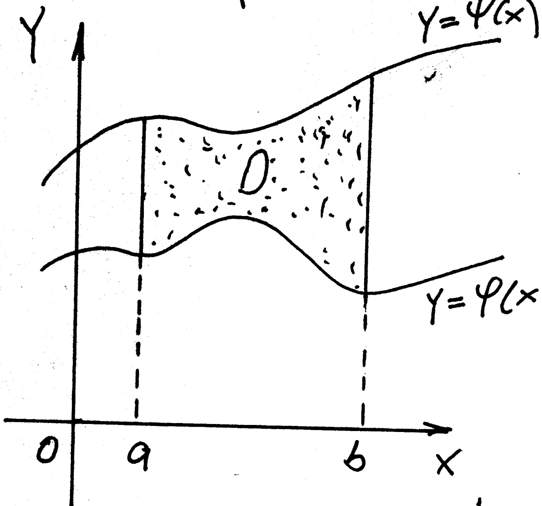
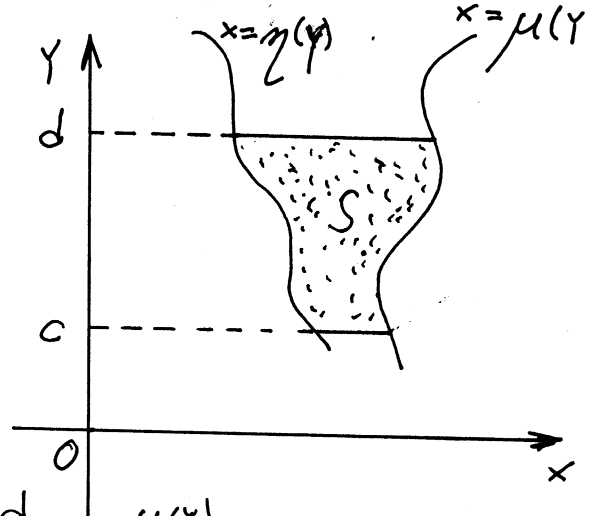
Dvostruki integral

$$\iint\limits_{D} f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right] dx$$



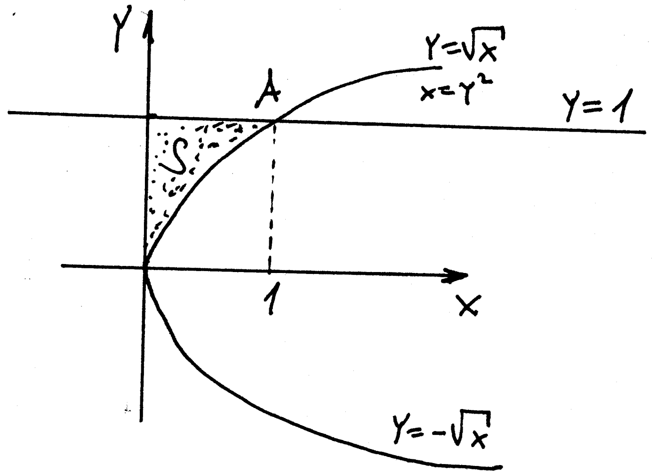
D-oblast integracije



$$\iint\limits_{S} f(x,y)\,dx\,dy = \int_c^d dy \int_{\eta(y)}^{\mu(y)} f(x,y)\,dx = \int_c^d \left[\int_{\eta(y)}^{\mu(y)} f(x,y)\,dx\right] dy$$

Izračunati integral $\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy$, gdje je S oblast omeđena parabolom $y^2=x$, te pravama $x=0$, $y=1$.

Rj. Nacrtajmo sliku



Tačka $A(1,1)$ je presjek parabole $y^2=x$ i prave $y=1$.

Moguća su dva načina:

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy=\int_0^1\left[\int_0^{y^2} e^{-\frac{x}{y}}\,dx\right]dy$$

ili

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy=\int_0^1\left[\int_{\sqrt{x}}^{1} e^{-\frac{x}{y}}\,dy\right]dx$$

Kako $\int e^{-\frac{1}{t}}\,dt=?$ imamo:

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy=\int_0^1\left[\int_0^{y^2} e^{-\frac{x}{y}}\,dx\right]dy=\int_0^1\left(-y\,e^{-\frac{x}{y}}\Big|_0^{y^2}\right)dy=-\int_0^1 y\left(e^{-y}-1\right)dy\overset{(*)}{=}$$

$$\int e^{-\frac{x}{y}}\,dx=\begin{vmatrix}-\frac{x}{y}=t\\ -\frac{1}{y}dx=dt\end{vmatrix}=\int e^{t}(-y)\,dt=-y\,e^{t}+C=-y\,e^{-\frac{x}{y}}+C$$

$$\overset{(*)}{=}\int_0^1\left(y-y\,e^{-y}\right)dy=\int_0^1 y\,dy+\int_0^1(-y)\,e^{-y}\,dy=\frac{1}{2}y^2\Big|_0^1+(y+1)\,e^{-y}\Big|_0^1\overset{(\Delta)}{=}$$

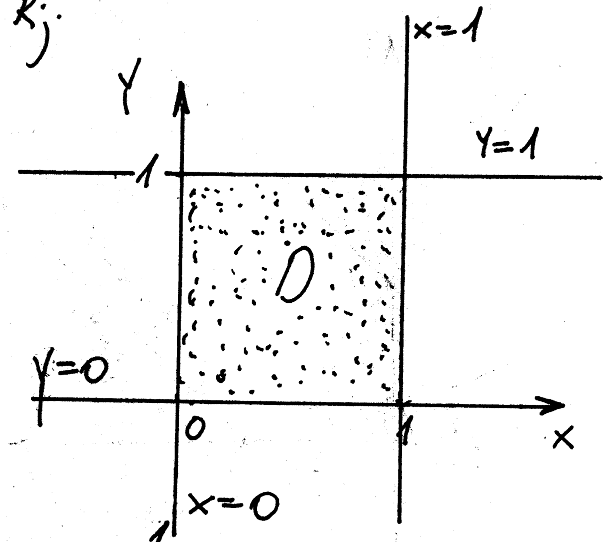
$$\int t\,e^{t}\,dt=\begin{vmatrix}u=t & dv=e^{t}dt\\ du=dt & v=e^{t}\end{vmatrix}=te^{t}-\int e^{t}\,dt=(t-1)e^{t}+C$$

$$\int(-t)\,e^{-t}\,dt=\begin{vmatrix}u=-t & dv=e^{-t}dt\\ du=-dt & v=-e^{-t}\end{vmatrix}=te^{-t}-\int e^{-t}\,dt=(t+1)e^{-t}+C$$

$$\overset{(\Delta)}{=}\frac{1}{2}+\left(2e^{-1}-1\right)=\frac{2}{e}-\frac{1}{2}$$

Izračunati vrijednost integrala $I=\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy$ gdje je $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1 \;;\; 0\le y\le 1\}$.

Rj.



I način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy=\int_0^1 dx\int_0^1 \frac{x^2}{1+y^2}\,dy=$$

$$=\int_0^1 x^2\,dx\int_0^1 \frac{dy}{1+y^2}=\int_0^1 x^2 \operatorname{arctg} y\Big|_0^1\,dx=$$

$$=\frac{\pi}{4}\int_0^1 x^2\,dx=\frac{\pi}{4}\cdot\frac{x^3}{3}\Big|_0^1=\frac{\pi}{12}$$

II način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy=\int_0^1 dy\int_0^1 \frac{x^2}{1+y^2}\,dx=\int_0^1 \frac{dy}{1+y^2}\int_0^1 x^2\,dx$$

$$=\int_0^1 \frac{1}{1+y^2}\cdot\frac{x^3}{3}\Big|_0^1\,dy=\frac{1}{3}\int_0^1 \frac{dy}{1+y^2}=\frac{1}{3}\operatorname{arctg} x\Big|_0^1=\frac{1}{3}\cdot\frac{\pi}{4}=\frac{\pi}{12}$$

Izračunati vrijednost integrala $I=\iint\limits_{|x|+|y|\le 1} x^2\,dx\,dy$

Rj: $x<0,\ y<0 \Rightarrow -x-y\le 1$

$y \ge -x-1$

$x<0,\ y>0 \Rightarrow -x+y\le 1$

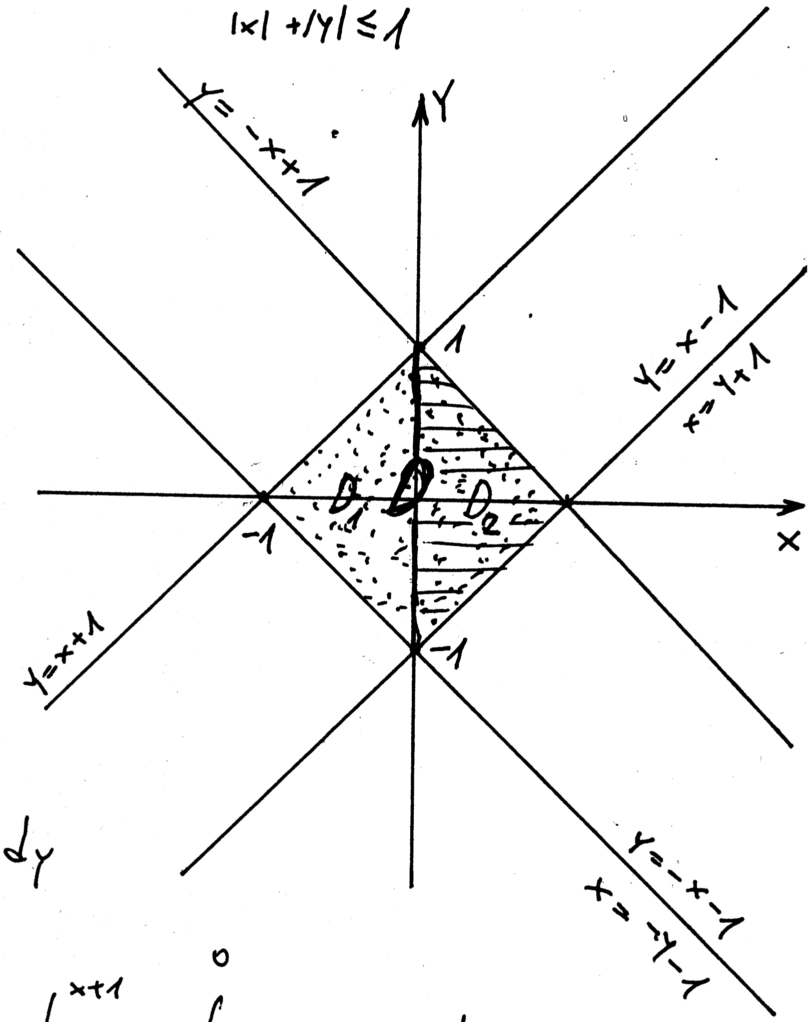
$y\le x+1$

$x\ge 0,\ y<0 \Rightarrow x-y\le 1$

$y\ge x-1$

$x\ge 0,\ y\ge 0 \Rightarrow x+y\le 1$

$y\le -x+1$



$$I=\iint\limits_{|x|+|y|\le 1} x^2\,dx\,dy = \iint\limits_{D_1} x^2\,dx\,dy + \iint\limits_{D_2} x^2\,dx\,dy$$

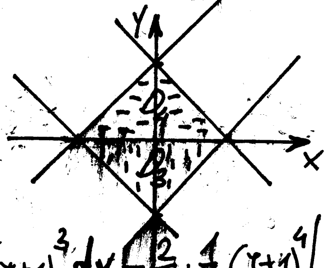
$$\iint\limits_{D_1} x^2\,dx\,dy = \int_{-1}^{0}dx\int_{-x-1}^{x+1}x^2\,dy = \int_{-1}^{0} x^2\,dx\ \underbrace{y\Big|_{-x-1}^{x+1}}_{x+1+x+1} = \int_{-1}^{0}x^2(2x+2)\,dx =$$

$$=\int_{-1}^{0}(2x^3+2x^2)\,dx = \frac{2}{4}x^4\Big|_{-1}^{0}+\frac{2}{3}x^3\Big|_{-1}^{0} = \frac{1}{2}\cdot(-1)+\frac{2}{3}\cdot 1 = \frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

$$\iint\limits_{D_2} x^2\,dx\,dy = \int_{0}^{1}dx\int_{x-1}^{-x+1}x^2\,dy = \int_{0}^{1}x^2\,dx\ \underbrace{y\Big|_{x-1}^{-x+1}}_{-x+1-x+1} = \int_{0}^{1}x^2(-2x+2)\,dx =$$

$$\int_{0}^{1}(-2x^3+2x^2)\,dx = -\frac{2}{4}x^4\Big|_0^1+\frac{2}{3}x^3\Big|_0^1 = -\frac{1}{2}+\frac{2}{3}=\frac{1}{6},$$

$$I=\iint\limits_{|x|+|y|\le 1} x^2\,dx\,dy = \frac{2}{6}$$



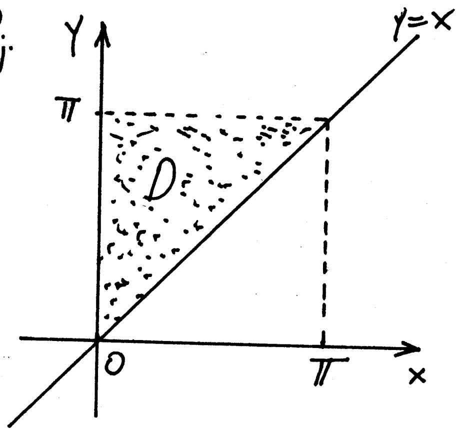
II način: $I=\iint\limits_{D_3} x^2\,dx\,dy + \iint\limits_{D_4} x^2\,dx\,dy$

$$\iint\limits_{D_3} x^2\,dx\,dy = \int_{-1}^{0}dy\int_{-y-1}^{y+1}x^2\,dx = \int_{-1}^{0}dy\,\frac{1}{3}x^3\Big|_{-y-1}^{y+1} = \frac{1}{3}\int_{-1}^{0}2(y+1)^3\,dy = \frac{2}{3}\int_{-1}^{0}(y+1)^3\,dy = \frac{2}{3}\cdot\frac{1}{4}(y+1)^4\Big|_{-1}^{0} = \frac{1}{6}\ \ldots$$

Izračunati dvostruki integral $\iint_D \cos(x+y)\,dx\,dy$

ako je $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq \pi \;;\; x \leq y \leq \pi\}$

Rj.



$$\iint_D \cos(x+y)\,dx\,dy = \int_0^{\pi} dx \int_x^{\pi} \cos(x+y)\,dy =$$

$$= \int_0^{\pi} dx \, \sin(x+y)\Big|_x^{\pi} = \int_0^{\pi} \left[\sin(x+\pi) - \sin 2x\right] dx \overset{(*)}{=}$$

$\boxed{\sin(x+\pi) = \sin x \cos\pi + \sin\pi \cos x = -\sin x}$

$$\overset{(*)}{=} \int_0^{\pi} (-\sin x - \sin 2x)\,dx = -\int_0^{\pi} \sin x\,dx - \int_0^{\pi} \sin 2x\,dx = \cos x\Big|_0^{\pi} + \frac{1}{2}\cos 2x\Big|_0^{\pi} =$$

$$= (-1-1) + \frac{1}{2}(1-1) = -2$$

II način

$$\iint_D \cos(x+y)\,dx\,dy = \int_0^{\pi} dy \int_0^{y} \cos(x+y)\,dx = \int_0^{\pi} dy \, \sin(x+y)\Big|_0^{y} =$$

$$\int_0^{\pi} (\sin 2y - \sin y)\,dy = -\frac{1}{2}\cos 2y\Big|_0^{\pi} + \cos y\Big|_0^{\pi} = -\frac{1}{2}(1-1) + (-1-1) = -2$$

(#) Izračunati dvostruki integral $\iint_D (x^2+y)\,dx\,dy$

gdje je D površ ograničena linijama $y=x^2$ i $y^2=x$.

Rj. Nađimo presječnu tačku i nacrtajmo ove dvije krive

$y=x^2$

$y^2=x$

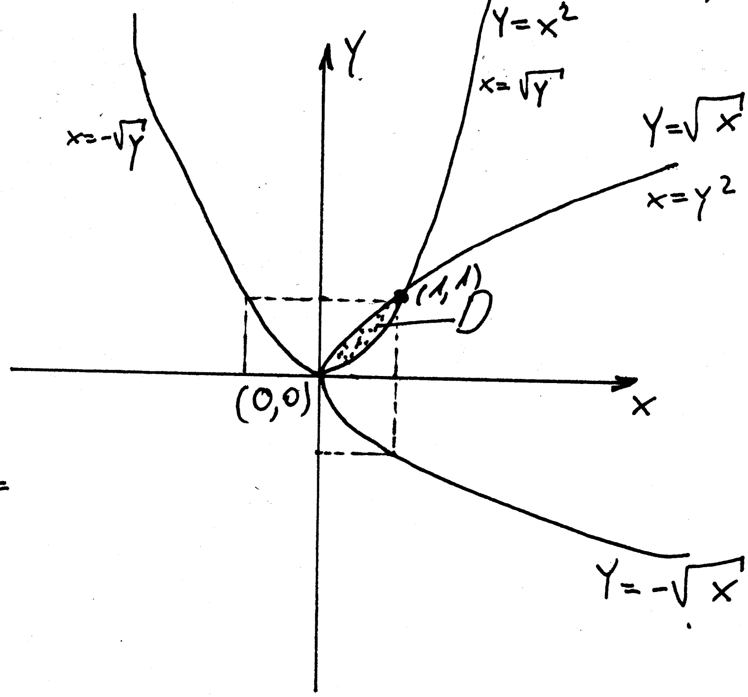
$x^4=x$

$x(x^3-1)=0$

$x(x-1)(x^2+x+1)=0$

$x=0$ ili $x=1$

Presječne tačke krivih su $(0,0)$ i $(1,1)$.



$$\iint_D (x^2+y)\,dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2+y)\,dy =$$

$$= \int_0^1 dx \left(x^2 y \Big|_{x^2}^{\sqrt{x}} + \frac{1}{2} y^2 \Big|_{x^2}^{\sqrt{x}} \right) = \int_0^1 \left[x^2(\sqrt{x}-x^2) + \frac{1}{2}(x-x^4) \right] dx$$

$$= \int_0^1 \left(x^2\sqrt{x} - \underline{x^4} + \frac{1}{2}x - \underline{\frac{1}{2}x^4} \right) dx = \int_0^1 \left(-\frac{3}{2}x^4 + x^{\frac{5}{2}} + \frac{1}{2}x \right) dx =$$

$$= -\frac{3}{2}\cdot\frac{1}{5}x^5\Big|_0^1 + \frac{2}{7}\cdot x^{\frac{7}{2}}\Big|_0^1 + \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^1 = -\frac{3}{10} + \frac{2}{7} + \frac{1}{4} = \frac{-3\cdot 14 + 2\cdot 20 + 1\cdot 35}{140}$$

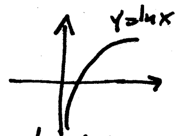
$$= \frac{-42+40+35}{140} = \frac{33}{140}$$

II način: $\iint_D (x^2+y)\,dx\,dy = \int_0^1 \left(\int_{y^2}^{\sqrt{y}} (x^2+y)\,dx \right) dy = \ldots = \int_0^1 \left(\frac{4}{3}\sqrt{y^3} - y^3 - \frac{1}{3}y^6 \right) dy$

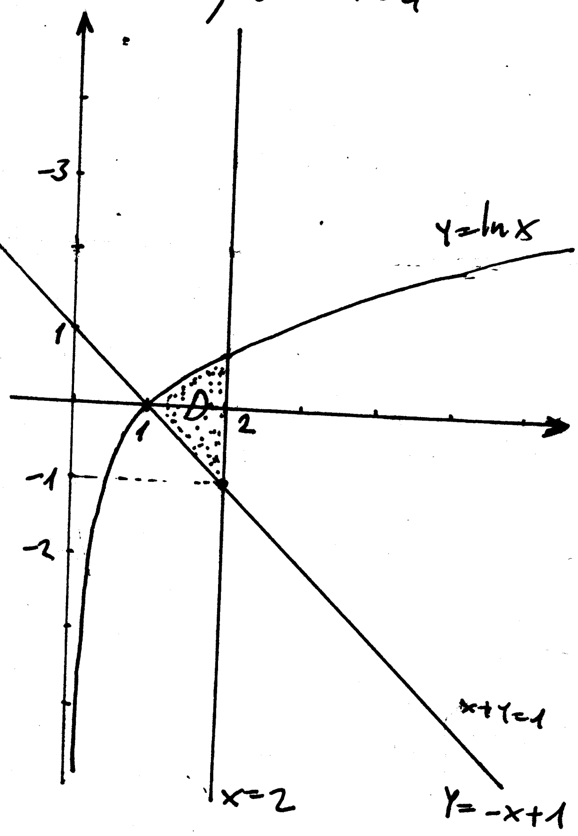
za vježbu

$$= \ldots = \frac{33}{140}$$

(#) Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$, gdje je $D:\ y=\ln x,\ \ x=2,\ \ x+y=1.$

Rj. Kriva $y=\ln x$ izgleda ovako 

Nacrtajmo sliku



Pronađimo presječne tačke datih krivi.

$$\begin{array}{c} y=\ln x \\ x=2 \\ \hline y=\ln 2\approx 0{,}69 \\ (2,\ln 2) \end{array} \qquad \begin{array}{c} y=\ln x \\ x+y=1 \\ \hline y=\ln x \\ y=-x+1 \\ \hline \ln x=-x+1 \\ x=1 \\ (1,0) \end{array} \qquad \begin{array}{c} x=2 \\ x+y=1 \\ \hline 2+y=1 \\ y=-1 \\ (2,-1) \end{array}$$

$$I=\iint\limits_D xy\,dx\,dy=\int_1^2 dx\int_{-x+1}^{\ln x} xy\,dy=\int_1^2 x\,dx\int_{-x+1}^{\ln x} y\,dy=$$

$$=\int_1^2 x\left(\frac12 y^2\Big|_{-x+1}^{\ln x}\right)dx=\frac12\int_1^2 x\big(\ln^2 x-\underbrace{(-x+1)^2}_{x^2-2x+1}\big)\,dx=$$

$$=\frac12\int_1^2 x\ln^2 x\,dx-\frac12\int_1^2 (x^3-2x^2+x)\,dx$$

$$\int_1^2 x\ln^2 x\,dx=\begin{vmatrix} u=\ln^2 x & dv=x\,dx \\ du=2\ln x\cdot\frac1x\,dx & v=\frac12 x^2\end{vmatrix}=\overbrace{\frac12 x^2\ln^2 x\Big|_1^2}^{\frac12 4(\ln 2)^2\ \ \ 2\cdot 0{,}48\approx 0{,}96}-\int_1^2 x\ln x\,dx=$$

$$=\begin{vmatrix} u=\ln x & dv=x\,dx \\ du=\frac1x\,dx & v=\frac12 x^2\end{vmatrix}=2\ln^2 2-\Big[\overbrace{\frac12 x^2\ln x\Big|_1^2}^{2\ln 2}-\frac12\int_1^2 x\,dx\Big]=2\ln^2 2-2\ln 2+\frac34$$

$$\frac12 x^2\Big|_1^2 \qquad \frac12\cdot 3$$

$$\int_1^2 (x^3-2x^2+x)\,dx=\frac14 x^4\Big|_1^2-\frac23 x^3\Big|_1^2+\frac12 x^2\Big|_1^2=$$

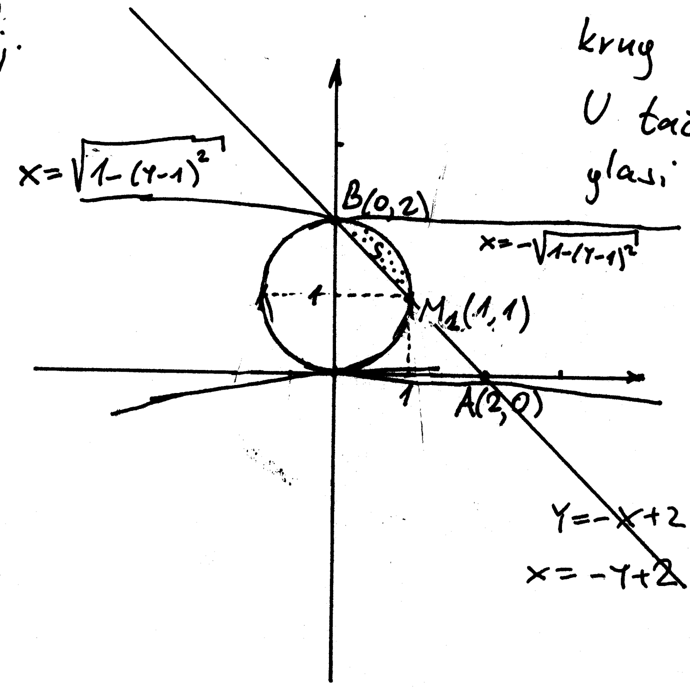
$$=\frac{15}{4}-\frac{14}{3}+\frac32=\frac{45-56+18}{12}=\frac{7}{12}$$

traženo rješenje

$$I=\frac12\left(2\ln^2 2-2\ln 2+\frac34\right)-\frac12\cdot\frac{7}{12}=\ln^2 2-\ln 2+\frac38-\frac{7}{24}=\ln^2 2-\ln 2+\frac{1}{12}$$

(#) Izračunati dvostruki integral $\iint\limits_S x\,dx\,dy$ gdje je područje integracije S ograničeno pravcem koji prolazi tačkama $A(2,0)$, $B(0,2)$ i lukom kruga poluprečnika 1 sa centrom u tački $C(0,1)$.

Rj.



$\overset{x_1,y_1}{A(2,0)}$, $\overset{x_2,y_2}{B(0,2)}$

krug $x^2+(y-1)^2=1$

U tačkama $A(2,0)$ i $B(0,2)$ jednačina prave glasi

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0=\frac{2-0}{0-2}(x-2)$$

$$y=-x+2$$

Nađimo presječne tačke prave i kruga

$$x^2+(y-1)^2=1$$
$$y=-x+2$$

$$x^2+(-x+2-1)^2=1$$

$$x^2+(-x+1)^2=1$$
$$x^2+x^2-2x+1=1$$
$$2x^2-2x=0$$
$$2x(x-1)=0$$

$x_1=0$, $x_2=1$

$x_1=0 \Rightarrow y=2$

$x_2=1 \Rightarrow y=1$

Presječne tačke prave i kruga su $M_1(0,2)$ i $M_2(1,1)$

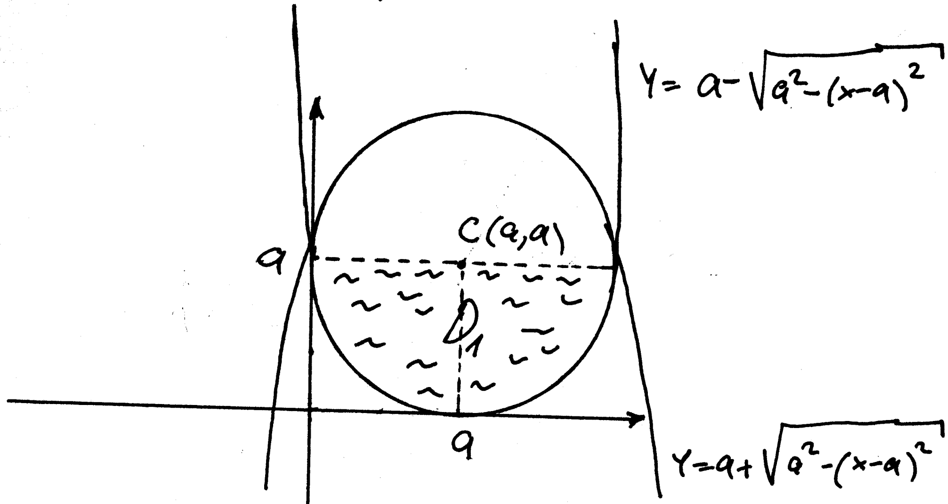
$$\iint\limits_S x\,dx\,dy=\int_1^2\left[\int_{-y+2}^{\sqrt{1-(y-1)^2}} x\,dx\right]dy=\int_1^2 \frac{1}{2}x^2\Big|_{-y+2}^{\sqrt{1-(y-1)^2}}dy=\frac{1}{2}\int_1^2\left[(1-(y-1)^2)-(2-y)^2\right]dy$$

$$=\frac{1}{2}\int_1^2\left[1-y^2+2y-1-4+4y-y^2\right]dy=\frac{1}{2}\int_1^2(-2y^2+6y-4)\,dy=\frac{1}{2}\cdot 2\int_1^2(-y^2+3y-2)\,dy$$

$$=-\frac{1}{3}y^3\Big|_1^2+\frac{3}{2}y^2\Big|_1^2-2y\Big|_1^2=-\frac{7}{3}+\frac{9}{2}-2=\frac{-14+27-12}{6}=\frac{1}{6}$$

(#) Izračunati dvostruki integral $\iint\limits_S \frac{dx\,dy}{\sqrt{2a-x}}$ gde je S krug poluprečnika a, koji dodiruje koordinatne ose i leži u prvom kvadrantu.

Rj.



krug $(x-a)^2+(y-a)^2=a^2$

$y - a = \pm\sqrt{a^2-(x-a)^2}$

$y = a \pm \sqrt{a^2-(x-a)^2}$

$$\iint\limits_S \frac{dx\,dy}{\sqrt{2a-x}} = 2D_1$$

$$D_1 = \int_0^{2a}\left[\int_{a-\sqrt{a^2-(x-a)^2}}^{a} \frac{1}{\sqrt{2a-x}}\,dy\right]dx =$$

$$= \int_0^{2a} \frac{dx}{\sqrt{2a-x}} \cdot y\Big|_{a-\sqrt{a^2-(x-a)^2}}^{a} = \int_0^{2a}\sqrt{\frac{a^2-(x^2-2ax+a^2)}{2a-x}}\,dx = \int_0^{2a}\sqrt{\frac{a^2-x^2+2ax-a^2}{2a-x}}\,dx$$

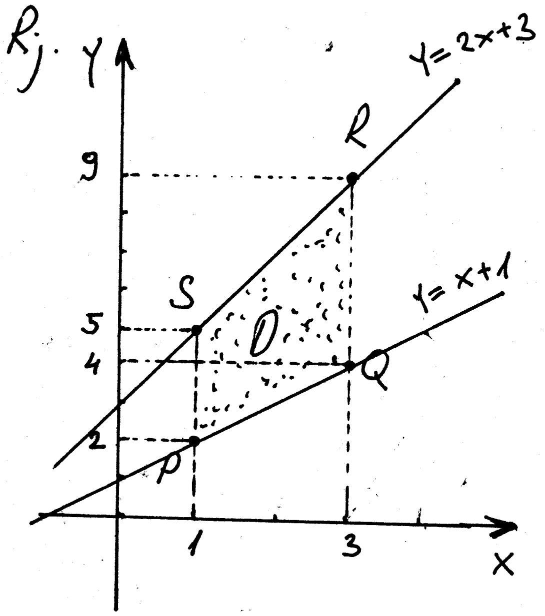
$$= \int_0^{2a}\sqrt{\frac{x(2a-x)}{2a-x}}\,dx = \int_0^{2a}\sqrt{x}\,dx = \int_0^{2a} x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2a} = \frac{2}{3}\cdot(2a)^{\frac{3}{2}} =$$

$$= \frac{2}{3}\sqrt{8a^3} = \frac{2}{3}\,2a\sqrt{2a} = \frac{4a}{3}\sqrt{2a}$$

Prema tome $\iint\limits_S \frac{dx\,dy}{\sqrt{2a-x}} = \frac{8a}{3}\sqrt{2a}$

(#) Izračunati $\iint_D x\,dx\,dy$ pri čemu je D četverougao $\square PQRS$ gdje su tačke $P(1,2)$, $Q(3,4)$, $R(3,9)$ i $S(1,5)$.

Rj.



$$\iint_D x\,dx\,dy = \int_1^3 \left[\int_{x+1}^{2x+3} x\,dy \right] dx =$$

$$y - y_1 = k(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$\overset{x_1\ y_1}{P(1,2)}$, $Q(3,4)$

$$y - 2 = \frac{2}{2}(x - 1)$$

$$y = x + 1$$

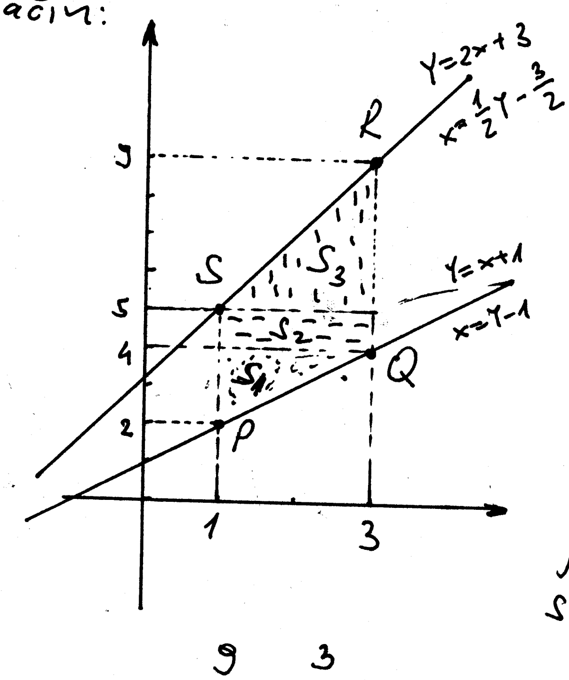
$\overset{x_1\ y_1}{R(3,9)}$, $S(1,5)$

$$y - 9 = \frac{-4}{-2}(x - 3) \Rightarrow y = 2x + 3$$

$$= \int_1^3 x\,y \Big|_{x+1}^{2x+3} dx = \int_1^3 x(2x + 3 - x - 1)\,dx = \int_1^3 (x^2 + 2x)\,dx = \frac{1}{3}x^3 \Big|_1^3 + x^2 \Big|_1^3 =$$

$$= \frac{1}{3}(27 - 1) + (9 - 1) = \frac{26}{3} + 8 = \frac{50}{3}$$

II način: S: $\square PQRS$



$$\iint_S x\,dx\,dy = \iint_{S_1} x\,dx\,dy + \iint_{S_2} x\,dx\,dy + \iint_{S_3} x\,dx\,dy$$

$$\iint_{S_1} x\,dx\,dy = \int_2^4 \left[\int_1^{y-1} x\,dx \right] dy = \ldots = \int_2^4 \left(\frac{1}{2}y^2 - y\right) dy = \ldots = \frac{10}{3}$$

$$\iint_{S_2} x\,dx\,dy = \int_4^5 \left[\int_1^3 x\,dx \right] dy = \ldots = \int_4^5 4\,dy = \ldots = 4$$

$$\iint_{S_3} x\,dx\,dy = \int_5^9 \left[\int_{\frac{1}{2}y - \frac{3}{2}}^3 x\,dx \right] dy = \ldots = \int_5^9 \left(\frac{9}{2} - \frac{(y-3)^2}{8} \right) dy = \ldots = \frac{28}{3}$$

Izračunati $\iint\limits_D (x+y)\,dx\,dy$ ako je D oblast ograničena linijama $y^2=2x$, $x+y=4$ i $x+y=12$.

Rj. Nađimo presječne tačke ovih linija.

$$\begin{array}{l} y^2=2x \\ x+y=4 \\ \hline y^2=2x \\ y=4-x \\ \hline \end{array} \qquad \begin{array}{l} (4-x)^2=2x \\ 16-8x+x^2=2x \\ x^2-10x+16=0 \\ D=100-64=36 \\ x_{1,2}=\frac{10\pm 6}{2} \quad x_1=2,\ x_2=8 \end{array}$$

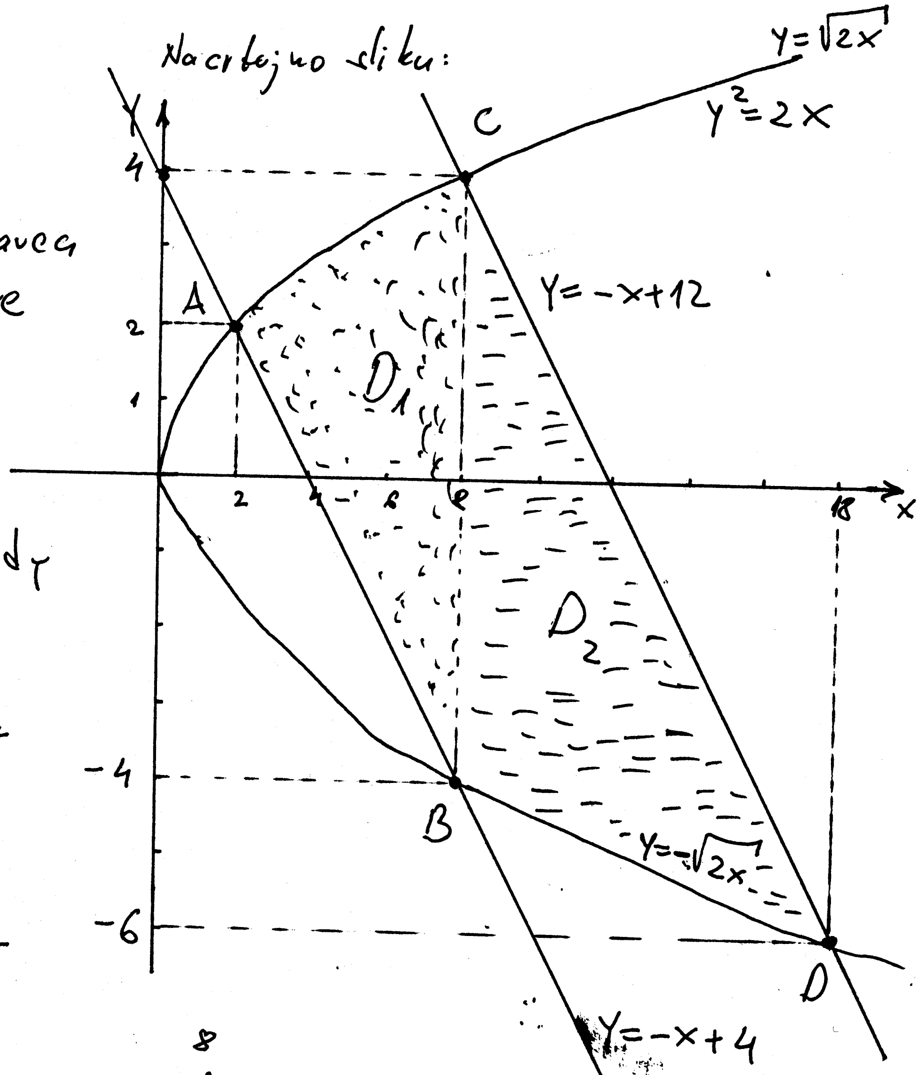
$A(2,2)$
$B(8,-4)$

$$\begin{array}{l} y^2=2x \\ x+y=12 \\ \hline y^2=2x \\ y=12-x \\ \hline (12-x)^2=2x \\ 144-24x+x^2=2x \\ x^2-26x+144=0 \end{array} \qquad \begin{array}{l} D=676-576 \\ D=100 \\ x_{1,2}=\frac{26\pm 10}{2} \\ x_1=8 \quad x_2=18 \\ C(8,4) \\ D(18,-6) \end{array}$$

$$\begin{array}{l} x+y=4 \\ x+y=12 \\ \hline y=-x+4 \\ y=-x+12 \end{array}$$

ove dvije prave imaju isti koeficijent pravca dvije paralelne prave

Nacrtajmo sliku:



$$I=\iint\limits_D (x+y)\,dx\,dy=\iint\limits_{D_1}(x+y)\,dx\,dy+\iint\limits_{D_2}(x+y)\,dx\,dy$$

$$\iint\limits_{D_1}(x+y)\,dx\,dy=\int_2^8\left[\int_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}(x+y)\,dy\right]dx=$$

$$=\int_2^8\left(xy\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}+\frac{1}{2}y^2\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}\right)dx=$$

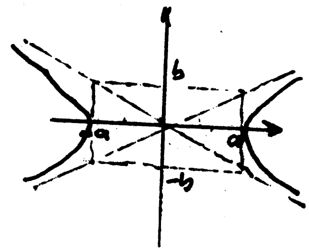
$$=\int_2^8\left[x\left(\sqrt{2}\sqrt{x}+x-4\right)+\frac{1}{2}\left(2x-(-x+4)^2\right)\right]dx=\ldots=\int_2^8\left(x+\sqrt{2}\,x^{\frac{3}{2}}+\frac{1}{2}x^2-8\right)dx=\ldots=\frac{826}{5}$$

$$\iint\limits_{D_2}(x+y)\,dx\,dy=\int_8^{18}\left[\int_{-\sqrt{2x}}^{-x+12}(x+y)\,dy\right]dx=\ldots=\int_8^{18}\left(\sqrt{2}\,x^{\frac{3}{2}}-x-\frac{1}{2}x+72\right)dx=\ldots=\frac{5678}{15}$$

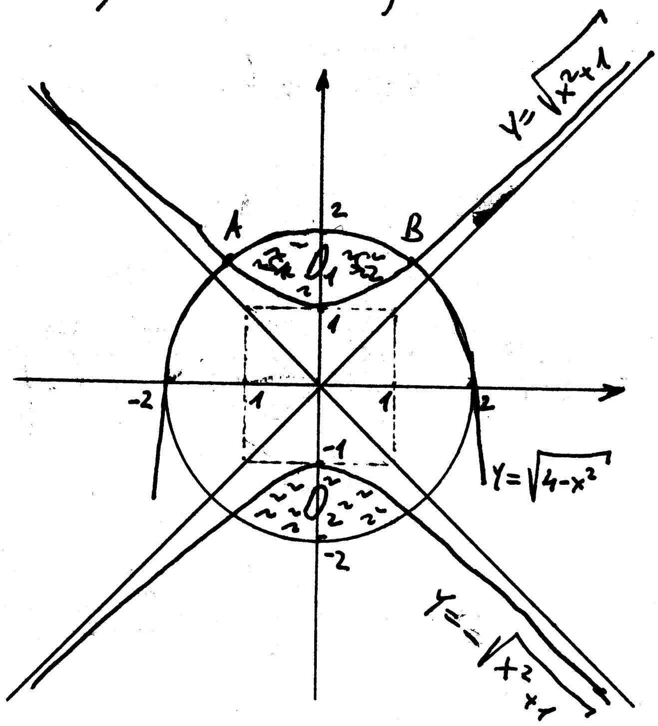
$I=543\frac{11}{15}$ vrijedn. dvostr. integr.

(#) Izračunati $I=\iint\limits_D dx\,dy$, ako je $D: y^2-x^2=1,\ x^2+y^2=4.$

Rj. Krive oblika $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo hiperbole i one su oblika



Skicirajmo naše dvije krive



$x^2+y^2=4$ je krug sa centrom u $(0,0)$ poluprečnika $r=2$

$D=D_1\cup D_2$

$$I=\iint\limits_D dx\,dy=\iint\limits_{D_1\cup D_2}dx\,dy=2\iint\limits_{D_1}dx\,dy$$

$y^2=4-x^2$ $\qquad$ $y^2=x^2+1$

$y=\pm\sqrt{4-x^2}$ $\qquad$ $y=\pm\sqrt{x^2+1}$

Nađimo presječnu tačku krivih $y=\sqrt{x^2+1}$ i $y=\sqrt{4-x^2}$

$\sqrt{x^2+1}=\sqrt{4-x^2}\quad /^2$

$x^2+1=4-x^2$

$2x^2-3=0$

$2x^2=3$

$x^2=\frac{3}{2}$

$x_1=-\sqrt{\frac{3}{2}}$

$x_2=\sqrt{\frac{3}{2}}$

$x_1=-\sqrt{\frac{3}{2}} \Rightarrow y=\sqrt{\frac{3}{2}+1}=\sqrt{\frac{5}{2}}$

Presječne tačke su $A\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$ i $B\left(\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$.

Primjetimo da je oblast D_1 simetrična

$$\iint\limits_{D_1}dx\,dy=\iint\limits_{S_1}dx\,dy+\iint\limits_{S_2}dx\,dy=2\iint\limits_{S_2}dx\,dy=2\int_0^{\sqrt{\frac{3}{2}}}dx\int_{\sqrt{x^2+1}}^{\sqrt{4-x^2}}dy=2\int_0^{\sqrt{\frac{3}{2}}}\left(\sqrt{4-x^2}-\sqrt{x^2+1}\right)dx$$

$$\int\sqrt{4-x^2}\,dx=2\arcsin\frac{x}{2}+\frac{x\sqrt{4-x^2}}{2}$$ (sami)

(metoda Ostrogradskog)

$$\int\sqrt{x^2+1}\,dx=\int\frac{x^2+1}{\sqrt{x^2+1}}dx=(ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}}\quad /'$$

$$\frac{x^2+1}{\sqrt{x^2+1}} = a\sqrt{x^2+1} + (ax+b)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} + \lambda\frac{1}{\sqrt{x^2+1}} \qquad /\cdot\sqrt{x^2+1}$$

$$x^2+1 = \underline{a}(x^2+1) + \underline{a}x^2 + \underline{\underline{b}}x + \lambda$$

$$2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$

$$b = 0$$

$$a + \lambda = 1 \qquad \lambda = \frac{1}{2}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\int\frac{dx}{\sqrt{x^2+1}}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| + C$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{4-x^2}\,dx = 2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} \qquad \text{(sami)}$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{x^2+1}\,dx = \frac{1}{2}\cdot\sqrt{\frac{3}{2}}\,\overbrace{\sqrt{\frac{3}{2}+1}}^{\sqrt{\frac{5}{2}}} + \frac{1}{2}\ln\left|\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}+1}\right| = \frac{\sqrt{15}}{4} + \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

$$I = \iint_D dx\,dy = 2\iint_{D_1} dx\,dy = 2\cdot 2\iint_{S_2} dx\,dy = 4\left(2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} - \right.$$

$$\left.\frac{\sqrt{15}}{4} - \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|\right) = 8\arcsin\frac{\sqrt{6}}{4} - 2\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

traženo mjerenje.

Smjena promjenjivih u dvostrukom integralu

Neka je dat integral $I = \iint_D f(x,y)\,dx\,dy$.

Ako uvodimo nove promjenjive u i v takve da je
$x = \varphi(u,v)$
$y = \psi(u,v)$ tada se oblast D preslikava u D'. Jakobijan transformacijom $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ i imamo

$$I = \iint_{D'} f(\varphi(u,v), \psi(u,v))\,|J|\,du\,dv,$$

$$dx\,dy = |J|\,du\,dv$$

Npr. smjena polarnim koordinatama izgleda
$x = r\cos\varphi$
$y = r\sin\varphi$, r i φ su polarne koordinate, $r \geq 0$, $0 \leq \varphi \leq 2\pi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r(\sin^2\varphi + \cos^2\varphi) = r$$

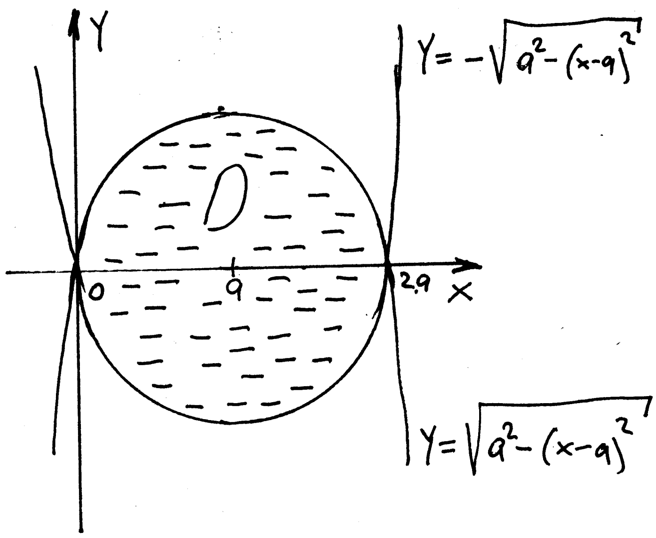
$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(r\cos\varphi, r\sin\varphi)\cdot|r|\,d\varphi\,dr$$

Polarne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u jednačinama koje opisuju oblast integracije pojavljuje izraz x^2+y^2.

Poopštene polarne koordinate izgledaju $x = ar\cos\varphi$ $(a>0)$
$y = br\sin\varphi$ $(b>0)$

$J = \ldots = abr$ (ostavljamo za vježbu kako doći do ovog rezultata)

Izračunati $\iint\limits_D (x^2+y^2)\,dx\,dy$ gdje je D unutrašnjost kruga $x^2+y^2=2ax$.



Rj.

$$x^2+y^2=2ax$$
$$x^2-2ax+y^2=0$$
$$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$$
$$(x-a)^2+y^2=a^2$$

$S(a,0)$ centar

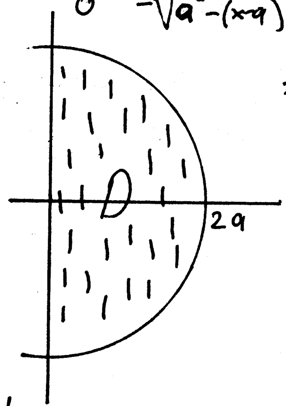
poluprečnik a

$$\iint\limits_D (x^2+y^2)\,dx\,dy=\int_0^{2a}\left(\int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}}(x^2+y^2)\,dy\right)dx=$$

$x=r\cos\varphi$

$y=r\sin\varphi$

$x^2+y^2=r^2$

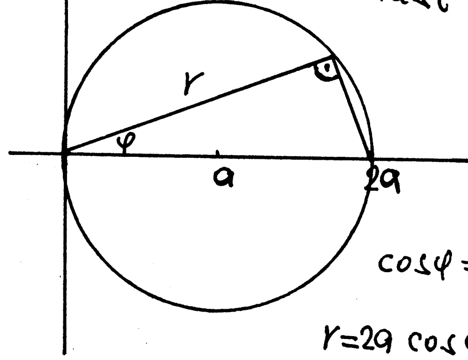


za ovakvu oblast imali bi

$0\le r\le 2a$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

za našu oblast



$\cos\varphi=\frac{r}{2a}$

$r=2a\cos\varphi$

$0\le r\le 2a\cos\varphi$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$dx\,dy=|J|\,dr\,d\varphi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$=$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2a\cos\varphi}r^2|r|\,dr\right]d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2a\cos\varphi}r^3\,dr\right]d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}r^4\Big|_0^{2a\cos\varphi}d\varphi=4a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,d\varphi$$

$$=\left|\cos^4\varphi=(\cos^2\varphi)^2=\left(\frac{1+\cos2\varphi}{2}\right)^2=\frac{1}{4}\left(\cos^2 2\varphi+2\cos2\varphi+1\right)\right|=$$

$1=\sin^2\varphi+\cos^2\varphi$

$\cos2\varphi=\cos^2\varphi-\sin^2\varphi$

$$= a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 2\varphi + 2\cos 2\varphi + 1)\, d\varphi = a^4 \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 4\varphi)\, d\varphi + 2\cdot \frac{1}{2} \sin 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right] \overset{(*)}{=}$$

$$1 = \sin^2 2\varphi + \cos^2 2\varphi$$

$$\cos 4\varphi = \cos^2 2\varphi - \sin^2 \varphi^2$$

$$1 + \cos 4\varphi = 2\cos^2 2\varphi$$

$$\cos^2 2\varphi = \frac{1}{2}(1+\cos 4\varphi)$$

$$\int \cos 2\varphi\, d\varphi = \left| \begin{array}{l} 2\varphi = t \\ 2d\varphi = dt \\ d\varphi = \frac{1}{2} dt \end{array} \right| = \frac{1}{2} \int \cos t\, dt$$

$$= \frac{1}{2} \sin t + c = \frac{1}{2} \sin 2\varphi + c$$

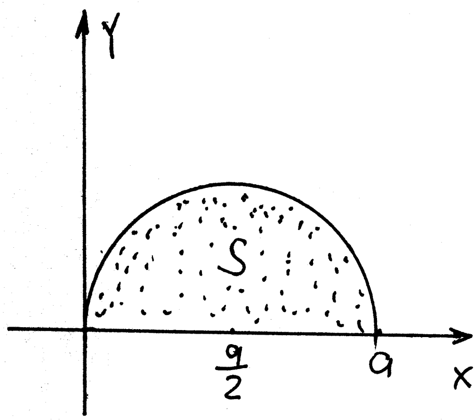
$$\overset{(*)}{=} a^4 \left[\frac{1}{2}\pi + \frac{1}{2}\cdot\frac{1}{4} \sin 4\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0 + \pi \right] = a^4 \left[\frac{3\pi}{2} + \frac{1}{8}\cdot 0 \right] = \frac{3\pi}{2} a^4$$

II način: Uvodimo smjenu $x = a + r\cos\varphi$, $y = r\sin\varphi$, $0 \le \varphi \le 2\pi$, $0 \le r \le a$

URADITI ZA VJEŽBU

Izračunati integral $\iint_S y\,dx\,dy$ gdje je S unutrašnjost gornjeg polukruga poluprečnika $\frac{a}{2}$ sa središtom u tački $(\frac{a}{2}, 0)$.

Rj.



I način:

$$\iint_S y\,dx\,dy = \dots = \int_0^{\frac{\pi}{2}}\left[\int_0^{a\cos\varphi} r\sin\varphi \cdot |r|\,dr\right]d\varphi$$

$$= \dots = \frac{a^3}{12}$$

OSTAVJAMO ZA VJEŽBU KAKO SMO OVO DOBILI

II način:

$$\iint_S y\,dx\,dy = \left| \begin{array}{l} x = \frac{a}{2} + r\cos\varphi \\ y = r\sin\varphi \\ 0 \le \varphi \le \pi \\ 0 \le r \le \frac{a}{2} \end{array} \right.$$

$dx\,dy = |J|\,dr\,d\varphi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$$J = r$$

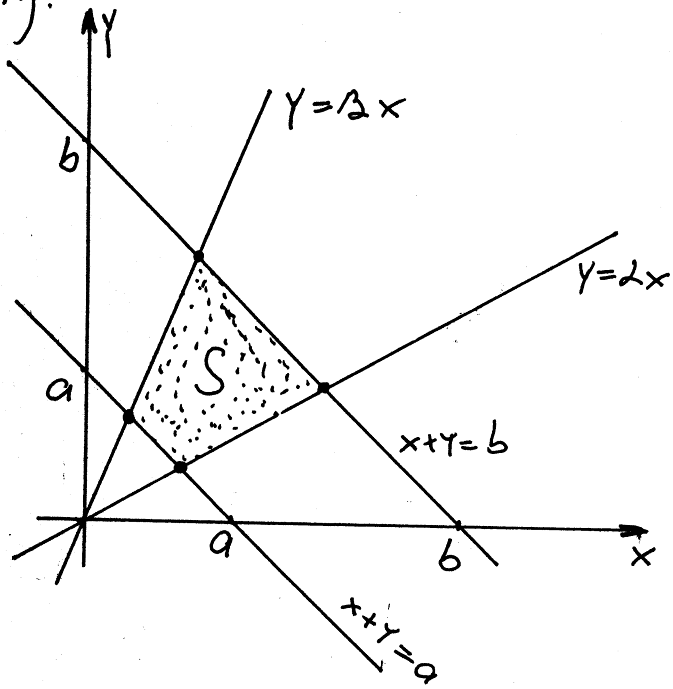
$$= \int_0^{\pi}\left[\int_0^{\frac{a}{2}} r\sin\varphi \cdot |r|\,dr\right]d\varphi = \int_0^{\pi} \sin\varphi \, \frac{1}{3} r^3 \Big|_0^{\frac{a}{2}} d\varphi = \frac{a^3}{24}\int_0^{\pi} \sin\varphi\,d\varphi =$$

$$= \frac{a^3}{24}(-\cos\varphi)\Big|_0^{\pi} = -\frac{a^3}{24}(-1-1) = \frac{a^3}{12}$$

Izračunati integral po oblasti S $\iint\limits_S \frac{1}{xy}\,dx\,dy$

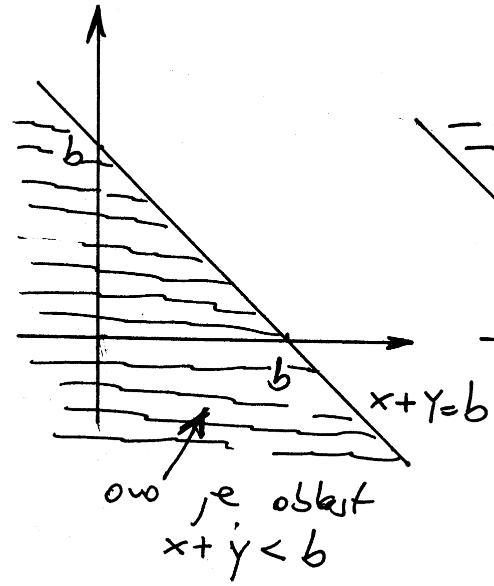
gdje je S oblast ograničena pravama $x+y=a$, $x+y=b$, $y=\alpha x$, $y=\beta x$ gdje su $0<a<b$ i $0<\alpha<\beta$.

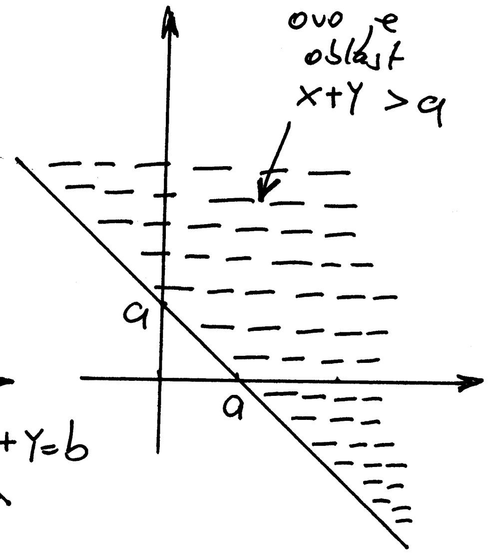
Rj.

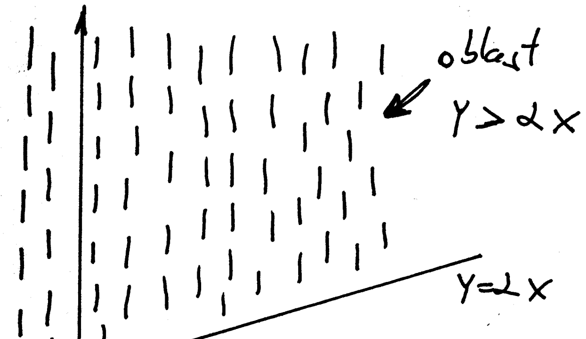


Na klasičan način ovaj zadatak nije lagano uraditi. Integral ćemo izračunati uvođenjem smjene.

Primjetimo







Iz ovoga možemo primjetiti da je S oblast gdje je $x+y$ između a i b a $\frac{y}{x}$ između α i β.

$$\iint\limits_S \frac{1}{xy}\,dx\,dy = \left| \begin{array}{ll} u=x+y & a\le u\le b \\ v=\frac{y}{x} & \alpha\le v\le\beta \end{array} \quad J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \begin{array}{l} y=vx \\ u=x+vx \\ u=x(1+v) \end{array} \right.$$

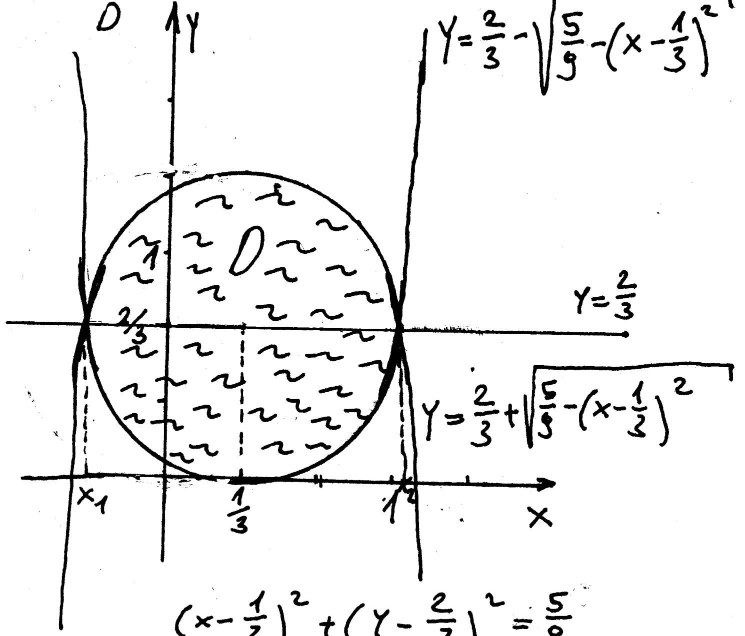
$$x=\frac{u}{1+v} \qquad y=\frac{uv}{1+v}$$

$$\frac{\partial x}{\partial u}=\frac{1}{1+v} \qquad \frac{\partial x}{\partial v}=u\cdot(-1)(1+v)^{-2}=\frac{-u}{(1+v)^2} \qquad dx\,dy=|J|\,du\,dv$$

$$\frac{\partial y}{\partial u}=\frac{v}{1+v} \qquad \frac{\partial y}{\partial v}=\frac{u(1+v)-uv\cdot 1}{(1+v)^2}=\frac{u}{(1+v)^2} \qquad J=\frac{u}{(1+v)^3}+\frac{uv}{(1+v)^3}=\frac{u}{(1+v)^2} \Bigg| =$$

$$=\int_a^b\left[\int_\alpha^\beta \frac{1}{\frac{u}{1+v}\cdot\frac{uv}{1+v}}\cdot\left|\frac{u}{(1+v)^2}\right|dv\right]du=\int_a^b\left[\int_\alpha^\beta\frac{1}{u}\cdot\frac{1}{v}\,dv\right]du=\int_a^b\frac{1}{u}\ln v\Big|_\alpha^\beta=\ln\frac{\beta}{\alpha}\cdot\ln u\Big|_a^b=\ln\frac{\beta}{\alpha}\ln\frac{b}{a}$$

(#) Izračunati dvostruki integral $I=\iint\limits_D (x^2+y^2)\,dx\,dy$ gdje je $D=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x^2+y^2\le \frac{2}{3}(x+2y)\right\}$.



Rj. Odredimo šta je oblast D.

$$x^2+y^2\le\frac{2}{3}(x+2y)$$
$$x^2+y^2\le\frac{2}{3}x+\frac{4}{3}y$$
$$x^2-\frac{2}{3}x+y^2-\frac{4}{3}y\le 0$$
$$x^2-2\cdot x\cdot\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+y^2-2\cdot y\cdot\frac{2}{3}+\frac{4}{9}-\frac{4}{9}\le 0$$
$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2\le\frac{5}{9}$$

D predstavlja unutrašnjost kruga s centrom u tački $\left(\frac{1}{3},\frac{2}{3}\right)$ poluprečnika $r=\frac{\sqrt{5}}{3}\approx 0,74$

$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2=\frac{5}{9}$$
$$y=\frac{2}{3}$$

$$\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$$
$$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{3}$$
$$x_{1,2}=\frac{1\pm\sqrt{5}}{3}$$

$$\left(y-\frac{2}{3}\right)^2=\frac{5}{9}-\left(x-\frac{1}{3}\right)^2$$
$$y=\frac{2}{3}\pm\sqrt{\frac{5}{9}-\left(x-\frac{1}{3}\right)^2}$$

I način: klasičan način

Nađimo presječnu tačku kruga i prave $y=\frac{2}{3}$

$$I=\iint\limits_D(x^2+y^2)\,dx\,dy=\int_{\frac{1-\sqrt{5}}{3}}^{\frac{1+\sqrt{5}}{3}}\left[\int_{\frac{2}{3}-\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}^{\frac{2}{3}+\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}(x^2+y^2)\,dy\right]dx=\ldots$$

NA KLASIČAN NAČIN OVO JE TEŠKO URADITI

II način: Uvedimo neku smjenu promjenjivih. Kako je dat krug uvedimo polarne koordinate.

$$x=a+r\cos\varphi \qquad y=b+r\sin\varphi$$

tj. $x=\frac{1}{3}+r\cos\varphi$, $y=\frac{2}{3}+r\sin\varphi$

$0\le\varphi\le 2\pi$, $0\le r\le\frac{\sqrt{5}}{3}$ ove vrijednosti čitamo sa slike

$dx\,dy=r\,dr\,d\varphi$.

Jakobijan

$$dx\,dy=|J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$$x^2+y^2=\left(\frac{1}{3}+r\cos\varphi\right)^2+\left(\frac{2}{3}+r\sin\varphi\right)^2=\underline{\underline{\frac{1}{9}}}+\frac{2}{3}r\cos\varphi+\underline{r^2\cos^2\varphi}+\underline{\underline{\frac{4}{9}}}+\frac{4}{3}r\sin\varphi+\underline{r^2\sin^2\varphi}=$$
$$=\frac{5}{9}+r^2+\frac{2}{3}r\cos\varphi+\frac{4}{3}r\sin\varphi$$

$$I=\iint\limits_D(x^2+y^2)\,dx\,dy=\iint\limits_{D'}\left(\frac{5}{9}+r^2+\frac{2}{3}r(\cos\varphi+2\sin\varphi)\right)r\,dr\,d\varphi=\iint\limits_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi+$$
$$+\frac{2}{3}\iint\limits_{D'}r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi,\quad \iint\limits_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{\sqrt{5}}{3}}\left(\frac{5}{9}r+r^3\right)dr\right]d\varphi=2\pi\cdot\left(\frac{5}{9}\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{5}}{3}}+\right.$$

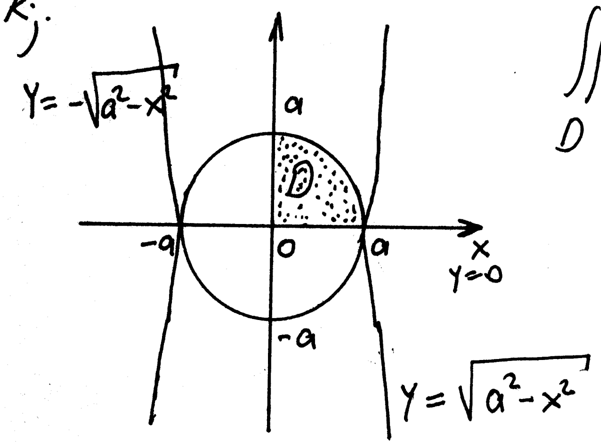
$$+\frac{1}{4}r^4\Big|_0^{\frac{\sqrt{5}}{3}}\Big) = 2\pi\left(\frac{5}{9\cdot 2}\cdot\frac{5}{9}+\frac{1}{4}\cdot\frac{5\cdot 5}{9\cdot 9}\right) = \pi\left(\frac{5^2}{9^2}+\frac{1}{2}\cdot\frac{5^2}{9^2}\right) = \frac{3}{2}\,\frac{5^2}{9^2}\pi = \frac{25}{54}\pi$$

$$\iint_{D'} r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi = \int_0^{\sqrt{5}/3} r^2\left[\int_0^{2\pi}(\cos\varphi+2\sin\varphi)\,d\varphi\right] = \frac{r^3}{3}\Big|_0^{\sqrt{5}/3}\left(\sin\varphi\Big|_0^{2\pi}-2\cos\varphi\Big|_0^{2\pi}\right)=0$$

Prema tome $\displaystyle\iint_D (x^2+y^2)\,dx\,dy = \frac{25}{54}\pi$

(#) Izračunati $I=\displaystyle\iint_D \sqrt{x^2+y^2}\,dx\,dy$ gdje je D četvrtina kruga $x^2+y^2\le a^2$.

R.:



$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \int_0^a\left(\int_0^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\,dy\right)dx =$$

$$= \left|\begin{array}{ll} x=r\cos\varphi & 0\le x\le a \\ y=r\sin\varphi & 0\le y\le\sqrt{a^2-x^2} \\ x^2+y^2=r^2(\cos^2\varphi+\sin^2\varphi)=r^2 & \Downarrow \\ & 0\le\varphi\le\frac{\pi}{2} \\ & 0\le r\le a \end{array}\right.$$

$$dx\,dy = |J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix} = \begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix} = r \quad\Bigg| \quad = \int_0^a\left[\int_0^{\frac{\pi}{2}}\sqrt{r^2}\,|r|\,d\varphi\right]dr$$

$$=\int_0^a\left[\int_0^{\frac{\pi}{2}} r^2\,d\varphi\right]dr = \int_0^a r^2\cdot\varphi\Big|_0^{\frac{\pi}{2}}\,dr = \int_0^a\frac{\pi}{2}r^2\,dr = \frac{\pi}{2}\cdot\frac{1}{3}r^3\Big|_0^a = \frac{a^3\pi}{6}$$

(#) Izračunati dvostruki integral

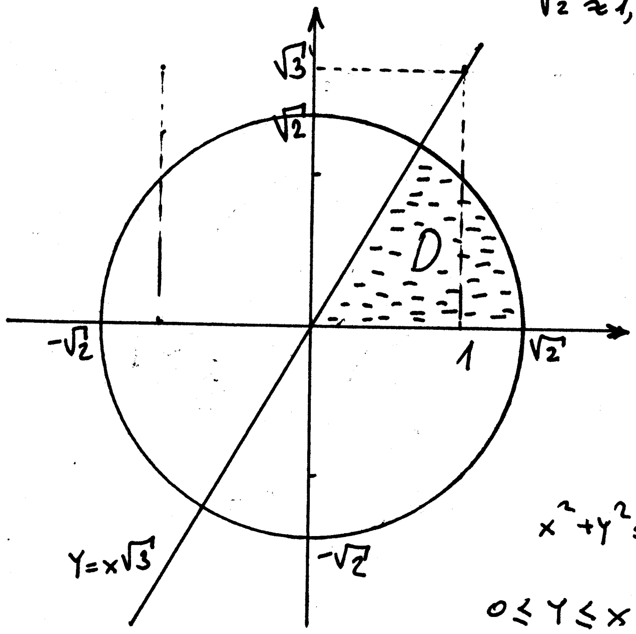
$$I=\iint\limits_D x^2y^2\sqrt{(x^2+y^2)^3+1}\,dx\,dy,\quad D:\ x^2+y^2\le 2,\ 0\le y\le x\sqrt{3}.$$

Rj. Nacrtajmo oblast integracije D

$\sqrt{2}\approx 1{,}41$

$y=x\sqrt{3}$

$x=1 \Rightarrow y=\sqrt{3}\approx 1{,}73$



Uvedimo polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$

$D \xrightarrow{\text{transformiše}} D'$

$x^2+y^2\le 2 \Rightarrow r^2\le 2$

$$D':\begin{cases}0\le r\le\sqrt{2}\\ 0\le\varphi\le\dfrac{\pi}{3}\end{cases}$$

$0\le y\le x\sqrt{3}$

$0\le r\sin\varphi\le r\cos\varphi\sqrt{3}\quad /:r$

$0\le\sin\varphi\le\cos\varphi\sqrt{3}$

$x^2y^2=r^2\cos^2\varphi\, r^2\sin^2\varphi$

$=r^4\cos^2\varphi\sin^2\varphi$

$$\left.\begin{array}{l}\sin\varphi\ge 0\\ \quad\varphi\in[0,\pi]\\ 0\le\sin\varphi\le\cos\varphi\sqrt{3}\quad /:\cos\varphi\\ 0\le \operatorname{tg}\varphi\le\sqrt{3}\\ \quad\varphi\in[0,\frac{\pi}{3}]\cup[\pi,\frac{4\pi}{3}]\end{array}\right\}\Rightarrow \varphi\in[0,\tfrac{\pi}{3}]$$

$1=\sin^2\varphi+\cos^2\varphi$

$\cos2\varphi=\cos^2\varphi-\sin^2\varphi$

$\cos^2\varphi=\dfrac{1+\cos2\varphi}{2}\qquad \sin^2\varphi=\dfrac{1-\cos2\varphi}{2}$

$$I=\iint\limits_D x^2y^2\sqrt{(x^2+y^2)^3+1}\,dx\,dy=\iint\limits_{D'} r^4\cos^2\varphi\sin^2\varphi\sqrt{r^6+1}\,r\,dr\,d\varphi=$$

$$=\int_0^{\sqrt{2}} r^5\sqrt{r^6+1}\,dr\int_0^{\pi/3}\cos^2\varphi\sin^2\varphi\,d\varphi=\left(\frac{\sqrt{3}}{64}+\frac{\pi}{24}\right)\int_0^{\sqrt{2}} r^5\sqrt{r^6+1}\,dr=\underline{\underline{\frac{26}{9}\left(\frac{\sqrt{3}}{64}+\frac{\pi}{24}\right)}}$$

$$\int_0^{\sqrt{2}} r^5\sqrt{r^6+1}\,dr=\left|\begin{array}{ll}r^6+1=t & r=0\Rightarrow t=1\\ 6r^5dr=dt & r=\sqrt{2}\Rightarrow t=9\end{array}\right|=\frac{1}{6}\int_1^9\sqrt{t}\,dt=\frac{1}{6}\cdot\left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^9=\left.\frac{\cancel{2}}{\cancel{6}_3\cdot 3}\sqrt{t^3}\right|_1^9=$$

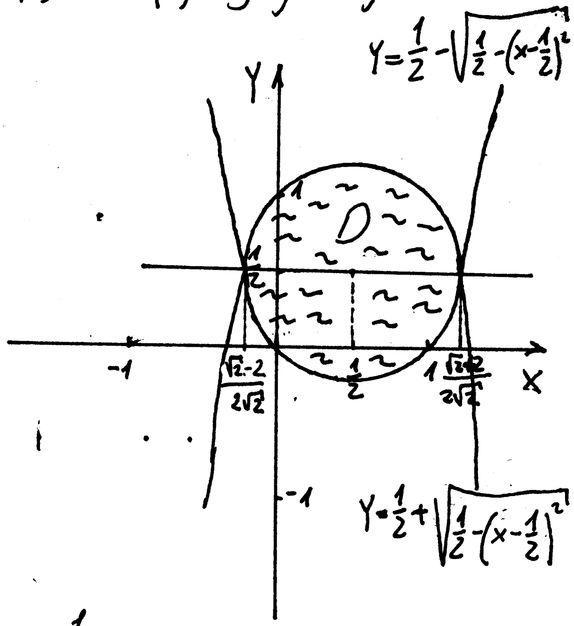
$$=\frac{26}{9}$$

$$\int_0^{\pi/3}\cos^2\varphi\sin^2\varphi\,d\varphi=\int_0^{\pi/3}\frac{1}{2}(1+\cos2\varphi)\cdot\frac{1}{2}(1-\cos2\varphi)\,d\varphi=\ldots=\frac{\sqrt{3}}{64}+\frac{\pi}{24}$$

tražena zapremina

Izračunati dvostruki integral: $I=\iint\limits_D (x+y)\,dx\,dy$, gdje je

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le x+y\}$.



Rj.
$$x^2+y^2\le x+y$$
$$x^2-x+y^2-y\le 0$$
$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}\le 0$$
$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{2}$$

Unutrašnjost kruga sa centrom u tački $S(\frac{1}{2},\frac{1}{2})$ poluprečnika $r=\frac{1}{\sqrt{2}}\approx 0,7$.

Nađimo presječne tačke kruga sa pravom $y=\frac{1}{2}$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2},\quad y=\frac{1}{2}$$
$$\left(x-\frac{1}{2}\right)^2=\frac{1}{2}\qquad x-\frac{1}{2}=\pm\frac{1}{\sqrt{2}}$$
$$x_1=\frac{1}{2}+\frac{1}{\sqrt{2}}=\frac{2+\sqrt{2}}{2\sqrt{2}}\qquad x_2=\frac{1}{2}-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-2}{2\sqrt{2}}$$

I način:

$$I=\iint\limits_D (x+y)\,dx\,dy=\int_{\frac{\sqrt{2}-2}{2\sqrt{2}}}^{\frac{\sqrt{2}+2}{2\sqrt{2}}}\left[\int_{\frac{1}{2}-\sqrt{\frac{1}{2}-(x-\frac{1}{2})^2}}^{\frac{1}{2}+\sqrt{\frac{1}{2}-(x-\frac{1}{2})^2}}(x+y)\,dy\right]dx=\ldots$$

KOMPLIKOVANO

$$\left(y-\frac{1}{2}\right)^2=\frac{1}{2}-\left(x-\frac{1}{2}\right)^2$$
$$y-\frac{1}{2}=\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}$$
$$y=\frac{1}{2}\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}$$

II način: Uvedimo neku smjenu promjenljivih.
Kako je u pitanju krug, uvedimo polarne koordinate.

$$x=a+r\cos\varphi\qquad y=b+r\sin\varphi$$

tj. $x=\frac{1}{2}+r\cos\varphi$, $y=\frac{1}{2}+r\sin\varphi$, $0\le\varphi\le 2\pi$, $0\le r\le\frac{\sqrt{2}}{2}$ — ove vrijednosti čitamo sa slike

Jakobijan: $dx\,dy=|J|\,dr\,d\varphi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r.$$

$$dx\,dy=r\,dr\,d\varphi$$

$$I=\iint\limits_D (x+y)\,dx\,dy=\iint\limits_{D'}\left(\frac{1}{2}+r\cos\varphi+\frac{1}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint\limits_{D'}(r+r^2(\cos\varphi+\sin\varphi))\,dr\,d\varphi$$

$$=\iint\limits_{D'}r\,dr\,d\varphi+\iint\limits_{D'}r^2(\cos\varphi+\sin\varphi)\,dr\,d\varphi,\qquad \iint\limits_{D'}r\,dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{\sqrt{2}}{2}}r\,dr\right]d\varphi=2\pi\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{2}}{2}}=\frac{\pi}{2}$$

$$\iint_{D'} r^2(\cos\varphi+\sin\varphi)\,dr\,d\varphi = \int_0^{2/\sqrt{2}} r^2\left[\int_0^{2\pi}(\cos\varphi+\sin\varphi)\,d\varphi\right]dr = \frac{1}{3}r^3\Big|_0^{\frac{2}{\sqrt{2}}}\cdot\left(\sin\varphi\Big|_0^{2\pi} - \cos\varphi\Big|_0^{2\pi}\right)$$

$$= \frac{1}{3}\cdot\frac{8}{2\sqrt{2}}\left(0-(1-1)\right) = 0$$

Prema tome:

$$\iint_D (x+y)\,dx\,dy = \frac{\pi}{2}$$

Zadaci za vježbu

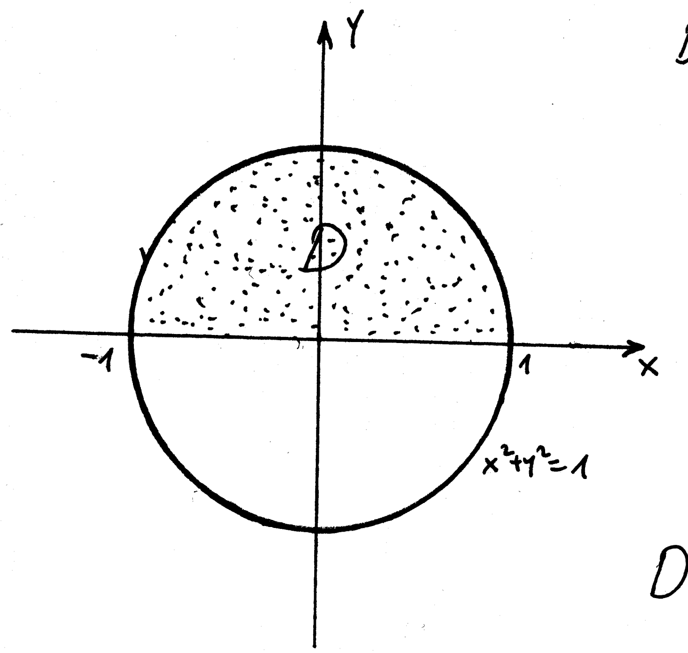
1. Izračunati dvostruki integral $\iint_S x\,dx\,dy$ gdje je područje integracije S ograničeno pravcem koji prolazi tačkama $A(2,0)$ $B(0,2)$ i lukom kruga poluprečnika 1 sa centrom u tački $C(0,1)$.
2. Izračunati dvostruki integral $\iint_S \frac{dx\,dy}{\sqrt{2a-x}}$ gdje je S krug poluprečnika a, koji dira koordinatne ose i leži u prvom kvadrantu
3. Izračunati dvostruki integral $\iint_S \sqrt{xy-y^2}\,dx\,dy$ gdje je S trokut sa vrhovima $O(0,0)$, $A(10,1)$ i $B(1,1)$.

Rješenja: 1. $\frac{1}{3}$ 2. $\frac{8}{3}a\sqrt{2a}$ 3. 6

(#) Izračunati integral $I=\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, ako je

D oblast data sa: $x^2+y^2 \le 1$, $y \ge 0$.

Rj. Skicirajmo oblast D



$$D:\begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$$

Uvedimo polarne koordinate

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

$$D \xrightarrow{\text{transformacije}} D', \quad D':\begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \end{cases}$$

$$1-x^2-y^2 = 1-(x^2+y^2) = 1-r^2$$
$$1+x^2+y^2 = 1+r^2$$

$$I=\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy = \iint\limits_{D'} \sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\varphi = \int_0^{\pi} d\varphi \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr$$

Izračunajmo posebno drugi integral

$$\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr = \int_0^1 \frac{\sqrt{1-r^2}}{\sqrt{1+r^2}}\,r\,dr = \int_0^1 \frac{1-r^2}{\sqrt{(1+r^2)(1-r^2)}}\cdot r\,dr = \int_0^1 \frac{r}{\sqrt{1-r^4}}\,dr - \int_0^1 \frac{r^3}{\sqrt{1-r^4}}\,dr$$

$$r\Big|_0^1 \Rightarrow t\Big|_0^1$$

$$\int_0^1 \frac{r}{\sqrt{1-r^4}}\,dr = \left| \begin{matrix} r^2 = t \\ 2r\,dr = dt \\ r\,dr = \frac{1}{2}dt \end{matrix} \right| = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\arcsin t\Big|_0^1 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

$$\int_0^1 \frac{r^3}{\sqrt{1-r^4}}\,dr = \left| \begin{matrix} 1-r^4 = s^2 & r^3dr = -\frac{1}{2}s\,ds \\ -4r^3dr = 2s\,ds & r\Big|_0^1 \Rightarrow s\Big|_1^0 \end{matrix} \right| = -\frac{1}{2}\int_1^0 \frac{s\,ds}{\sqrt{s^2}} = \frac{1}{2}$$

$$I = \int_0^{\pi} d\varphi \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr = \varphi\Big|_0^{\pi}\cdot\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi^2}{4}-\frac{\pi}{2}$$ traženo rješenje