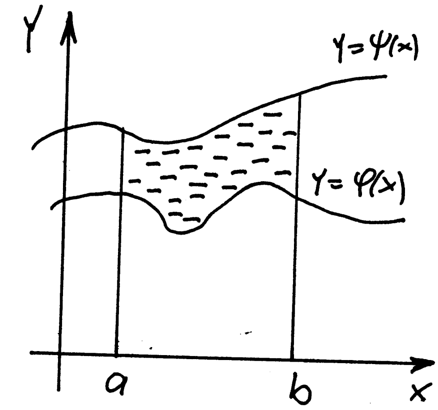
Trostruki integral

$$I = \iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz\,,$$ Ω oblast integracije u prostoru

ako je $\Omega: \begin{cases} a \le x \le b \\ \varphi(x) \le y \le \psi(x) \\ \alpha(x,y) \le z \le \beta(x,y) \end{cases}$ tada

$$I = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} dy \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz$$



Oblast Ω možemo projicirati na

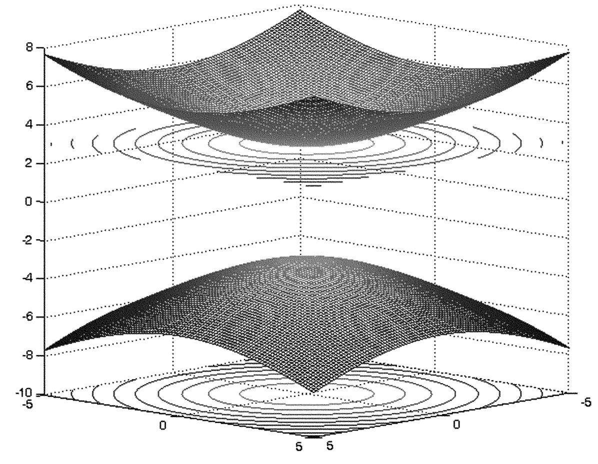
a) xoy ravan ili

b) yoz ravan ili

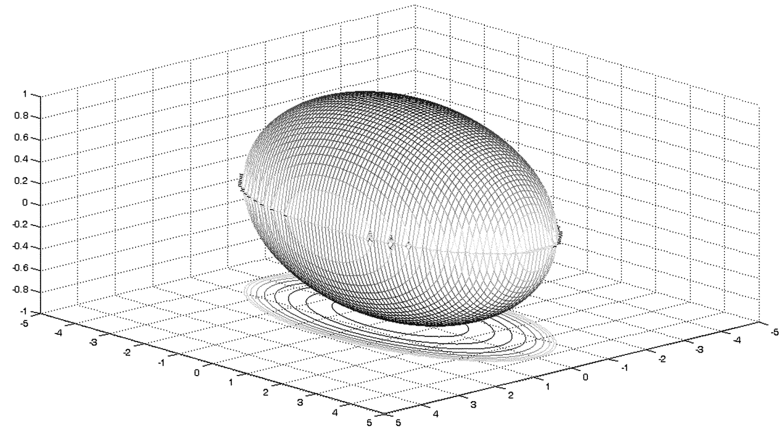
c) xoz ravan

U gornjem primjeru Ω smo prvo projicirali na xoy ravan.

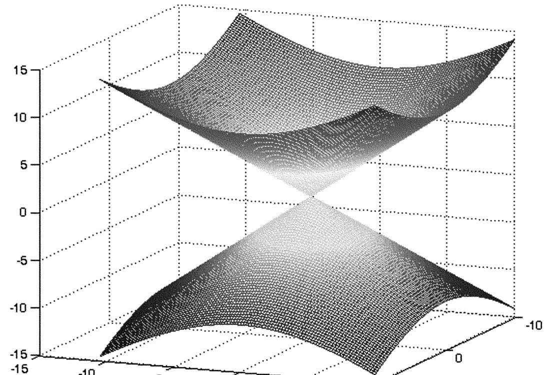
I se može izraziti na 6 načina.



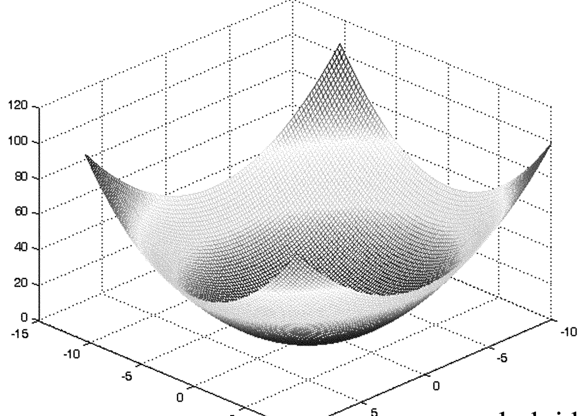
hiperboloid $x^2 + y^2 - z^2 = -9$



elipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$



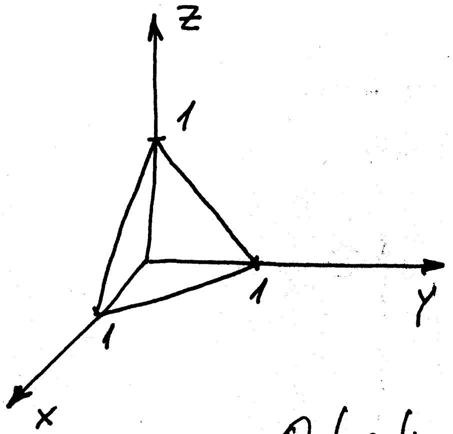
čunj $x^2 + y^2 = z^2$



paraboloid $2z = x^2 + y^2$

Izračunajte $\iiint\limits_{\Omega} (1-x)yz\,dx\,dy\,dz$ gdje je Ω oblast ograničena ravnima $x=0$, $y=0$, $z=0$ i $x+y+z=1$

Rj.



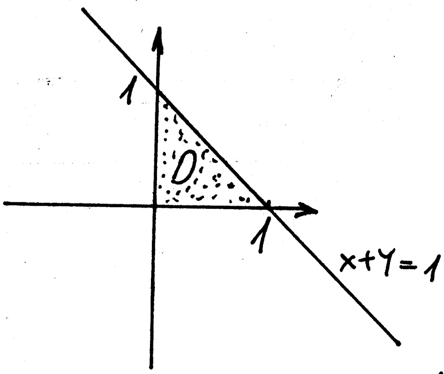
$x+y+z=1 \Leftrightarrow \frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1$ segmentni oblik jedn. ravni

$x=0$ je yoz ravan

$y=0$ je xoz ravan

$z=0$ je xoy ravan

Odredimo projekciju oblasti na xoy ravan



$$\begin{array}{c} x+y+z=1 \\ z=0 \\ \hline x+y=1 \end{array}$$

$z=1-x-y$

Sa slike odredimo granice

$0 \le x \le 1$

$0 \le y \le 1-x$

$0 \le z \le 1-x-y$

$$\iiint\limits_{\Omega}(1-x)yz\,dx\,dy\,dz = \int_0^1 (1-x)\,dx \int_0^{1-x} y\,dy \int_0^{1-x-y} z\,dz = \int_0^1 (1-x)\,dx \int_0^{1-x} y\cdot\frac{1}{2}z^2\Big|_0^{1-x-y} dy$$

$$= \frac{1}{2}\int_0^1 (1-x)\,dx \int_0^{1-x} y\cdot(\underbrace{1-x}_{a}-\underbrace{y}_{b})^2\,dy = \frac{1}{2}\int_0^1 (1-x)\,dx \int_0^{1-x} y\left[(1-x)^2-2y(1-x)+y^2\right]dy$$

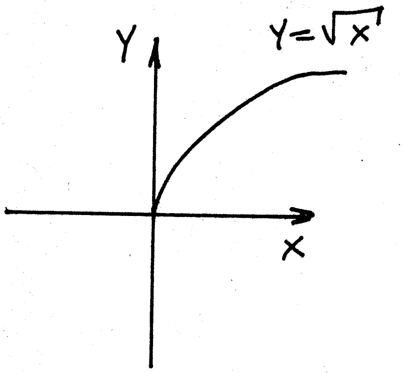
$$= \frac{1}{2}\int_0^1 (1-x)\,dx \int_0^{1-x}\left[(1-x)^2y-2y^2(1-x)+y^3\right]dy = \frac{1}{2}\int_0^1 (1-x)\Big[(1-x)^2\frac{1}{2}y^2\Big|_0^{1-x} -$$

$$- 2\cdot\frac{1}{3}y^3\Big|_0^{1-x}\cdot(1-x)+\frac{1}{4}y^4\Big|_0^{1-x}\Big]dx = \frac{1}{2}\int_0^1 (1-x)\left[(1-x)^4\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)\right]dx$$

$$= \frac{1}{2}\cdot\frac{1}{12}\int_0^1 (1-x)^5\,dx = \left|\begin{array}{ll} 1-x=t & x=0 \Rightarrow t=1 \\ -dx=dt & x=1 \Rightarrow t=0 \\ dx=-dt & \end{array}\right| = -\frac{1}{24}\int_1^0 t^5\,dt = -\frac{1}{24}\cdot\frac{1}{6}t^6\Big|_1^0 = \frac{1}{144}$$

(#) Izračunati $I=\iiint\limits_{\Omega} y\cos(x+z)\,dx\,dy\,dz$ gde je Ω oblast ograničena plohom $y=\sqrt{x}$ i ravnima $y=0$, $z=0$ i $x+z=\frac{\pi}{2}$.

Rj.

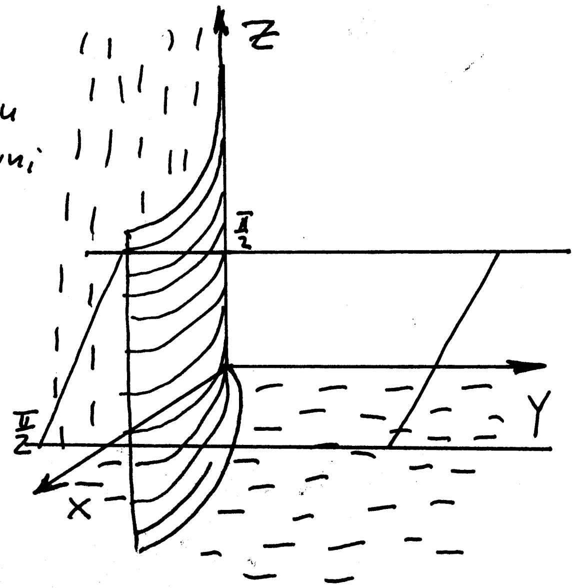


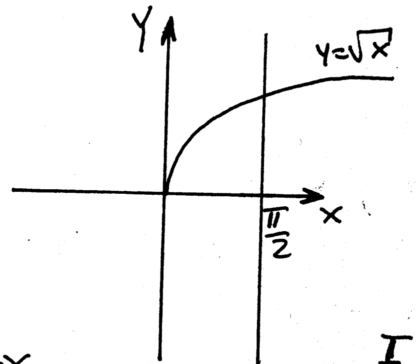
$y=0$ je xOz ravan

$z=0$ je xOy ravan

$x+z=\frac{\pi}{2}$

za $z=0$ dobiću projekciju ove ravni na xOy ravan





$0\le x\le \frac{\pi}{2}$

$0\le y\le \sqrt{x}$

$0\le z\le \frac{\pi}{2}-x$

$$I=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\,dy\int_0^{\frac{\pi}{2}-x}\cos(x+z)\,dz=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\sin(x+z)\Big|_{z=0}^{z=\frac{\pi}{2}-x}dy \overset{(*)}{=}$$

$$\boxed{\int\cos(x+a)\,dx=\left|\begin{matrix}x+a=t\\ dx=dt\end{matrix}\right|=\int\cos t\,dt=\sin t+c=\sin(x+a)+c}$$

$$\overset{(*)}{=}\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\Big[\underbrace{\sin\left(x+\tfrac{\pi}{2}-x\right)}_{1}-\sin x\Big]dy=\int_0^{\frac{\pi}{2}}\underbrace{\frac{1}{2}y^2\Big|_0^{\sqrt{x}}}_{x-0}(1-\sin x)\,dx=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}x\,dx-\frac{1}{2}\int_0^{\frac{\pi}{2}}x\sin x\,dx=\left|\begin{matrix}u=x & dv=\sin x\,dx\\ du=dx & v=-\cos x\end{matrix}\right|=\frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}}-$$

$$-\frac{1}{2}\left[-x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,dx\right]=\frac{1}{4}\cdot\frac{\pi^2}{4}-\frac{1}{2}\sin x\Big|_0^{\frac{\pi}{2}}=\frac{\pi^2-8}{16}$$

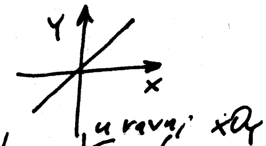
(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega} z\,dx\,dy\,dz$, ako je

$\Omega:\ y=x,\ \ y=2x,\ \ 2x=1,\ \ x^2+y^2+z^2=1,\ \ z\geq 0$

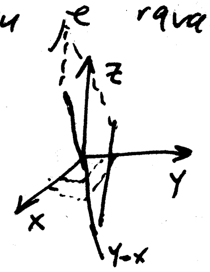
(oblast Ω je ograničena ovim površinama).

Rj: Komentarišimo površi koje čine Ω.

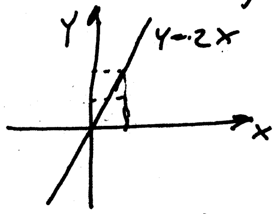
$y=x$ u ravni je prava



$y=x$ u prostoru je ravan koja sadrži pravu $y=x$

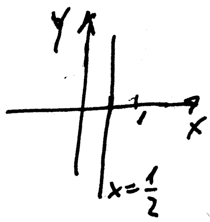


$y=2x$ u ravni je prava

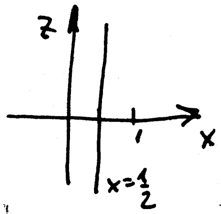


$y=2x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=2x$

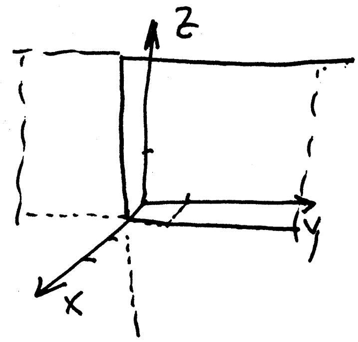
$2x=1$ u ravni xOy je prava



u ravni xOz je isto prava



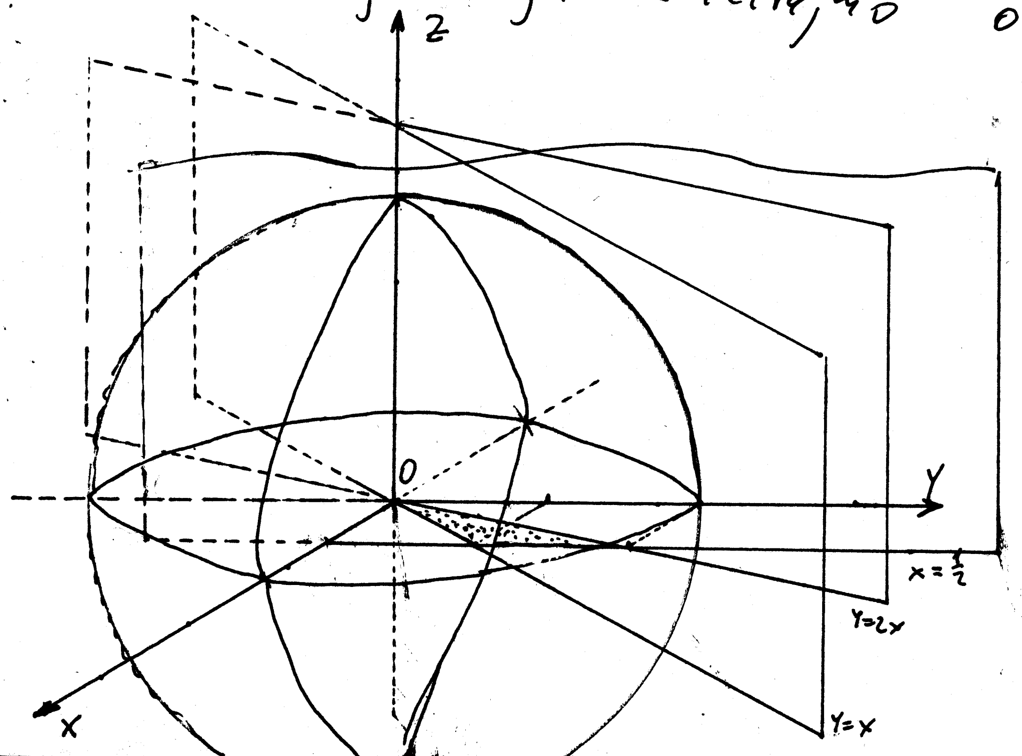
U prostoru to je ravan koja sadrži u xOz ravni pravu $x=\frac{1}{2}$ i u xOy ravni pravu $x=\frac{1}{2}$



$x=\frac{1}{2}$ je ravan koja je paralelna sa yOz osom

$x^2+y^2+z^2=1$ je jednačina kružnice

Na osnovu svega ovoga skicirajmo oblast Ω.



Oblast Ω je kružni isječak čija projekcija na xOy ravan je predstavljena tačkama na slici. Možemo zaključiti

$$\Omega:\begin{cases} 0\leq x\leq \frac{1}{2}\\ x\leq y\leq 2x\\ 0\leq z\leq \sqrt{1-x^2-y^2}\end{cases}$$

$$I=\iiint\limits_{\Omega} z\,dx\,dy\,dz=\int_{0}^{\frac12}dx\int_{x}^{2x}dy\int_{0}^{\sqrt{1-x^2-y^2}} z\,dz=\int_{0}^{\frac12}dx\int_{x}^{2x}\frac12 z^2\Big|_{0}^{\sqrt{1-x^2-y^2}}dy=$$

$$=\frac12\int_{0}^{\frac12}dx\int_{x}^{2x}(1-x^2-y^2)\,dy=\frac12\int_{0}^{\frac12}\left(y\Big|_{x}^{2x}-x^2y\Big|_{x}^{2x}-\frac13 y^3\Big|_{x}^{2x}\right)dx=$$

$$=\frac12\int_{0}^{1/2}\left(x-x^3-\frac13 7x^3\right)dx=\frac12\int_{0}^{1/2}\left(x-\frac{10}{3}x^3\right)dx=\frac12\left(\frac12 x^2\Big|_{0}^{\frac12}-\frac{\overset{5}{\cancel{10}}}{3}\cdot\frac{1}{\underset{2}{\cancel{4}}}x^4\Big|_{0}^{\frac12}\right)$$

$$=\frac12\left(\frac12\cdot\frac14-\frac56\cdot\frac{1}{16}\right)=\frac12\left(\frac18-\frac{5}{}\right)=\frac12\cdot\frac{12-5}{96}=\frac{7}{192}$$

Zadaci za vježbu

1. Nađite površinu ograničenu parabolama $y^2 = 10x + 25$ i $y^2 = -6x + 9$.

2. Prijelazom na polarne koordinate nađite površinu ograničenu krivima $x^2 + y^2 = 2x$, $x^2 + y^2 = 4x$, $y = x$ i $y = 0$.

3. Nađite površinu ograničenu pravcem $r\cos\varphi = 1$ i kružnicom $r = 2$. (Imajte u vidu područje u kojem se nalazi pol).

4. Nađite površinu omeđenu krivuljama $r = a(1+\cos\varphi)$ i $r = a\cos\varphi$ $(a > 0)$.

5. Nađite zapreminu tijela ograničenog paraboloidom $2az = x^2 + y^2$ i kuglom $x^2 + y^2 + z^2 = 3a^2$ (Misli se na volumen unutar paraboloida).

6. Izračunati volumen tijela ograničenog ravninom xOy, valjkom $x^2 + y^2 = 2ax$ i čunjem $x^2 + y^2 = z^2$.

7. Izračunajte $\iiint\limits_{(V)} z^2\,dx\,dy\,dz$ gdje je V zajednički dio kugli $x^2 + y^2 + z^2 \le R^2$ i $x^2 + y^2 + z^2 \le 2Rz$

8. Odredite granice integracije u trostrukom integralu $\iiint\limits_{(V)} f(x,y,z)\,dx\,dy\,dz$ gdje je V čunj omeđen plohama

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, \quad z = c.$$

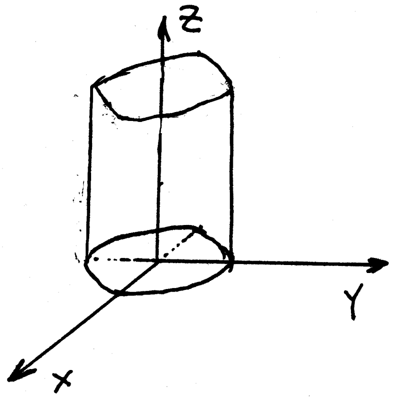
Rješenja:

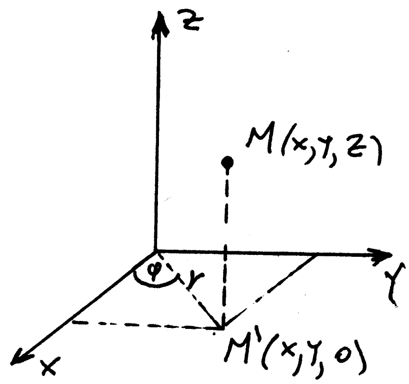
1. $\frac{16}{3}\sqrt{15}$ 2. $3\left(\frac{\pi}{4} + \frac{1}{2}\right)$ 3. $\frac{4\pi}{3} - \sqrt{3}$ 4. $\frac{5}{4}\pi a^2$

5. $\frac{\pi a^3}{3}(6\sqrt{3} - 5)$ 6. $\frac{32}{9}a^3$ 7. $\frac{59}{480}\pi R^5$ 8. $\int\limits_{-a}^{a} dx \int\limits_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy \int\limits_{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}^{c} f(x,y,z)\,dz$

Računanje trostrukih integrala uvođenjem cilindričnih i sfernih koordinata

cilindrične koordinate





uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

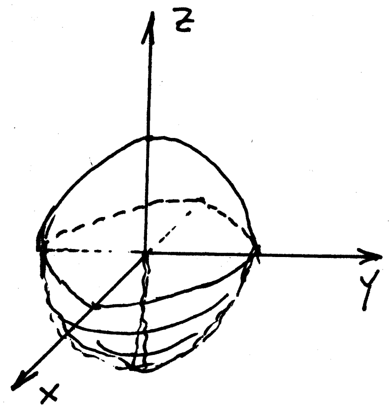
$z = z$

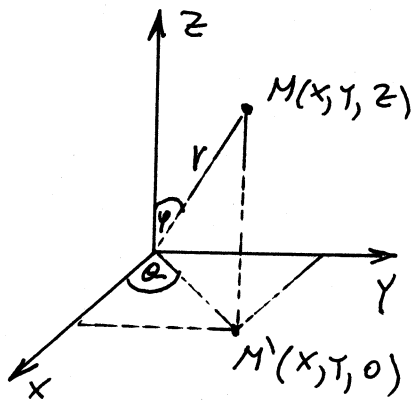
$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

cilindrične koordinate obično uvedemo ako se pojavi izraz x^2+y^2 $\left(x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2(\sin^2\varphi+\cos^2\varphi) = r^2\right)$

$(r \geq 0, \quad 0 \leq \varphi \leq 2\pi)$

sferne koordinate





uvodimo smjenu

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$r \geq 0$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

$$x^2+y^2+z^2 = r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi = \ldots = r^2$$

sferne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u opisu oblasti integracije pojavljuje izraz $x^2+y^2+z^2$.

#) Izračunati integral $\iiint\limits_{\Omega} \sqrt{z\cdot(x^2+y^2)}\, dx\,dy\,dz$ gdje

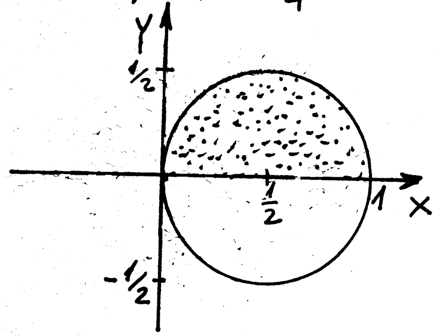
je Ω oblast $x^2+y^2 \le 0$, $y \ge 0$, $z \ge 0$, $z \le 3$.

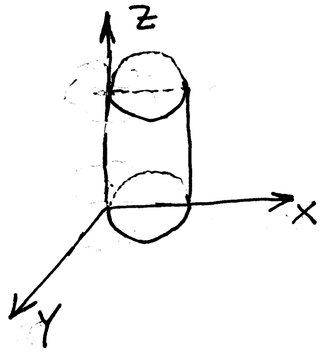
Rj. U ravni xOy kako izgleda $x^2+y^2 \le x$?

$x^2 - x + y^2 = 0$

$x^2 - 2\cdot\frac{1}{2}\cdot x + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$





Uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$x^2+y^2 \le x$

$r^2\cos^2\varphi + r^2\sin^2\varphi \le r\cos\varphi$

$r^2 \le r\cos\varphi \quad /:r \quad (r \ne 0)$

$r \le \cos\varphi$

kako je $r \ge 0$ to je $\cos\varphi \ge 0$

$y \ge 0$

$r\sin\varphi \ge 0 \quad /:r$

$\sin\varphi \ge 0$

imam $0 \le r \le \cos\varphi$

$\sin\varphi \ge 0$

$\cos\varphi \ge 0$

$0 \le z \le 3$

$\Rightarrow \quad \Omega' : \begin{cases} 0 \le r \le \cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le 3 \end{cases}$ $\quad dx\,dy\,dz = J\,d\varphi\,dr\,dz$

$$\iiint\limits_{\Omega} \sqrt{z(x^2+y^2)}\, dx\,dy\,dz = \iiint\limits_{\Omega'} \sqrt{z r^2}\, r\, d\varphi\, dr\, dz = \int_0^3 dz \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} \sqrt{z}\, r^2\, dr =$$

$$= \int_0^3 \sqrt{z}\, dz \int_0^{\frac{\pi}{2}} \frac{1}{3} r^3 \Big|_0^{\cos\varphi} d\varphi = \frac{1}{3} \int_0^3 \sqrt{z}\, dz \int_0^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi = \frac{1}{3} \int_0^3 \sqrt{z}\, dz \int_0^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)\, d\varphi$$

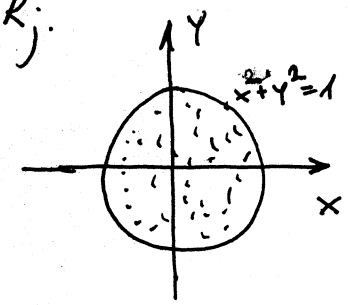
$$= \left| \begin{matrix} \sin\varphi = t \\ \cos\varphi\, d\varphi = dt \end{matrix} \quad \begin{matrix} \varphi = 0 \Rightarrow t = 0 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{matrix} \right| = \frac{1}{3} \int_0^3 \sqrt{z}\, dz \int_0^1 (1-t^2)\, dt = \frac{1}{3} \int_0^3 \sqrt{z} \left(t\Big|_0^1 - \frac{1}{3} t^3 \Big|_0^1 \right) dz$$

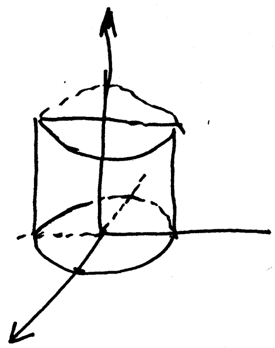
$$= \frac{1}{3}\left(1 - \frac{1}{3}\right) \frac{z^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^3 = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}\, 3^{\frac{3}{2}} = \frac{4}{27} \sqrt{3^3} = \frac{4}{9}\sqrt{3}$$

(#) Izračunati $I=\iiint\limits_{\Omega}(x^2+y^2+z)^3\,dx\,dy\,dz$ gdje je

Ω oblast ograničena sa $x^2+y^2=1$, $z=0$ i $z=1$.

Rj.





uvodimo smjene:

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$

$x^2+y^2=1$ $\qquad 0\le z\le 1$

$r^2=1$ $\qquad 0\le\varphi\le 2\pi$

$r\ge 0$

$0\le r\le 1$

$$\Omega'=\begin{cases}0\le r\le 1\\ 0\le\varphi\le 2\pi\\ 0\le z\le 1\end{cases}\qquad dx\,dy\,dz= r\,dr\,d\varphi\,dz$$

$$I=\iiint\limits_{\Omega}(x^2+y^2+z)^3\,dx\,dy\,dz=\iiint\limits_{\Omega'}(r^2+z)^3 r\,dr\,d\varphi\,dz=$$

$$=\int_0^1 dz\int_0^{2\pi}d\varphi\int_0^1(r^2+z)^3\cdot r\,dr=\left|\begin{matrix}r^2+z=t & r=0\Rightarrow t=z\\ 2r\,dr=dt & r=1\Rightarrow t=z+1\\ r\,dr=\frac{1}{2}dt & \end{matrix}\right|$$

$$=\frac{1}{2}\int_0^1 dz\int_0^{2\pi}d\varphi\int_z^{z+1}t^3\,dt=\frac{1}{2}\int_0^1 dz\int_0^{2\pi}\frac{1}{4}t^4\Big|_z^{z+1}d\varphi=\frac{1}{8}\int_0^1\left[(z+1)^4-z^4\right]\varphi\Big|_0^{2\pi}dz$$

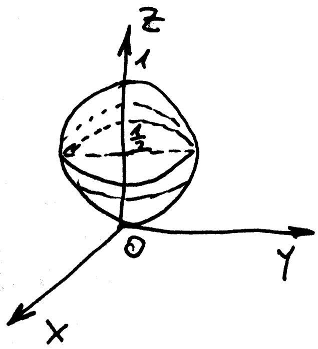
$$=\frac{1}{8}\cdot 2\pi\int_0^1\left[(z+1)^4-z^4\right]dz=\frac{\pi}{4}\cdot\left(\frac{1}{5}(z+1)^5\Big|_0^1-\frac{1}{5}z^5\Big|_0^1\right)=$$

$$\boxed{\int(z+1)^4\,dz=\left|\begin{matrix}t=z+1\\ dt=dz\end{matrix}\right|=\int t^4\,dt=\frac{1}{5}t^5+c=\frac{1}{5}(z+1)^5+c}$$

$$=\frac{\pi}{20}(31-1)=\frac{30\pi}{20}=\frac{3\pi}{2}$$

(#) Izračunati $I=\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$ gdje je Ω oblast ograničena sferom $x^2+y^2+z^2=z$.

Rj.



$$x^2+y^2+z^2=z$$
$$x^2+y^2+z^2-z=0$$
$$x^2+y^2+z^2-2\cdot z\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2=0$$
$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$$

centar sfere u tački $S(0,0,\frac{1}{4})$

poluprečnik sfere $r=\frac{1}{2}$

uvodimo smjene

$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$

Odredimo granice za r, φ i θ nove oblasti

$$x^2+y^2+z^2=r^2$$

$r^2=r\cos\varphi$ ← iz $x^2+y^2+z^2=z$ $\quad /:r \quad (r\neq 0)$

$r=\cos\varphi$ kako je $r>0 \Rightarrow \cos\varphi>0$ tj. $0\le\varphi\le\frac{\pi}{2}$

$$\Omega': \begin{cases} 0\le r\le\cos\varphi \\ 0\le\varphi\le\frac{\pi}{2} \\ 0\le\theta\le 2\pi \end{cases} \qquad dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$I=\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz=\iiint\limits_{\Omega'}\sqrt{r^2}\,r^2\sin\varphi\,dr\,d\varphi\,d\theta=$$

$$\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{\cos\varphi}r^3\,dr=\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\varphi\,\frac{1}{4}r^4\Big|_0^{\cos\varphi}d\varphi=$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\varphi\cos^4\varphi\,d\varphi=\left|\begin{matrix}\cos\varphi=t & \varphi=0\Rightarrow t=1\\ -\sin\varphi\,d\varphi=dt & \varphi=\frac{\pi}{2}\Rightarrow t=0\\ \sin\varphi\,d\varphi=-dt & \end{matrix}\right|=$$

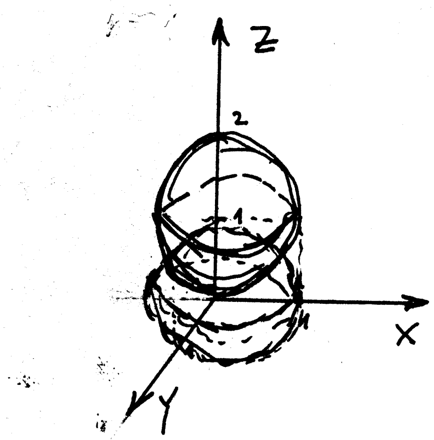
$$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^1 t^4\,dt=\frac{1}{4}\int_0^{2\pi}\frac{1}{5}t^5\Big|_0^1 d\theta=\frac{1}{20}\,\theta\Big|_0^{2\pi}=\frac{1}{20}\cdot 2\pi=\frac{\pi}{10}$$

(#) Izračunati $I=\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,dx\,dy\,dz$ gdje je Ω

oblast $x^2+y^2+z^2\le 1$ i $x^2+y^2+z^2\le 2z$.

Rj. $x^2+y^2+z^2=1$

sfera sa centrom u tački $(0,0,0)$ poluprečnika $r=1$



$x^2+y^2+z^2=2z$

$x^2+y^2+z^2-2z=0$

$x^2+y^2+z^2-2\cdot z\cdot 1+1-1=0$

$x^2+y^2+(z-1)^2=1$

sfera sa centrom u tački $S(0,0,1)$ poluprečnika $r=1$

uvodimo smjene

$x=r\sin\varphi\cos\theta$

$y=r\sin\varphi\sin\theta$

$z=r\cos\varphi$

$x^2+y^2+z^2\le 1$

$r^2\le 1$

$0\le r\le 1$

$x^2+y^2+z^2\le 2z$

$r^2\le 2r\cos\varphi \quad /:r$

$r\le 2\cos\varphi$

$0\le r\le 2\cos\varphi$

$0\le \cos\varphi\le 1 \quad \forall\varphi$

pa može biti $2\cos\varphi<1$ i $2\cos\varphi>1$

Ako je $2\cos\varphi<1 \Rightarrow \cos\varphi<\frac{1}{2}$ (kako $\cos\varphi>0$) $\Rightarrow \varphi\in(\frac{\pi}{3},\frac{\pi}{2})$

Ako je $2\cos\varphi>1 \Rightarrow \cos\varphi>\frac{1}{2}$ (kako $\cos\varphi>0$) $\Rightarrow \varphi\in(0,\frac{\pi}{3})$

Pa je oblast Ω unija dvije oblasti Ω_1 i Ω_2 gdje $\Omega_1\cap\Omega_2=\phi$.

$$\Omega_1:\begin{cases}0\le r\le 1\\ 0\le\theta\le 2\pi\\ 0\le\varphi\le\frac{\pi}{3}\end{cases}\qquad \Omega_2:\begin{cases}0\le r\le 2\cos\varphi\\ 0\le\theta\le 2\pi\\ \frac{\pi}{3}\le\varphi\le\frac{\pi}{2}.\end{cases}$$

$dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$x^2+y^2=r^2\sin^2\varphi\cos^2\theta+r^2\sin^2\varphi\sin^2\theta=r^2\sin^2\varphi$

$$I=I_1+I_2,\quad I_1=\iiint\limits_{\Omega_1}\sqrt{r^2\sin^2\varphi}\cdot r^2\sin\varphi\,dr\,d\varphi\,d\theta=\int_0^{2\pi}d\theta\int_0^1 r^3\,dr\int_0^{\frac{\pi}{3}}\sin^2\varphi\,d\varphi=$$

$$=\int_0^{2\pi}d\theta\int_0^1 r^3dr\int_0^{\frac{\pi}{3}}\frac{1}{2}(1-\cos 2\varphi)\,d\varphi=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^1 r^3\left(\varphi\Big|_0^{\frac{\pi}{3}}-\frac{1}{2}\sin 2\varphi\Big|_0^{\frac{\pi}{3}}\right)=\ldots=\frac{\sqrt{3}\pi}{32}+\frac{\pi^2}{12}$$

$$I_2=\int_0^{2\pi}d\theta\int_{\pi/3}^{\pi/2}\sin^2\varphi\int_0^{2\cos\varphi}r^3\,dr=\ldots=\frac{\pi^2}{12}-\frac{\pi\sqrt{3}}{8},\qquad I=I_1+I_2=\frac{2\pi^2}{12}-\frac{3\pi\sqrt{3}}{32}$$