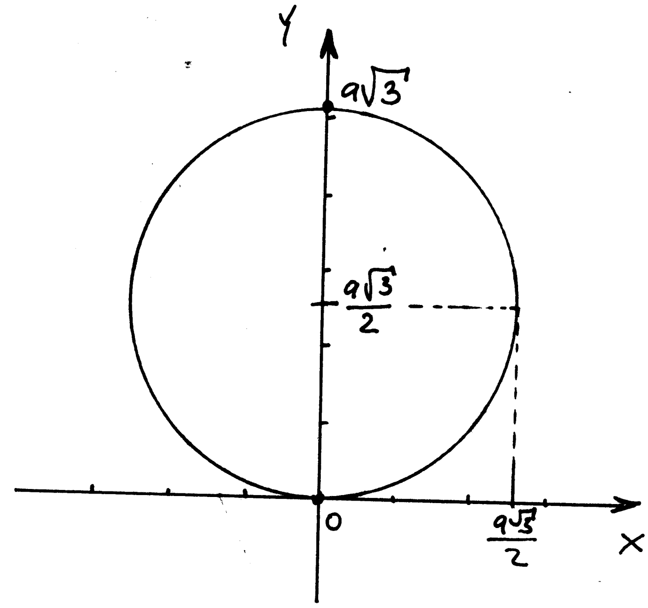
valjak $x^2 + y^2 = 2x$



funkcija $z = x^2 - 2xy + 3y + 2$

Izračunati površinu figure koja je ograničena linijom $x^2+y^2=a\sqrt{3}\,y$.

Rj. $P=\iint_D dx\,dy$



$x^2+y^2=a\sqrt{3}\,y$

$x^2+y^2-a\sqrt{3}\,y=0$

$x^2+y^2-2\cdot\frac{a\sqrt{3}}{2}y+\frac{a^2\cdot 3}{4}-\frac{3a^2}{4}=0$

$x^2+\left(y-\frac{a\sqrt{3}}{2}\right)^2=\left(\frac{a\sqrt{3}}{2}\right)^2$

krug s centrom u tački $C\left(0,\frac{a\sqrt{3}}{2}\right)$ poluprečnika $\frac{a\sqrt{3}}{2}$.

Uvodim smjene $x=r\cos\varphi$, $y=\frac{a\sqrt{3}}{2}+r\sin\varphi$, $0\le r\le\frac{a\sqrt{3}}{2}$, $0\le\varphi\le 2\pi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}$$

$\frac{\partial x}{\partial r}=\cos\varphi$ $\quad\frac{\partial x}{\partial\varphi}=-r\sin\varphi$

$\frac{\partial y}{\partial r}=\sin\varphi$ $\quad\frac{\partial y}{\partial\varphi}=r\cos\varphi$

$dx\,dy=|J|\,dr\,d\varphi$

$J=r$

$$P=\iint_D dx\,dy=\iint_{D'}|r|\,dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{a\sqrt{3}}{2}} r\,dr\right]d\varphi=$$

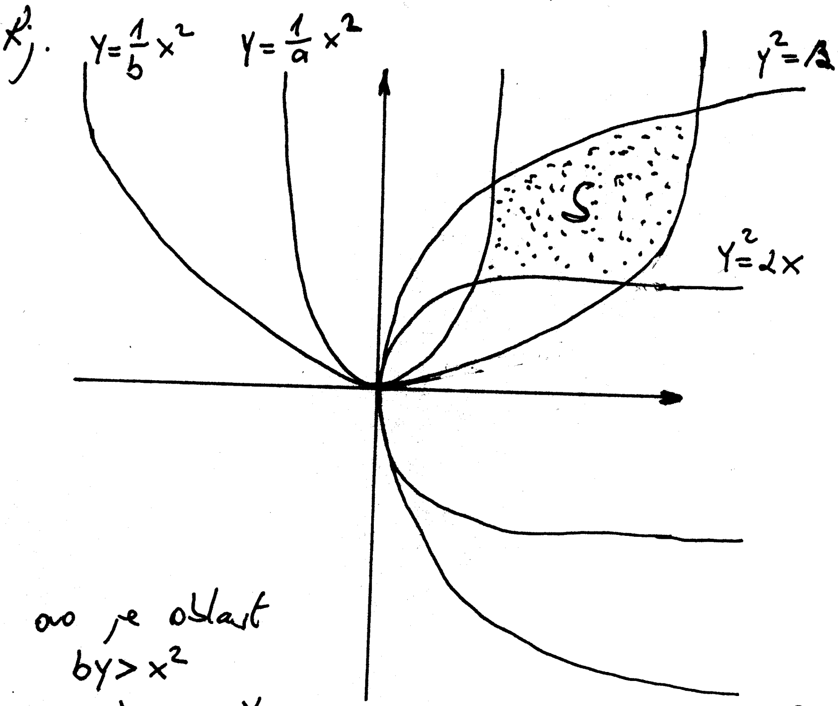
$$=\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^{\frac{a\sqrt{3}}{2}}d\varphi=\frac{a^2\cdot 3}{4}\cdot\frac{1}{2}\varphi\Big|_0^{2\pi}=\frac{3a^2}{4}\cdot\pi$$

površina figure koja je ograničena linijom

Izračunati površinu krivolinijskog 4-ugla omeđenog lukovima parabola $x^2=ay$, $x^2=by$, $y^2=\alpha x$ i $y^2=\beta x$ $(0<a<b,\ 0<\alpha<\beta)$.

$$P=\iint dx\,dy$$

Rj.



$x^2=ay$, S $x^2=by$

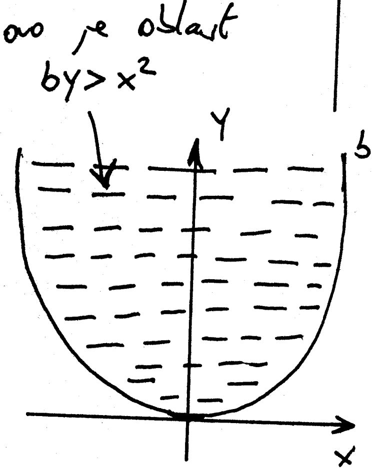
$y=\frac{1}{a}x^2$ $\qquad y=\frac{1}{b}x^2$

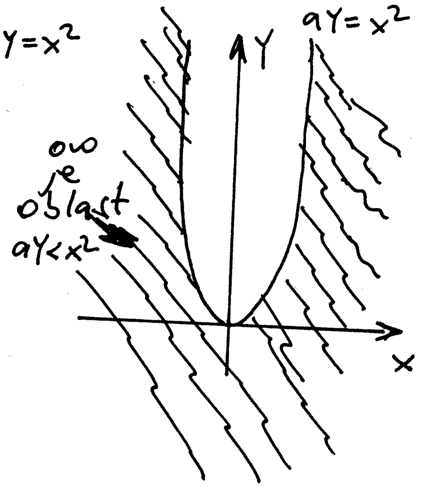
$a<b$

$\frac{1}{a}>\frac{1}{b}$ $\qquad \frac{1}{a}x^2>\frac{1}{b}x^2$

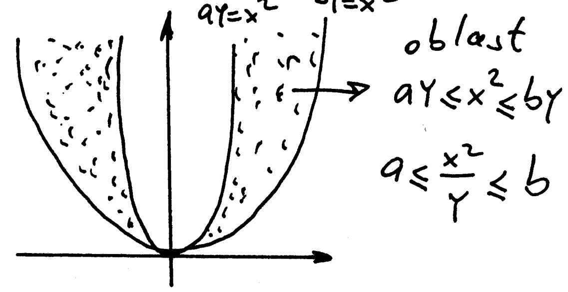
Na klasičan način površinu $\iint_S dx\,dy$ je teško izračunati.

Primjetimo sljedeće:





$\Longrightarrow$



Slično $y^2\ge\alpha x$ i $y^2\le\beta x$

imamo $\alpha x\le y^2\le\beta x$ $\qquad \alpha\le\frac{y^2}{x}\le\beta$

Vidimo da možemo uvesti smjene $u=\frac{x^2}{y}$ i $v=\frac{y^2}{x}$ gdje

$a\le u\le b$ $\qquad u=\frac{x^2}{y}$ $\qquad v=\frac{y^2}{x}$

$\alpha\le v\le\beta$ $\qquad y=\frac{x^2}{u}$, $\qquad x=\frac{y^2}{v}$ $\Longrightarrow$ $x=\frac{\left(\frac{x^2}{u}\right)^2}{v}=\frac{x^4}{u^2v}$ $\Longrightarrow$ $x^3=u^2v$, $x=\sqrt[3]{u^2v}$

$$y=\frac{x^2}{u}=\frac{\sqrt[3]{(u^2v)^2}}{u}=\sqrt[3]{\frac{u^4v^2}{u^3}}=\sqrt[3]{uv^2}$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\qquad dx\,dy=|J|\,du\,dv$$

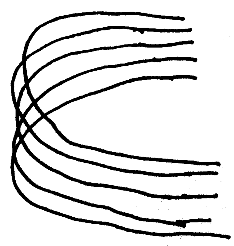
$x=u^{\frac{2}{3}}v^{\frac{1}{3}}$, $\qquad \frac{\partial x}{\partial u}=\frac{2}{3}u^{-\frac{1}{3}}v^{\frac{1}{3}}$ $\qquad \frac{\partial x}{\partial v}=u^{\frac{2}{3}}\frac{1}{3}v^{-\frac{2}{3}}$

$y=u^{\frac{1}{3}}v^{\frac{2}{3}}$ $\qquad \frac{\partial y}{\partial u}=\frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}}$ $\qquad \frac{\partial y}{\partial v}=u^{\frac{1}{3}}\frac{2}{3}v^{-\frac{1}{3}}$ $\qquad J=\frac{4}{9}-\frac{1}{9}=\frac{3}{9}=\frac{1}{3}$

$$\iint_S dx\,dy=\int_a^b\left[\int_\alpha^\beta \frac{1}{3}\,dv\right]du=\frac{1}{3}\int_a^b v\Big|_\alpha^\beta du=\frac{1}{3}(\beta-\alpha)u\Big|_a^b=\frac{1}{3}(b-a)(\beta-\alpha)$$

(#) Izračunati zapreminu tijela koje je ograničeno površinama $x=2y^2$, $x+2y+z=4$ i $z=0$.

Rj. $x=2y^2$ cilindar u prostoru

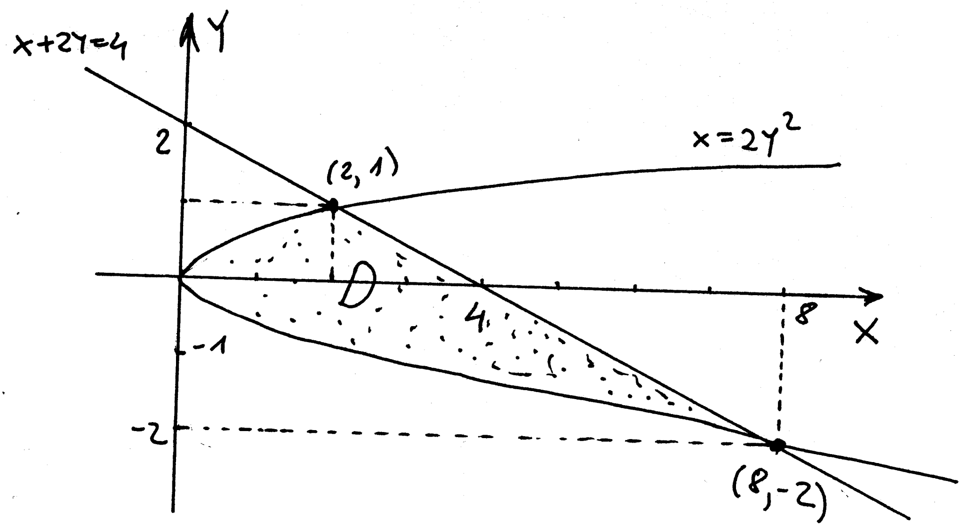


Pronađimo projekciju površina na xOy ravan:

$$x=2y^2$$
$$x+2y=4$$
$$\overline{\qquad\qquad}$$
$$2y^2+2y=4 \quad /:2$$
$$y^2+y-2=0$$
$$(y-1)(y+2)=0$$
$$y_1=1 \Rightarrow x_1=2$$
$$y_2=-2 \Rightarrow x_2=8$$

$$x+2y=4 \quad /:4$$
$$\frac{x}{4}+\frac{y}{2}=1$$

Nacrtajmo sliku



$$D: \begin{cases} -2 \le y \le 1 \\ 2y^2 \le x \le 4-2y \end{cases}$$

$$x+2y+z=4$$
$$z=4-x-2y$$

$$V=\iint\limits_D (4-x-2y)\,dx\,dy$$

$$V=\int_{-2}^{1}\left[\int_{2y^2}^{4-2y}(4-x-2y)\,dx\right]dy=\int_{-2}^{1}\left[4x\Big|_{2y^2}^{4-2y}-\frac{1}{2}x^2\Big|_{2y^2}^{4-2y}-2y\cdot x\Big|_{2y^2}^{4-2y}\right]dy=$$

$$=\int_{-2}^{1}\left[4(4-2y-2y^2)-\frac{1}{2}\left((4-2y)^2-(2y^2)^2\right)-2y(4-2y-2y^2)\right]dy$$

$$16-8y-8y^2 \qquad\qquad 16-16y+4y^2-4y^4$$

$$=\int_{-2}^{1}\left[16-\cancel{8y}-8y^2-8+\cancel{8y}-2y^2+2y^4-8y+4y^2+4y^3\right]dy=\int_{-2}^{1}(2y^4-6y^2+4y^3-8y+8)\,dy$$

$$=\frac{2}{5}y^5\Big|_{-2}^{1}-\frac{6}{3}y^3\Big|_{-2}^{1}+\frac{4}{4}y^4\Big|_{-2}^{1}-\frac{8}{2}y^2\Big|_{-2}^{1}+8y\Big|_{-2}^{1}=\frac{2}{5}\cdot 33-2\cdot 9+1\cdot(-15)-\frac{8}{2}(-3)$$

$$+8\cdot 3=\frac{66}{5}-18-15+12+24=\frac{66}{5}+36-33=\frac{66}{5}+\frac{15}{5}=\frac{81}{5}$$