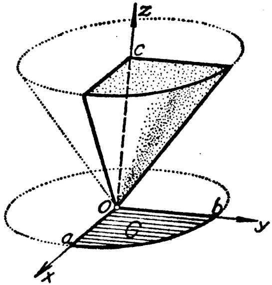
\# Izračunati integral

$$\iiint_{\Phi} \frac{xy}{\sqrt{z}}\,dx\,dy\,dz$$

gde je ϕ: $c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} \leqslant z \leqslant c$, $x \geqslant 0$, $y \geqslant 0$.

Rešenje.



$$I=\iint_G xy\,dx\,dy \int_{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}^{c} z^{-1/2}\,dz=$$

$$=2\iint_G xy\sqrt{c}\left[1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{1/4}\right]dx\,dy=$$

$$=2ab\sqrt{c}\int_0^{\pi/2} d\varphi \int_0^1 \rho^2 ab \sin\varphi\cos\varphi\,(1-\rho^{1/2})\,\rho\,d\rho=$$

$$=a^2b^2\sqrt{c}\int_0^{\pi/2}\sin 2\varphi\,d\varphi\int_0^1 (1-\rho^{1/2})\,\rho^3\,d\rho=$$

$$=\frac{a^2b^2\sqrt{c}}{36}.$$

$$\frac{x}{a}=\rho\cos\varphi$$

$$\frac{y}{b}=\rho\sin\varphi$$

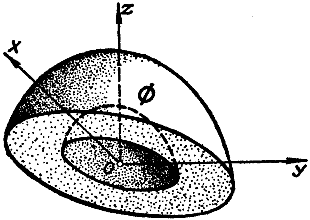
$$J=ab\rho$$

Izračunati integral

$$\iiint\limits_{\Phi} (x^2+y^2)\,dx\,dy\,dz$$

gde je ϕ: $z \geqslant 0, \quad r^2 \leqslant x^2+y^2+z^2 \leqslant R^2,$

$0 < r < R.$



Rešenje: Prelaskom na sferne koordinate: $x=\rho\cos\varphi\sin\psi$, $y=\rho\sin\varphi\sin\psi$, $z=\rho\cos\psi$, $J=\rho^2\sin\psi$, biće

$$I=\iiint\limits_{\Phi'} \rho^2\sin^2\psi\,\rho^2\,|\sin\psi|\,d\rho\,d\varphi\,d\psi \qquad \left|\begin{array}{l} x^2+y^2+z^2=R^2,\quad \rho=R \\ x^2+y^2+z^2=r^2,\quad \rho=r \end{array}\right.$$

$$=\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi/2} \sin^3\psi\,d\psi \int\limits_r^R \rho^4\,d\rho = \frac{4\pi}{15}(R^5-r^5).$$

$\#$ Izračunati integral

$$\iiint\limits_{\Phi} \ln (x^2+y^2+z^2)\, dxdydz$$

gde je ϕ: $x^2+y^2+z^2 \leqslant R^2$. $\qquad (R \leqslant 1)$

Rešenje. Podintegralna funkcija nije definisana u koordinatnom početku, pa imamo uopšteni integral druge vrste.

$$I = \lim_{\to 0} \iiint\limits_{\Phi'} \ln (x^2+y^2+z^2)\, dx\, dy\, dz$$

$$\Phi': \ \varepsilon \leqslant x^2+y^2+z^2 \leqslant R^2.$$

Prelaskom na sferne koordinate biće

$$I = \lim_{\varepsilon \to 0} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\psi\, d\psi \int_{\varepsilon}^{R} 2\rho^2 \ln \rho\, d\rho =$$

$$= 8\pi \lim_{\varepsilon \to 0} \left[\frac{R^3}{3}\left(\ln R - \frac{1}{3}\right) - \frac{\varepsilon^3}{3}\left(\ln \varepsilon - \frac{1}{3}\right)\right] = \frac{8\pi}{9} R^3 (3 \ln R - 1).$$

\# Izračunati integral

$$\iiint\limits_{\Phi} (x^2+y^2+z^2)\, dxdydz$$

gde je ϕ: $x^2+y^2+z^2 \leqslant R^2$

$x^2+y^2 \leqslant z^2, \quad z \geqslant 0.$

Rešenje. Prelaskom na sferne koordinate biće:

$$x^2+y^2+z^2=R^2 \quad \Rightarrow \quad \rho=R$$

$$x^2+y^2=z^2 \quad \Rightarrow \quad \operatorname{tg}\psi=1 \quad \Rightarrow \quad \psi=\frac{\pi}{4}.$$

pa je

$$I=\int_0^{2\pi} d\varphi \int_0^{\pi/4} \sin\psi\, d\psi \int_0^R \rho^4\, d\rho = \frac{\pi R^5}{5}(2-\sqrt{2}).$$

\# Izračunati integral

$$\iiint\limits_{\Phi} (xy+yz+zx)\,dxdydz$$

gde je ϕ: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\leqslant 1.$

Rezultat i uputstvo.

$I=0$. Koristiti uopštene sfere koordinate

$$\begin{cases} x=a\,\rho\cos\varphi\sin\psi \\ y=b\,\rho\sin\varphi\sin\psi \\ z=c\,\rho\cos\psi. \end{cases}$$

\# Izračunati integral

$$\iiint\limits_{\Phi} z^2\,dxdydz$$

gde je ϕ: $x^2+y^2+z^2\leqslant 2$

$$\sqrt{x^2+y^2}\leqslant z.$$

Rezultat. $I=\dfrac{4\pi}{15}(2\sqrt{2}-1).$

\# Odrediti srednju vrednost funkcije

$$f(x,y,z)=e^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}$$

u unutrašnjosti elipsoida

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Rezultat. $\mu=3(e-2).$