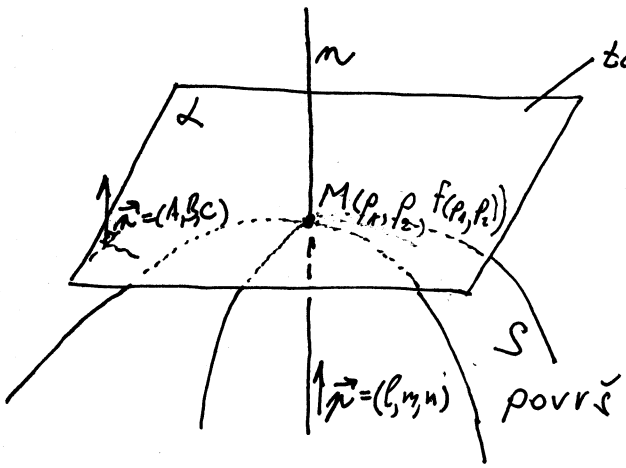
Jednačina tangentne ravni i jednačina normale na površ

Jednačina tangentne ravni (hiperravni) na površ S, čija je jednačina $z=f(x_1,x_2)$, u tački $M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ (ako je f diferencijabilna u tački (ρ_1,ρ_2) glasi:

$$z-f(\rho_1,\rho_2)=f'_{x_1}(\rho_1,\rho_2)(x_1-\rho_1)+f'_{x_2}(\rho_1,\rho_2)(x_2-\rho_2)$$

Može li se uspostaviti sličnost sa jednačinom tangente na krivu liniju $y=f(x)$ u ravni?



tangentna ravan

$$Ax+By+Cz+D=0$$

$M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ tačka dodira

n - normala na površ $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

Jednačina normale na površ $z=f(x,y)$ u tački $M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ (ako je f diferencijabilna u (ρ_1,ρ_2)) glasi:

$$\frac{x-\rho_1}{f'_x(\rho_1,\rho_2)}=\frac{y-\rho_2}{f'_y(\rho_1,\rho_2)}=\frac{z-f(\rho_1,\rho_2)}{-1}$$

sličnost sa krivom $y=f(x)$ u ravni: $k_1\cdot k_2=-1$, $M(\rho_1,\rho_2)$ $y-\rho_2=f'(\rho_1)(x-\rho_1)$

$k_2=\frac{-1}{k_1}$, $y-\rho_2=\frac{-1}{f'(\rho_1)}(x-\rho_1)$

$$\frac{x-\rho_1}{f'(\rho_1)}=\frac{y-\rho_2}{-1}$$

Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$

$$\alpha:\ F'_x(\rho_1,\rho_2,f(\rho_1,\rho_2))(x-\rho_1)+F'_y(\rho_1,\rho_2,f(\rho_1,\rho_2))(y-\rho_2)+F'_z(\rho_1,\rho_2,f(\rho_1,\rho_2))(z-f(\rho_1,\rho_2))=0$$

$$n:\ \frac{x-\rho_1}{F'_x(\rho_1,\rho_2,f(\rho_1,\rho_2))}=\frac{y-\rho_2}{F'_y(\rho_1,\rho_2,f(\rho_1,\rho_2))}=\frac{z-f(\rho_1,\rho_2)}{F'_z(\rho_1,\rho_2,f(\rho_1,\rho_2))}$$

(#) Naći jednačinu tangentne ravni i normale na površ

a) $z=\frac{x^2}{2}-y^2$ u tački $M(2,-1,1)$

b) $3xyz-z^3=a^3$ u tački za koju je $x=0,\ y=a$

c)✓ $z=x^2+2y^2$ u tački $A(1,1,3)$

d)✓ $z=\operatorname{arc\,tg}\frac{y}{x}$ u tački $(1,1,\frac{\pi}{4})$ → rj. $\alpha: z=\frac{\pi}{4}-\frac{1}{2}(x-y)$, $n: \frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-\frac{\pi}{4}}{2}$

e)✓ $z=\sqrt{169-x^2-y^2}$ → rj. u tački $(3,4,z(3,4))$ $\alpha: 3x+4y+12z-169=0$, $n: \frac{x-3}{3}=\frac{y-4}{4}=\frac{z-12}{12}$

f)✓ $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{8}=0$ u tački $M(4,3,4)$ → rj. $\alpha: 3x+4y-6z=0$, $n: \frac{x-4}{3}=\frac{y-3}{4}=\frac{z-4}{-6}$

g)✓ $x^2+y^2+z^2=2Rz$ u tački $(R\cos\alpha, R\sin\alpha, R)$ $(R>0)$.

Rj. a) $z=f(x,y)$, $z-f(p_1,p_2)=f'_x(p_1,p_2)(x-p_1)+f'_y(p_1,p_2)(y-p_2)$ jedn. tang. ravni

$z=\frac{x^2}{2}-y^2$, $z'_x=x$, $z'_x(2,-1)=2$, $\frac{\partial z}{\partial y}=-2y$, $z'_y(2,-1)=2$

$M(2,-1,1)$, $f(2,-1)=1$ $\qquad z-1=2(x-2)+2(y+1)$

$2x+2y-z-1$ jednačina tangentne ravni

$\frac{x-p_1}{f'_x(p_1,p_2)}=\frac{y-p_2}{f'_y(p_1,p_2)}=\frac{z-f(p_1,p_2)}{-1} \Rightarrow \frac{x-2}{2}=\frac{y+1}{2}=\frac{z-1}{-1}$ jedn. normale

b) Nađimo tačku dodira tangentne ravni i površi

$x=0,\ y=a$, $3xyz-z^3=a^3 \Rightarrow -z^3=a^3 \Rightarrow z=-a$

Tačka dodira je $M(0,a,-a)$

$F'_x=3yz \Rightarrow F'_x(0,a,-a)=-3a^2$

$F'_y=3xz \Rightarrow F'_y(0,a,-a)=0$

$F'_z=3xy-3z^2 \Rightarrow F'_z(0,a,-a)=-3a^2$

$\alpha: F'_x(p_1,p_2,f(p_1,p_2))(x-p_1)+F'_y(p_1,p_2,f(p_1,p_2))(y-p_2)+F'_z(p_1,p_2,f(p_1,p_2))(z-f(p_1,p_2))=0$

$-3a^2(x-0)+0(y-a)+(-3a^2)(z-(-a))=0 \Rightarrow -3a^2x-3a^2z-3a^3=0$

tj. $x+z+a=0$ jedn. tang. ravni

$\frac{x-0}{-3a^2}=\frac{y-a}{0}=\frac{z+a}{-3a^2} \Rightarrow \frac{x}{1}=\frac{y-a}{0}=\frac{z+a}{1}$ jednačina normale

c)✓ rj. $\alpha: 2x+4y-z-3=0$, $n: \frac{x-1}{2}=\frac{y-1}{4}=\frac{z-3}{-1}$

g)✓ rj. $\alpha: x\cos\alpha+y\sin\alpha-R=0$, $n: \frac{x-R\cos\alpha}{\cos\alpha}=\frac{y-R\sin\alpha}{\sin\alpha}=\frac{z-R}{0}$

(#) Na površ $x^2+2y^2+3z^2=21$ postaviti tangentnu ravan paralelnu ravni $x+4y+6z=0$.

Rj.

$\beta: Ax+By+Cz+D=0$

$\beta: ?$ $\qquad \alpha \parallel \beta$

$\alpha: x+4y+6z=0$

$\vec{n}_\alpha=(1,4,6)$, $\vec{n}_\beta \parallel \vec{n}_\alpha$

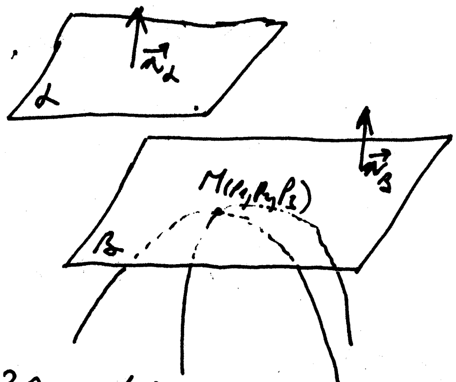
Treba nam tačka dodira tražene tangentne ravni sa površi $x^2+2y^2+3z^2=21$.

$$F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

$F'_x=2x$

$F'_y=4y$

$F'_z=6z$



$$n: \frac{x-p_1}{F'_x(p_1,p_2,p_3)}=\frac{y-p_2}{F'_y(p_1,p_2,p_3)}=\frac{z\,z_0}{F'_z(p_1,p_2,p_3)}$$

Vektor normale tražene tangentne ravni je

$$\vec{n}_\beta=(2p_1,4p_2,6p_3)$$

$$\vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \frac{2p_1}{1}=\frac{4p_2}{4}=\frac{6p_3}{6} \Rightarrow 2p_1=p_2=p_3$$

odredimo p_1, p_2 i p_3

$p_1^2+2\cdot 4p_1^2+3\cdot 4p_1^2=21$

$21p^2=21$

$p_1=\pm 1 \Rightarrow p_2=p_3=\pm 2$

I rješenje:

$p_1=-1,\ p_2=p_3=-2$

$-2(x+1)-8(y+2)-12(z+2)=0$

$-2x-8y-12z=42$

$x+4y+6z=-21$

II rješenje $p_1=1,\ p_2=p_3=2$

$2(x-1)+8(y-2)+12(z-2)=0$

$2x+8y+12z-42=0 \quad /:2$

$x+4y+6z=21$

jednačin tražene tangentne ravni

(#) Odrediti jednačine normale i jednačinu tangentne ravni površi $z=\sqrt{169-x^2-y^2}$ u tački $(3,4,\ z(3,4))$.

Rj:

$$z(3,4)=\sqrt{169-9-16}=\sqrt{144}=12$$

$$M(3,4,12)$$

jednačina tangentne ravni i normale na površ $z=f(x,y)$ u tački $M(p_1,p_2,p_3)$: $\quad z-p_3=z'_x(p_1,p_2)(x-p_1)+z'_y(p_1,p_2)(y-p_2)$

$$\frac{x-p_1}{z'_x(p_1,p_2)}=\frac{y-p_2}{z'_y(p_1,p_2)}=\frac{z-p_3}{-1}$$

$$=\frac{1}{2\sqrt{169-x^2-y^2}}(-2x)=\frac{-x}{\sqrt{169-x^2-y^2}} \Rightarrow z'_x(3,4)=\frac{-3}{\sqrt{169-25}}=\frac{-3}{12}=-\frac{1}{4}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{169-x^2-y^2}}(-2y)=\frac{-y}{\sqrt{169-x^2-y^2}} \Rightarrow z'_y(3,4)=\frac{-4}{12}=-\frac{1}{3}$$

$$z-12=-\frac{1}{4}(x-3)-\frac{1}{3}(y-4) \quad /\cdot 12$$

$$12z-144=-3(x-3)-4(y-4)$$

$$3x+4y+12z-144-9-16=0$$

$$3x+4y+12z-169=0$$ jednačina tangentne ravni na površ z

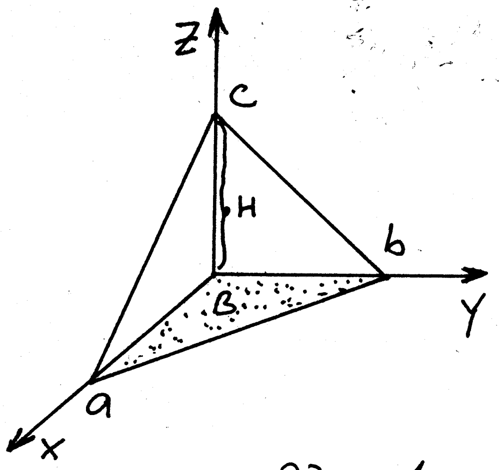
$$\frac{x-3}{-\frac{1}{4}}=\frac{y-4}{-\frac{1}{3}}=\frac{z-12}{-1} \quad /\cdot\left(\frac{1}{-12}\right)$$

$$\frac{x-3}{3}=\frac{y-4}{4}=\frac{z-12}{12}$$ jednačina normale na površ z

(#) Dokazati da tangentne ravni površi $z = \frac{1}{xy}$ tvore s koordinatnim ravnima piramide konstantne zapremine.

Rj.

Jednačina tangentne ravni na površ $z = f(x,y)$ u tački $M(\rho_1, \rho_2, \rho_3)$: $z - \rho_3 = z'_x(\rho_1,\rho_2)(x-\rho_1) + z'_y(\rho_1,\rho_2)(y-\rho_2)$



$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ kanonični oblik jednačine ravni gdje su a, b i c odsječci koje ravan odsjeca na koordinatnim osama

$$V_{piramide} = \frac{B \cdot H}{3} = \frac{\frac{a \cdot b}{2} \cdot c}{3} = \frac{a \cdot b \cdot c}{6}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y} \cdot \frac{-1}{x^2} = \frac{-1}{x^2 y} \Rightarrow z'_x(\rho_1,\rho_2) = \frac{-1}{\rho_1^2 \rho_2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x} \cdot \frac{-1}{y^2} = \frac{-1}{x y^2} \Rightarrow z'_y(\rho_1,\rho_2) = \frac{-1}{\rho_1 \rho_2^2}$$

$$\rho_3 = f(\rho_1,\rho_2) = \frac{1}{\rho_1 \cdot \rho_2}$$

$$z - \frac{1}{\rho_1 \cdot \rho_2} = \frac{-1}{\rho_1^2 \rho_2}(x-\rho_1) + \frac{-1}{\rho_1 \rho_2^2}(y-\rho_2) \quad / \cdot \rho_1^2 \rho_2^2$$

$$\rho_1^2 \rho_2^2 z - \rho_1 \rho_2 = -\rho_2(x-\rho_1) - \rho_1(y-\rho_2)$$

$$\rho_1^2 \rho_2^2 z + \rho_2 x + \rho_2 y = \rho_1\rho_2 + \rho_1\rho_2 + \rho_1\rho_2 \quad / \cdot \frac{1}{\rho_1 \rho_2}$$

$$\frac{x}{\rho_1} + \frac{y}{\rho_1} + \rho_1 \rho_2 z = 3 \quad / \cdot \frac{1}{3}$$

$$\frac{x}{3\rho_1} + \frac{y}{3\rho_2} + \frac{z}{\frac{3}{\rho_1\rho_2}} = 1 \Rightarrow V_{piramide} = \frac{3\rho_1 \cdot 3\rho_2 \cdot \frac{3}{\rho_1\rho_2}}{6} = \frac{9}{2}$$

zapremina piramide za sve tangentne ravni na površ

(#) Nađite udaljenost ishodišta koordinatnog sistema od tangentne ravni (helikoida) $y = x\,tg\frac{z}{a}$ u tački $(a, a, \frac{\pi a}{4})$.

Rj: $F'_x(p_1,p_2,p_3)(x-p_1) + F'_y(p_1,p_2,p_3)(y-p_3) + F'_z(p_1,p_2,p_3)(z-p_3) = 0$

jednačina tangentne ravni na površ $F(x,y,z)=0$

$y - x\,tg\frac{z}{a} = 0$

$\frac{\partial F}{\partial x} = -tg\frac{z}{a} \quad \Rightarrow \quad F'_x(a,a,\frac{\pi a}{4}) = -tg\frac{\frac{\pi a}{4}}{a} = -tg\frac{\pi}{4} = -1$

$\frac{\partial F}{\partial y} = 1 \quad \Rightarrow \quad F'_y(a,a,\frac{\pi a}{4}) = 1$

$\frac{\partial F}{\partial z} = \frac{-x}{\cos^2\frac{z}{a}} \cdot \frac{1}{a} = \frac{-x}{a\cos^2\frac{z}{a}} \quad \Rightarrow \quad F'_z(a,a,\frac{\pi a}{4}) = \frac{-a}{a\cos^2\frac{\pi}{4}} = \frac{-1}{\left(\frac{\sqrt{2}}{2}\right)^2}$

$F'_z(a,a,\frac{\pi a}{4}) = -2$

$-1(x-a) + 1\cdot(y-a) + (-2)(z-\frac{\pi a}{4}) = 0$

$-x + y - 2z + a - a + \frac{\pi a}{2} = 0$

$-x + y - 2z + \frac{\pi a}{2} = 0$

jednačina tangentne ravni helikoida u tački $(a,a,\frac{\pi a}{4})$.

$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2+B^2+C^2}}$, $O(0,0,0)$

$d = \frac{0+0+0+\frac{\pi a}{2}}{\sqrt{1+1+4}} = \frac{\pi a}{2\sqrt{6}}$

udaljenost početka koordinatnog sistema od tangentne ravni

(#) Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ u tački $M(2,2,1)$.

Rj. Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$ tada jednačina tangentne ravni i normale na površ S u tački $M(p_1,p_2,p_3)$ se računaju po formuli:

$$\alpha: F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

$$n: \frac{x-p_1}{F'_x(p_1,p_2,p_3)} = \frac{y-p_2}{F'_y(p_1,p_2,p_3)} = \frac{z-p_3}{F'_z(p_1,p_2,p_3)}$$

$$2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$$

$$\left(\frac{x}{z}\right)'_z = (xz^{-1})'_z = (-1)x\,z^{-2}$$

$$F(x,y,z) = 2^{\frac{x}{z}} + 2^{\frac{y}{z}} - 8 = 0$$

$$F'_x = 2^{\frac{x}{z}} \ln 2 \cdot \frac{1}{z} \Rightarrow F'_x(2,2,1) = 4\ln 2.$$

$$F'_y = 2^{\frac{y}{z}} \ln 2 \cdot \frac{1}{z} \Rightarrow F'_y(2,2,1) = 4\ln 2$$

$$F'_z = 2^{\frac{x}{z}} \ln 2 \cdot \left(\frac{x}{z}\right)'_z + 2^{\frac{y}{z}} \ln 2 \left(\frac{y}{z}\right)'_z = -\frac{x}{z^2} 2^{\frac{x}{z}} \ln 2 - \frac{y}{z^2} 2^{\frac{y}{z}} \ln 2$$

$$= -\frac{1}{z^2} \ln 2 \left(x 2^{\frac{x}{z}} + y 2^{\frac{y}{z}}\right)$$

$$F'_z(2,2,1) = -\ln 2 (2\cdot 4 + 2\cdot 4) = -16\ln 2$$

$$4\ln 2(x-2) + 4\ln 2(y-2) + (-16\ln 2)(z-1) = 0.$$

$$4x\ln 2 + 4y\ln 2 - 16z\ln 2 + 8\ln 2 = 0$$ jednačina tangentne ravni

$$\frac{x-2}{4\ln 2} = \frac{y-2}{4\ln 2} = \frac{z-1}{-16\ln 2} \Rightarrow \frac{x-2}{1} = \frac{y-2}{1} = \frac{z-1}{-4}$$

jednačina normale na površ

Naći jednačinu tangentne ravni elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ koja na koordinatnim osama odsjeca jednake pozitivne odsječke.

Rj. Jednačina tangentne ravni na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$ ima jednačinu $F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)$

Nađimo jednačinu tangentne ravni na elipsoid u proizvoljnoj tački $M(p_1,p_2,p_3)$: (U našem slučaju $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$)

$$F'_x=\frac{1}{a^2}\cdot 2x=\frac{2x}{a^2},\quad F'_y=\frac{2y}{b^2},\quad F'_z=\frac{2z}{c^2}$$

$$F'_x(M)=\frac{2p_1}{a^2},\quad F'_y(M)=\frac{2p_2}{b^2},\quad F'_z(M)=\frac{2p_3}{c^2}$$

$$\frac{2p_1}{a^2}(x-p_1)+\frac{2p_2}{b^2}(y-p_2)+\frac{2p_3}{c^2}(z-p_3)=0\quad /\cdot\frac{1}{2}$$

$$\frac{p_1}{a^2}x+\frac{p_2}{b^2}y+\frac{p_3}{c^2}z=\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}$$ Napišimo jednačinu ravni u kanonskom obliku

$$\frac{x}{\frac{a^2}{p_1}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{y}{\frac{b^2}{p_2}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{z}{\frac{c^2}{p_3}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}=1$$

Odavdje možemo primjetiti da ako želimo da jednačina tangentne ravni na kordanatnim osama odsjeca jednake odsječke, potrebno i dovoljno je da $\frac{a^2}{p_1}=\frac{b^2}{p_2}$, $\frac{a^2}{p_1}=\frac{c^2}{p_3}$ i $\frac{b^2}{p_2}=\frac{c^2}{p_3}$. ...(*)

Isto tako primjetimo da je $\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}=1$ (ZAŠTO?) ...(**)

$(*)\Rightarrow p_1=\frac{a^2}{b^2}p_2,\ p_3=\frac{c^2}{b^2}p_2$ ...(1) Sad imamo:

$$\frac{x}{\frac{a^2}{\frac{a^2}{b^2}p_2}}+\frac{y}{\frac{b^2}{p_2}}+\frac{z}{\frac{c^2}{\frac{c^2}{b^2}p_2}}=1\quad /:p_2$$

$$\frac{x}{b^2}+\frac{y}{b^2}+\frac{z}{b^2}=\frac{1}{p_2}$$

Kada (1) stavimo u (**) dobićemo da je $p_2=\frac{b^2}{\sqrt{a^2+b^2+c^2}}$

Prema tome:

$x+y+z=\sqrt{a^2+b^2+c^2}$ je jednačina tražene tangente