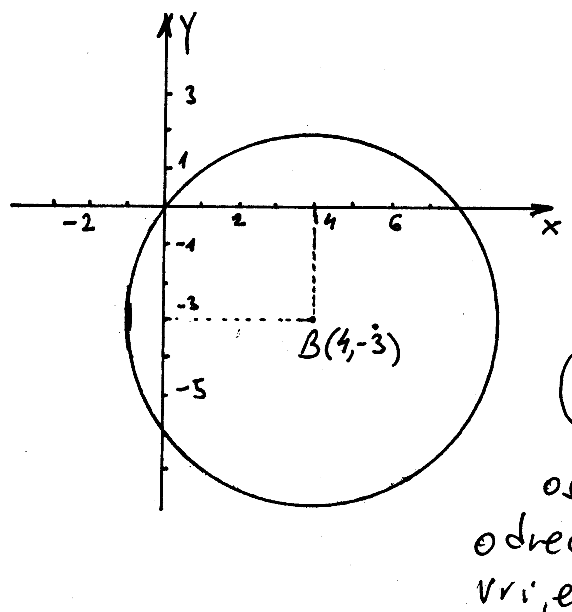
Funkcija dvije nezavisne promjenjive

Neka je S neprazan podskup prostora $\mathbb{R}^2$ i $T \subseteq \mathbb{R}$. Ako svakoj tački $M(x,y) \in S$ možemo unaprijed po datom pravilu f pridružiti jednu i samo jednu realnu vrijednost $z \in T$, tada kažemo da je data realna f-ja dvije realne promjenjive f iz $\mathbb{R}^2$ u $\mathbb{R}$ (sa skupa $S \subseteq \mathbb{R}$ u skup $T \subseteq \mathbb{R}$) i pišemo $z = f(x,y)$. Skup S na kojem je određena f-ja f naziva se domen ili definiciono područje f-je f (označavat ćemo ga sa $\mathcal{D}(f)$), a skup $f(A)$ skup vrijednosti f-je f ili kodomen (označavat ćemo ga sa $\mathcal{R}(f)$). Ako za f-ju, zadanu analitički (formulom) nije data oblast njene definisanosti, onda se pod njom podrazumjeva skup svih tačaka $M \in \mathbb{R}^2$ u kojoj f-ja, odnosno njen analitički izraz imaju određenu realnu vrijednost.

1. Opisati geometrijski skup tačaka ravni čije koordinate zadovoljavaju nejednakost $(x-4)^2+(y+3)^2<25$.

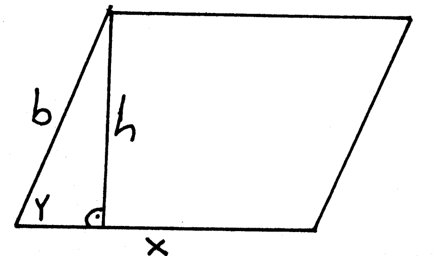
Rj.
$$\sqrt{(x-4)^2+(y+3)^2} < 5$$

S obzirom da $\sqrt{(x-4)^2+(y+3)^2}$ predstavlja rastojanje tačke $A(x,y)$ od tačke $B(4,-3)$ to nejednakost $(x-4)^2+(y+3)^2<25$ zadovoljavaju koordinate tačaka koje leže unutar kružnice sa centrom u tački $B(4,-3)$ čiji je poluprečnik jednak 5. Tačke kružnice ne pripadaju zadanom skupu.



2. Zadana je površina P paralelograma. Odrediti obim kao f-ju njegove osnovice x i oštrog ugla y. Zatim odredite i nacrtajte područje mogućih vrijednosti x i y.

Rj.

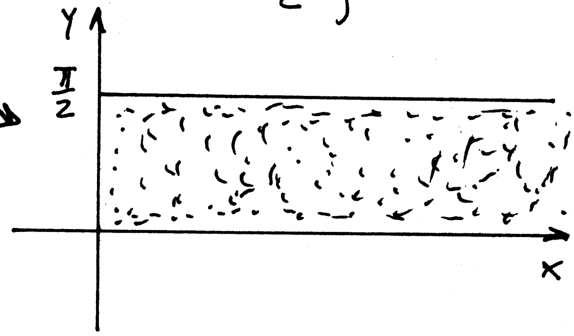


$P = x \cdot h$, $\sin y = \dfrac{h}{b}$ (h-visina, b-druga stranica)

$$h = \frac{P}{x}, \quad b = \frac{h}{\sin y} = \frac{\frac{P}{x}}{\sin y} = \frac{P}{x \sin y}$$

$S(x,y) = 2x + 2b = 2\left(x + \frac{p}{x \sin y}\right)$

F-ja $S(x,y)$ definisana je za $x \neq 0$ i $y \neq k\pi$ $(k = 0, \pm 1, \pm 2, \dots)$. Zbog prirode zadatka mora biti $x > 0$ i $0 < y < \frac{\pi}{2}$ pa je domen f-je S: $D(S) = \{(x,y) \mid x > 0 \wedge 0 < y < \frac{\pi}{2}\}$



3. Odrediti i grafički predstaviti oblast definisanosti sljedećih f-ja:

a) $f(x,y) = \frac{2x-3y}{3x-2y}$

b) $f(x,y) = \sqrt{(x+2)(y-1)}$

c) $f(x,y) = \frac{1}{\ln(-x-y)}$

d) $f(x,y) = \ln(x+2)(y-3)$

e) $f(x,y) = y^2 + \arccos\frac{x}{x+y}$

Rj. a) $3x - 2y \neq 0$ Oblast definisanosti je cijela ravan bez prave $3x-2y=0$.

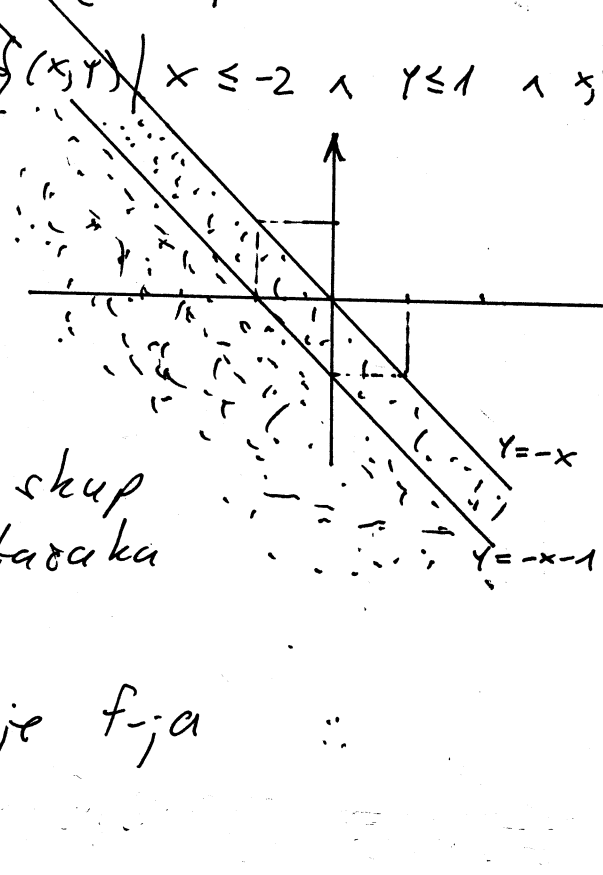
b)
$$\begin{array}{c} x+2 \geq 0 \\ y-1 \geq 0 \\ \hline x \geq -2 \\ y \geq 1 \end{array} \quad \text{ili} \quad \begin{array}{c} x+2 \leq 0 \\ y-1 \leq 0 \\ \hline x \leq -2 \\ y \leq 1 \end{array}$$

$D(f(x,y)) = \{(x,y) \mid x \geq -2 \wedge y \geq 1 \wedge x,y \in \mathbb{R}\}$
$\cup \{(x,y) \mid x \leq -2 \wedge y \leq 1 \wedge x,y \in \mathbb{R}\}$

c) $\ln(-x-y) \neq 0 \Rightarrow -x-y \neq 1$

$-x-y > 0$

$D(f(x,y)) = \{(x,y) \mid y \neq -x-1 \wedge y < -x \wedge x,y \in \mathbb{R}\}$



Geometrički ovo predstavlja skup tačaka poluravni $y < -x$ bez tačaka prave $y = -x - 1$ (slika).

4. Odrediti definiciono područje f-ja

a) $z = 2x + \frac{1}{y}$

b) $z = \sqrt{xy}$

c) $z = \sqrt{x} + y$

Tablica izvoda

1. $c' = 0$, c - konst.
2. $(x^{\alpha})' = \alpha x^{\alpha-1}$, $\alpha \in \mathbb{R}$

   $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, $x > 0$
3. $(a^x)' = a^x \ln a$

   $(e^x)' = e^x$
4. $(\log_a x)' = \frac{1}{x \ln a}$

   $(\ln x)' = \frac{1}{x}$
5. $(\sin x)' = \cos x$
6. $(\cos x)' = -\sin x$
7. $(tg x)' = \frac{1}{\cos^2 x}$
8. $(ctg x)' = -\frac{1}{\sin^2 x}$
9. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $|x|<1$
10. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $|x|<1$
11. $(arc tg x)' = \frac{1}{1+x^2}$
12. $(arc ctg x)' = -\frac{1}{1+x^2}$

$$\boxed{sh x = \frac{e^x - e^{-x}}{2} \qquad ch x = \frac{e^x + e^{-x}}{2}}$$

13. $(sh x)' = ch x$
14. $(ch x)' = sh x$
15. $h x)' = \frac{1}{ch^2 x}$
16. $(cth x)' = -\frac{1}{sh^2 x}$

Pravila izvoda:

a) $(f \pm g)'(x) = f'(x) \pm g'(x)$

b) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

c) $(\alpha f)'(x) = \alpha f'(x)$

d) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$, $g(x) \neq 0$

Izvod složene f-je: $y = f(g(x))$, $y'_x = f'_{g(x)} \cdot g'_x$

Parcijalni (djelimični) izvodi f-ja dviju i više realnih promjenjivih

$z=f(x,y)$

$$z'_x=\frac{\partial z}{\partial x}=f'_x(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$z'_y=\frac{\partial z}{\partial y}=f'_y(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

parcijalni izvodi po x-u i parcijalni izvod po y-u u tački (x,y)

$u=f(x,y,z)$

$$u'_x=\frac{\partial u}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x},\quad u'_y=\frac{\partial u}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y}$$

$$u'_z=\frac{\partial u}{\partial z}=\lim_{\Delta z\to 0}\frac{f(x,y,z+\Delta z)-f(x,y,z)}{\Delta z}$$

1. Naći sve parcijalne izvode prvog reda f-je

a) $z=x^2y^5+3x^3y-7$

b) $z=x^y$

c) $z=(2x^2y^2-x+1)^3$

d) $z=\dfrac{x+y^2}{x^2+y^2+1}$

e) $z=\operatorname{arctg}\dfrac{y}{x}$

f) $u=\sqrt{x^2+y^2+z^2}$

g) $u=\ln(x^3-y^2+z^4)$

R: a) $z'_x=2xy^5+9x^2y$

$z'_y=x^2\cdot 5y^4+3x^3=5x^2y^4+3x^3$

b) $z'_x=yx^{y-1}$

$z'_y=x^y\ln x$

c) $z'_x=3(2x^2y^2-x+1)^2(4xy^2-1)$

$z'_y=3(2x^2y^2-x+1)^2(4x^2y)=12x^2y(2x^2y^2-x+1)^2$

d) $$z'_x=\frac{1\cdot(x^2+y^2+1)-(x+y^2)\cdot 2x}{(x^2+y^2+1)^2}=\frac{x^2+y^2+1-2x^2-2xy^2}{(x^2+y^2+1)^2}=\frac{-x^2+y^2+1-2xy^2}{(x^2+y^2+1)^2}$$

$$z'_y=\frac{2y(x^2+y^2+1)-(x+y^2)(2y)}{(x^2+y^2+1)^2}=\frac{2x^2y+\cancel{2y^3}+2y-2xy-\cancel{2y^3}}{(x^2+y^2+1)^2}=\frac{2y(x^2-x+1)}{(x^2+y^2+1)^2}$$

e) $z=\operatorname{arctg}\frac{y}{x}$

$$z'_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{(-1)\cdot y}{\left(1+\frac{y^2}{x^2}\right)\cdot x^2}=\frac{-y}{x^2+y^2}$$

$$z'_y=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{1}{\left(1+\frac{y^2}{x^2}\right)\cdot x}=\frac{x}{x^2+y^2}$$

f) $u=\sqrt{x^2+y^2+z^2}$, $\quad u'_x=\dfrac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$

$$u'_y=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y=\frac{y}{\sqrt{x^2+y^2+z^2}},\quad u'_z=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

g) $u=\ln(x^3-y^2+z^4)$, $u'_x=\dfrac{3x^2}{x^3-y^2+z^4}$, $u'_y=\dfrac{-2y}{x^3-y^2+z^4}$, $u'_z=\dfrac{4z^3}{x^3-y^2+z^4}$

2. Ako je $z=x^y\cdot y^x$ dokazati da je

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=z\cdot(x+y+\ln z).$$

Rj.

$$\frac{\partial z}{\partial x}=yx^{y-1}\cdot y^x+x^y\cdot y^x\ln y$$

$$\frac{\partial z}{\partial y}=x^y\ln x\cdot y^x+x^y\cdot x\,y^{x-1}$$

$$x\cdot\frac{\partial z}{\partial x}=xy\,x^{y-1}y^x+x\ln y\,x^y y^x$$

$$=y\,x^y y^x+\underline{x\ln y\,x^y y^x}$$

$$y\cdot\frac{\partial z}{\partial y}=\underline{y\ln x\cdot x^y y^x}+x\cdot x^y y^x$$

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=y\,x^y y^x+\ln y^x\cdot x^y y^x+x^y y^x\ln x^y+x\,x^y y^x=$$

$$=x^y y^x\left(y+\ln(x^y\cdot y^x)+x\right)=z\cdot(x+y+\ln z)$$

što je i trebalo dobiti

Totalni i parcijalni diferencijal f-ja dviju i više realnih promjenjivih

$z = f(x,y)$

$dz = f'_x(a,b)\,dx + f'_y(a,b)\,dy$ totalni diferencijal f-je $z=f(x,y)$ u tački (a,b)

Ako f-ja $f(x,y)$ ima konačne i određene parcijalne izvode po svim argumentima u okolini tačke (a,b) i ako su ti izvodi neprekidne f-je u tački (a,b), tada je f-ja $f(x,y)$ diferencijabilna u tački (a,b).

1. Odrediti totalni diferencijal f-je $z=\arcsin\frac{x}{y}$ u tački $(4,5)$.

Rj. f-ja je definisana za $\left|\frac{x}{y}\right|<1$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(\frac{x}{y}\right)'_x = \frac{1}{y\sqrt{1-\frac{x^2}{y^2}}} = \frac{1}{\sqrt{y^2-x^2}}, \quad \frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right) = \frac{-x}{y\sqrt{y^2-x^2}}$$

$$dz = \frac{1}{\sqrt{y^2-x^2}}\,dx + \frac{-x}{y\sqrt{y^2-x^2}}\,dy = \frac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$$

Stavljajući u dobijeni izraz $x=4$ i $y=5$ dobijemo $dz=\frac{1}{15}(5dx-4dy)$

2. Pomoću totalnog diferencijala približno izračunati $\ln(\sqrt[3]{1,03}+\sqrt[4]{0,98}-1)$.

Rj. Neka je $z=\ln(\sqrt[3]{x}+\sqrt[4]{y}-1)$ gdje je $x=a+\epsilon=1+0,03$ i $y=b+\omega=1-0,02$

Tada je $z(a,b)=\ln(\sqrt{1}-\sqrt{1}-1)=\ln 1=0$ i $z=z(a,b)+\Delta z$.

($\Delta z = f(a+\epsilon,\, b+\omega)-f(a,b)$ totalni priraštaj f-je u tački (a,b)).

Kako je $\Delta z \approx dz = \frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy = \frac{1}{\sqrt[3]{x}+\sqrt[4]{y}-1}\left(\frac{1}{3\sqrt[3]{x^2}}dx+\frac{1}{4\sqrt[4]{y^3}}dy\right) =$

$= \frac{1}{1}\left(\frac{1}{3}\cdot 0,03-\frac{1}{4}\cdot 0,02\right)=0,005$. Pa $z=z_0+\Delta z\approx 0,0005$.

3. Naći totalni diferencijal i totalni priraštaj f-je $z=x^2+y^2+xy$ pri prelazu od tačke $(1,1)$ u tačku $(1,1;\,0,9)$.

Rj. po definiciji totalnog priraštaja dobijemo

$\Delta z = f(x+\Delta x,\, y+\Delta y)-f(x,y) = (x+\Delta x)^2+(y+\Delta y)^2+(x+\Delta x)(y+\Delta y)-(x^2+y^2+xy) =$

$= x^2+2x\Delta x+\Delta x^2+y^2+2y\Delta y+\Delta y^2+xy+x\Delta y+y\Delta x+\Delta x\Delta y-x^2-y^2-xy$

$= 2x\Delta x+\Delta x^2+y\Delta x+2y\Delta y+\Delta y^2+x\Delta y+\Delta x\Delta y=(2x+y+\Delta x)\Delta x+(2y+x+\Delta x+\Delta y)\Delta y$

Ako ubacimo u formulu vrijednosti $x=1$, $y=1$, $\Delta x=1,1-1=0,1$, $\Delta y=0,9-1=-0,1$ dobićemo totalni priraštaj date f-je u tački $(1,1)$

$\Delta z=(2+1+0,1)0,1+(2+1+0,1-0,1)(-0,1)=3,1\cdot 0,1+3\cdot(-0,1)=0,31-0,3=0,01$

$dz=(2x+y)dx+(2y+x)dy \qquad dz=(2+1)\cdot 0,1+(2+1)\cdot(-0,1)=0,3-0,3=0$

$d_x f(x,y) = \dfrac{\partial f(x_0,y_0)}{\partial x} dx$ parcijalni diferencijal f-je $f(x,y)$ po promjenjivoj x u tački (x_0, y_0)

$d_y f(x,y) = \dfrac{\partial f(x_0,y_0)}{\partial y} dy$ parcijalni diferencijal f-je $f(x,y)$ po promjenjivoj y u tački (x_0, y_0)

4. Odrediti parcijalne diferencijale f-je $z = \sqrt[3]{x^3+y^3}$.

Rj.:

$$z'_x = \frac{\partial z}{\partial x} = \frac{1}{3}(x^3+y^3)^{-\frac{2}{3}} \cdot 3x^2 = \frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}$$

$$z'_y = \frac{\partial z}{\partial y} = \frac{1}{3}(x^3+y^3)^{-\frac{2}{3}} \cdot 3y^2 = \frac{y^2}{\sqrt[3]{(x^3+y^3)^2}}$$

dobijeni izrazi za parcijalne izvode nisu definisani u tački $(0,0)$. Izvode u toj tački treba odrediti po definiciji

$$z'_x(0,0) = \lim_{\varepsilon \to 0} \frac{z(0+\varepsilon,0) - z(0,0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\sqrt[3]{\varepsilon^3+0^3} - 0}{\varepsilon} = \lim_{\varepsilon \to 0} 1 = 1$$

$$z'_y(0,0) = \lim_{\varepsilon \to 0} \frac{z(0,0+\varepsilon) - z(0,0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\sqrt[3]{0^3+\varepsilon^3} - 0}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} = 1$$

f-ja f ima parcijalne izvode u svim tačkama iz oblasti definisanosti. Parcijalni diferencijali su

$$d_x z = \frac{\partial z}{\partial x} dx = \begin{cases} \dfrac{x^2}{\sqrt[3]{(x^3+y^3)^2}} dx, & (x,y) \neq (0,0) \\ dx, & (x,y) = (0,0) \end{cases}$$

$$d_y z = \frac{\partial z}{\partial y} dy = \begin{cases} \dfrac{y^2}{\sqrt[3]{(x^3+y^3)^2}} dy, & (x,y) \neq (0,0) \\ dy, & (x,y) = (0,0) \end{cases}$$

Parcijalni izvodi i diferencijali višeg reda f-ja dvije i više promjenjivih

Parcijalnim izvodima drugog reda f-je $z = f(x,y)$ nazivamo parcijalnim izvodima njenih parcijalnih izvoda prvog reda. Za parcijalne izvode drugog reda upotrebljavamo ove oznake

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x,y)$$

$\boxed{\partial \text{ DELTA}}$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x,y) \quad \text{itd.}$$

Analogno se definiraju i označavaju izvodi viših redova.

Diferencijalom drugog reda f-je $z = f(x,y)$ nazivamo diferencijal diferencijala prvog reda te f-je za fiksirane priraste nezavisnih varijabli.

$$d^2 z = d(dz)$$

Analogno se određuju diferencijali f-je z višega nego drugog reda, na primjer $d^3 z = d(d^2 z)$ i općenito $d^n z = d(d^{n-1} z)$ $(n = 2, 3, \ldots)$

Ako je $z = f(x,y)$ gdje su x i y nezavisne varijable i f-ja f ima neprekidne parcijalne izvode drugog reda, tada se diferencijal drugog reda f-je z računa po formuli

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx\,dy + \frac{\partial^2 z}{\partial y^2} dy^2.$$

Općenito, kada postoje neprekidne odgovarajuće derivacije, vrijedi simbolička formula

$$d^n z = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^n z,$$

koja se formalno razvije po binomnom zakonu.

1. Naći parcijalne izvode drugog reda f-je

a) $z=e^{-xy}$ c) $u=x^3y+y^3x+z^3y$ e)$^{\vee}$ $z=\ln tg\frac{x}{y}$

b) $z=x^3+y^3-xy$ d)$^{\vee}$ $u=\ln(x+y-z)$ f)$^{\vee}$ $u=\sin(x^2+y+z^3)$

Rj. a) $z=e^{-xy}$

$$\frac{\partial z}{\partial x}=e^{-xy}\cdot(-y)=-ye^{-xy} \qquad \frac{\partial^2 z}{\partial x^2}=(-y)e^{-xy}\cdot(-y)=y^2e^{-xy}$$

$$\frac{\partial z}{\partial y}=e^{-xy}\cdot(-x)=-xe^{-xy} \qquad \frac{\partial^2 z}{\partial y^2}=(-x)e^{-xy}\cdot(-x)=x^2e^{-xy}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=-e^{-xy}-ye^{-xy}(-x)=$$
$$=e^{-xy}(xy-1)$$

b) $z=x^3+y^3-xy$

$$\frac{\partial z}{\partial x}=3x^2-y \qquad \frac{\partial^2 z}{\partial x^2}=6x, \quad \frac{\partial^2 z}{\partial y^2}=6y, \quad \frac{\partial^2 z}{\partial x\partial y}=-1$$

$$\frac{\partial z}{\partial y}=3y^2-x$$

c) $u=x^3y+y^3x+z^2y$

$$\frac{\partial u}{\partial x}=3x^2y+y^3 \qquad \frac{\partial^2 u}{\partial x^2}=6xy, \quad \frac{\partial^2 u}{\partial y^2}=6xy, \quad \frac{\partial^2 u}{\partial z^2}=6yz$$

$$\frac{\partial u}{\partial y}=x^3+3y^2x+z^3 \qquad \frac{\partial^2 u}{\partial x\partial y}=3x^2+3y^2, \quad \frac{\partial^2 u}{\partial x\partial z}=0$$

$$\frac{\partial u}{\partial z}=3z^2y \qquad \frac{\partial^2 u}{\partial y\partial z}=3z^2$$

d) $u=\ln(x+y-z)$

$$\frac{\partial u}{\partial x}=\frac{1}{x+y-z} \qquad \frac{\partial u}{\partial z}=\frac{-1}{x+y-z} \qquad \frac{\partial^2 u}{\partial y^2}=\frac{-1}{(x+y-z)^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x+y-z} \qquad \frac{\partial^2 u}{\partial x^2}=\frac{-1}{(x+y-z)^2} \qquad \frac{\partial^2 u}{\partial z^2}=\frac{-1}{(x+y-z)^2}$$

završiti sami ...

③ Provjeriti da li vrijedi

a) $u=\ln(x^2+y^2) \implies \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

b) $u=e^{-\alpha x}\cdot\varphi(x-y) \implies \frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\alpha\cdot\frac{\partial u}{\partial y}=\alpha^2 u$

Rj. a) $\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2}$ $\qquad \frac{\partial^2 u}{\partial x^2}=\frac{2(x^2+y^2)-2x\cdot(2x)}{(x^2+y^2)^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2}$ $\qquad \frac{\partial^2 u}{\partial y^2}=\frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2}=\frac{2x^2-2y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}+\frac{2x^2-2y^2}{(x^2+y^2)^2}=\frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}=0$$ što je i trebalo dobiti

b) $u=e^{-\alpha x}\cdot\varphi(x-y)$

$$\frac{\partial u}{\partial x}=e^{-\alpha x}\cdot(-\alpha)\varphi(x-y)+e^{-\alpha x}\cdot\varphi'_x=e^{-\alpha x}\left[-\alpha\varphi(x-y)+\varphi'_x\right]$$

$$\frac{\partial^2 u}{\partial x^2}=e^{-\alpha x}\cdot(-\alpha)\left(-\alpha\varphi(x-y)+\varphi'_x\right)+e^{-\alpha x}\left[-\alpha\varphi'_x+\varphi''_{xx}\right]$$
$$=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-\alpha\varphi'_x-\alpha\varphi'_x+\varphi''_{xx}\right)=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}\right)$$

$$\frac{\partial u}{\partial y}=e^{-\alpha x}\cdot\varphi'_y\cdot(-1)=-e^{\alpha x}\varphi'_y$$

$$\frac{\partial^2 u}{\partial y^2}=-e^{\alpha x}\varphi''_{yy}\cdot(-1)=e^{\alpha x}\cdot\varphi''_{yy}$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\alpha\frac{\partial u}{\partial y}=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}-\varphi''_{yy}+2\alpha\varphi'_y\right)=$$ (u slučaju da je $\varphi'_x=\varphi'_y$ i $\varphi''_{xx}=\varphi''_{yy}$) $=\alpha^2e^{-\alpha x}\varphi(x-y)=\alpha^2 u$

Parcijalni izvodi i diferencijal složenih f-ja

Neka je $F=F(u,v)$ f-ja su dvije nezavisne promjenjive gdje su u i v f-je. Dalje, neka su $u=u(x,y)$ i $v=v(x,y)$ f-je koje zavise od x i y.

Znači imamo $F=F(u(x,y),v(x,y))$. Sad možemo tražiti izvode po x i po y

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial F}{\partial y}=\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}$$

1. Naći diferencijal f-je u (naći du) ako je $u=f(\sqrt{x^2+y^2})$.

Rj. $u=f(\sqrt{x^2+y^2})$, uvedimo oznaku $t=\sqrt{x^2+y^2}$.

$u=f(t)=f(t(x,y))$, $\quad du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$

$$\frac{\partial u}{\partial x}=f'_t\cdot\frac{\partial t}{\partial x}=f'_t\,\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x f'_t}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y}=f'_t\cdot\frac{\partial t}{\partial y}=f'_t\cdot\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y\cdot f'_t}{\sqrt{x^2+y^2}}$$

$$du=\frac{f'_{\sqrt{x^2+y^2}}(\sqrt{x^2+y^2})(x\,dx+y\,dy)}{\sqrt{x^2+y^2}}$$

(#) Ako je $z = \frac{y}{f(x^2-y^2)}$ tada je $\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{z}{y^2}$

Dokazati.

Rj.

$z = \frac{y}{f(\xi)}$ gdje je $\xi = x^2 - y^2$

$$\frac{\partial z}{\partial x} = \frac{0\cdot f(\xi) - y\cdot\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}}{f^2(\xi)} = \frac{-2xy\frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

$$\frac{\partial z}{\partial y} = \frac{1\cdot f(\xi) - y\cdot\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}}{f^2(\xi)} = \frac{f(\xi) + 2y^2\frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

(#) Ako je $u = f(x-y,\ y-z)$ dokazati da je $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Rj.

$u = f(\eta, \mu)$, $\eta = x - y$, $\mu = y - z$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \mu}\cdot\frac{\partial \mu}{\partial x} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \mu}\cdot 0 = \frac{\partial f}{\partial \eta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial \mu}\cdot\frac{\partial \mu}{\partial y} = \frac{\partial f}{\partial \eta}(-1) + \frac{\partial f}{\partial \mu}\cdot 1 = \frac{\partial f}{\partial \mu} - \frac{\partial f}{\partial \eta}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial z} + \frac{\partial f}{\partial \mu}\cdot\frac{\partial \mu}{\partial z} = \frac{\partial f}{\partial \eta}\cdot 0 + \frac{\partial f}{\partial \mu}\cdot(-1) = -\frac{\partial f}{\partial \mu}$$

Odavde vidimo da je $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

q.e.d.

(#) Ako je $x^2 = v\cdot w$, $y^2 = u\cdot w$, $z^2 = u\cdot v$ i $f(x,y,z) = F(u,v,w)$

dokazati $x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} + z\cdot\frac{\partial f}{\partial z} = u\cdot\frac{\partial F}{\partial u} + v\cdot\frac{\partial F}{\partial v} + w\cdot\frac{\partial F}{\partial w}$.

Rj. $F(u,v,w) = f(x,y,z) = f(\sqrt{v\cdot w}, \sqrt{u\cdot w}, \sqrt{u\cdot v})$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= f'_x\cdot 0 + f'_y\cdot\frac{\sqrt{w}}{2\sqrt{u}} + f'_z\cdot\frac{\sqrt{v}}{2\sqrt{u}} = f'_y\cdot\frac{\sqrt{w}}{2\sqrt{u}} + f'_z\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{w}}{2\sqrt{v}} + \frac{\partial f}{\partial y}\cdot 0 + \frac{\partial f}{\partial z}\cdot\frac{\sqrt{u}}{2\sqrt{v}} = f'_x\cdot\frac{\sqrt{w}}{2\sqrt{v}} + f'_z\cdot\frac{\sqrt{u}}{2\sqrt{v}}$$

$$\frac{\partial F}{\partial w} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{v}}{2\sqrt{w}} + \frac{\partial f}{\partial y}\cdot\frac{\sqrt{u}}{2\sqrt{w}} + \frac{\partial f}{\partial z}\cdot 0 = f'_x\cdot\frac{\sqrt{v}}{2\sqrt{w}} + f'_y\cdot\frac{\sqrt{u}}{2\sqrt{w}}$$

$$u\cdot\frac{\partial F}{\partial u} = \frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2} + \frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$v\cdot\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2} + \frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$w\cdot\frac{\partial F}{\partial w} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2} + \frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2}$$

Prema tome $x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} + z\cdot\frac{\partial f}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\cdot\frac{\partial F}{\partial w}$

q.e.d.

Nađi du ako je $u=f(\xi,\eta)$, $\xi=x\cdot y$, $\eta=\frac{x}{y}$.

Rj. $u=f(\xi(x,y),\eta(x,y))=f(xy,\frac{x}{y})$

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}=\frac{\partial f}{\partial \xi}\cdot y+\frac{\partial f}{\partial \eta}\cdot\frac{1}{y}.$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=\frac{\partial f}{\partial \xi}\cdot x+\frac{\partial f}{\partial \eta}\cdot\left(-\frac{x}{y^2}\right)$$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\left(\frac{\partial f}{\partial \xi}y+\frac{\partial f}{\partial \eta}\cdot\frac{1}{y}\right)dx+\left(\frac{\partial f}{\partial \xi}x+\frac{\partial f}{\partial \eta}\left(-\frac{x}{y^2}\right)\right)dy$$

$$du=\frac{\partial f}{\partial \xi}(y\,dx+x\,dy)+\frac{\partial f}{\partial \eta}\left(\frac{1}{y}dx-\frac{x}{y^2}dy\right)$$

II način: $du=\frac{\partial f}{\partial \xi}d\xi+\frac{\partial f}{\partial \eta}d\eta$, $d\xi=\frac{\partial \xi}{\partial x}dx+\frac{\partial \xi}{\partial y}dy$ ....

Ako je $x^2=v\cdot w$, $y^2=u\cdot w$, $z^2=u\cdot v$ i $f(x,y,z)=F(u,v,w)$

Dokazati $x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}+z\cdot\frac{\partial f}{\partial z}=u\cdot\frac{\partial F}{\partial u}+v\cdot\frac{\partial F}{\partial v}+w\cdot\frac{\partial F}{\partial w}$.

uputa: $F(u,v,w)=f(\sqrt{v\cdot w},\sqrt{u\cdot w},\sqrt{u\cdot v})$

$$\frac{\partial F}{\partial u}=f'_x\cdot x'_u+f'_y\cdot y'_u+f'_z\cdot z'_u$$
$$=f'_y\sqrt{w}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{u}}+f'_z\sqrt{v}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{u}}$$
⋮

ISPITNI ZADATAK

Ako je $z=z(x,y)$ i $x+y+z=f(x^2+y^2+z^2)$ dokazati da je

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y.$$

Rj. $z=f(x^2+y^2+z^2)-x-y$

$z=f(t)-x-y$, $t=x^2+y^2+z^2$

$$\frac{\partial z}{\partial x}=f'_t\cdot\left(2x+2z\frac{\partial z}{\partial x}\right)-1 \Rightarrow \frac{\partial z}{\partial x}=2xf'_t+2zf'_t\frac{\partial z}{\partial x}-1$$

$$(1-2zf'_t)\frac{\partial z}{\partial x}=2xf'_t-1$$

$$\frac{\partial z}{\partial x}=\frac{2xf'_t-1}{1-2z\cdot f'_t}$$

analogno $\frac{\partial z}{\partial y}=\frac{2yf'_t-1}{1-2zf'_t}$

$$(y-z)\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=(y-z)\frac{2xf'_t-1}{1-2z\cdot f'_t}+(z-x)\frac{2yf'_t-1}{1-2zf'_t}=$$

$$=\frac{\cancel{2xyf'_t}-y-2xzf'_t+\cancel{z}+2yzf'_t-\cancel{z}-\cancel{2xyf'_t}+x}{1-2zf'_t}=\frac{x-y-2zf'_t(x-y)}{1-2zf'_t}=\frac{(x-y)(1-2zf'_t)}{(1-2zf'_t)}=x-y$$

(#) Ako je $z=\dfrac{y}{f(x^2-y^2)}$, gdje je f diferencijabilna f-ja, izračunati $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y}$.

Rj. $z = y f^{-1}(x^2-y^2) = y f^{-1}(u)$, gdje je $u=x^2-y^2$

$$\frac{\partial z}{\partial x} = y(-1)f_u^{-2}(x^2-y^2)\cdot 2x = \frac{-2xy}{f_u^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = \left(y f^{-1}(u)\right)'_y = 1\cdot f^{-1}(u) + y\cdot(-1)f_u^{-2}(u)\cdot(-2y) =$$

$$= \frac{1}{f(x^2-y^2)} + \frac{2y^2}{f_u^2(x^2+y^2)}$$

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{-2y}{f_u^2(x^2+y^2)} + \frac{1}{y\,f(x^2-y^2)} + \frac{2y}{f_u^2(x^2+y^2)} =$$

$$= \frac{1}{y\,f(x^2-y^2)} = \frac{1}{y^2}\cdot\frac{y}{f(x^2-y^2)} = \frac{z}{y^2}$$

prema tome $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y} = \dfrac{z}{y^2}$.

(#) Ako je $z = e^{y}\varphi\left(y e^{\frac{x^2}{2y^2}}\right)$ gdje je φ diferencijabilna f-ja, dokazati da je $(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz$.

Rj. $z = e^{y}\varphi(\xi)$, gdje je $\xi(x,y) = y e^{\frac{x^2}{2y^2}}$

$$\frac{\partial \xi}{\partial x} = y e^{\frac{x^2}{2y^2}} \cdot 2 \cdot \frac{x}{2y^2} = \frac{x}{y} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial y} = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2y^{-2}\right)'_y = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2\cdot(-2)y^{-3}\right)$$

$$= e^{\frac{x^2}{2y^2}} - \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial z}{\partial x} = e^{y}\cdot\frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \varphi}{\partial x} = \frac{x}{y} e^{y} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial z}{\partial y} = e^{y}\varphi(\xi) + e^{y}\cdot\frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} = e^{y}\varphi(\xi) + e^{y}e^{\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi} -$$

$$- e^{y}\cdot\frac{x^2}{y^2} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = (x^2-y^2)\cdot\frac{x}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi} +$$

$$+ xy\left(e^{y}\varphi(\xi) + e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi} - \frac{x^2}{y^2} e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi}\right)$$

$$= \underline{\frac{x^3}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}} \underline{\underline{- yx e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} + xy e^{y}\varphi(\xi) +$$

$$\underline{\underline{+ xy e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} \underline{- \frac{x^3}{y} e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi}} =$$

$$= xy e^{y}\varphi(\xi) = xy e^{y}\varphi\left(y e^{\frac{x^2}{2y^2}}\right) = xyz$$

Parcijalni izvodi višeg reda složenih f-ja

(#) Ako je $u=\varphi(\xi,\eta)$ pričemu je $\xi = x+y$, $\eta = x-y$ izračunati izvode $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x\partial y}$, $\frac{\partial^2 u}{\partial y^2}$.

Rj.

$$\frac{\partial u}{\partial x} = \frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \varphi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial \eta}\right) = \frac{\partial^2 \varphi}{\partial \xi^2}\,\frac{\partial \xi}{\partial x} + \frac{\partial^2 \varphi}{\partial \xi\partial \eta}\,\frac{\partial \eta}{\partial x} +$$

$$+ \frac{\partial^2 \varphi}{\partial \eta\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 \varphi}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial^2 \varphi}{\partial \xi^2} + 2\frac{\partial^2 \varphi}{\partial \xi\partial \eta} + \frac{\partial^2 \varphi}{\partial \eta^2}$$

$$= \left(\frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}\right)^2$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial \xi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial \eta}\right) = \frac{\partial^2 \varphi}{\partial \xi^2}\,\frac{\partial \xi}{\partial y} + \frac{\partial^2 \varphi}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial y} +$$

$$+ \frac{\partial^2 \varphi}{\partial \eta\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 \varphi}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial^2 \varphi}{\partial \xi^2} - \frac{\partial^2 \varphi}{\partial \eta^2}.$$

(#) Ako je $u = \dfrac{\varphi(x-y) + \psi(x+y)}{x}$, gdje su φ i ψ diferencijabilne f-je izračunati $\dfrac{\partial}{\partial x}\left(x^2 \dfrac{\partial u}{\partial x}\right) - x^2 \dfrac{\partial^2 u}{\partial y^2}$.

Rj.

$$u = \frac{1}{x}(\varphi(x-y) + \psi(x+y)) = x^{-1}(\varphi(x-y) + \psi(x+y))$$

$$u'_x = \frac{\partial u}{\partial x} = (-1)x^{-2}(\underbrace{\varphi(x-y)}_{s} + \underbrace{\psi(x+y)}_{t}) + \frac{1}{x}(\varphi'_s \cdot s'_x + \psi'_t \cdot t'_x) =$$

$$= \frac{-1}{x^2}[\varphi(x-y) + \psi(x+y)] + \frac{1}{x}(\varphi'_s \cdot 1 + \psi'_t \cdot 1)$$

$$x^2 \frac{\partial u}{\partial x} = -\varphi(x-y) - \psi(x+y) + x(\varphi'_s + \psi'_t)$$ gdje su $s = x-y$ i $t = x+y$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = -\varphi'_s \cdot 1 - \psi'_t \cdot 1 + 1 \cdot (\varphi'_s + \psi'_t) + x(\varphi''_{ss} \cdot 1 + \psi''_{tt} \cdot 1)$$

$$= x(\varphi''_{ss} + \psi''_{tt}) \quad \dots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}(\varphi'_s \cdot s'_y + \psi'_t \cdot t'_y) = \frac{1}{x}(\varphi'_s \cdot (-1) + \psi'_t \cdot 1) = \frac{1}{x}(-\varphi'_s + \psi'_t)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}(-\varphi''_{ss} \cdot s'_y + \psi''_{tt} \cdot t'_y) = \frac{1}{x}(\varphi''_{ss} + \psi''_{tt})$$

$$x^2 \frac{\partial^2 u}{\partial y^2} = x(\varphi''_{ss} + \psi''_{tt}) \quad \dots (2)$$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2} \overset{(1)\, i\, (2)}{=\!=} 0$$ traženo rješenje

(#) Naći potpuni diferencijal prvog i drugog reda f-je $u = f(\sqrt{x^2+y^2})$ (x i y su nezavisne promjenjive).

Rj. $u = f(\xi)$, $\xi = \sqrt{x^2+y^2}$ $\qquad \xi$ KSI

$$du = \frac{\partial f}{\partial \xi} d\xi, \quad d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy$$

$$\frac{\partial \xi}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial \xi}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$du = \frac{\partial f}{\partial \xi}\left(\frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy\right) = \frac{\partial f}{\partial \xi} \cdot \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$ potpuni diferencijal prvog reda

$$d^2u = d\left(\frac{\partial f}{\partial \xi}\right) \cdot \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \xi} d\left(\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}\right)$$

$$d\left(\frac{\partial f}{\partial \xi}\right) = \frac{\partial^2 f}{\partial \xi^2} d\xi = \frac{\partial^2 f}{\partial \xi^2} \cdot \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}$$

$$d\left(\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}\right) = \frac{\partial\left(\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}\right)}{\partial x} dx + \frac{\partial\left(\frac{dx+y\,dy}{\sqrt{x^2+y^2}}\right)}{\partial y} dy$$

$$\frac{\partial\left(\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}\right)}{\partial x} = \frac{dx \cdot \sqrt{x^2+y^2} - (x\,dx+y\,dy)\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{(x^2+y^2)^2}} = \frac{\cancel{x^2 dx} + y^2 dx - \cancel{x^2 dx} - xy\,dy}{\sqrt{(x^2+y^2)}}$$

$$\frac{\partial\left(\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}\right)}{\partial y} = \frac{dy(\sqrt{x^2+y^2}) - (x\,dx+y\,dy) \cdot \frac{y}{\sqrt{x^2+y^2}}}{\sqrt{(x^2+y^2)^2}} = \frac{x^2 dy + \cancel{y^2 dy} - xy\,dx \ \cancel{y^2 dy}}{\sqrt{(x^2+y^2)^3}}$$

$$d\left(\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}\right) = \frac{y^2 dx^2 - xy\,dx\,dy + x^2 dy^2 - xy\,dx\,dy}{\sqrt{(x^2+y^2)^3}} = \frac{(y\,dx - x\,dy)^2}{\sqrt{(x^2+y^2)^3}}$$

$$d^2u = \frac{\partial^2 f}{\partial \xi^2}\left(\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}\right)^2 + \frac{\partial f}{\partial \xi} \cdot \frac{(y\,dx-x\,dy)^2}{\sqrt{(x^2+y^2)^3}}$$

$$d^2u = f'' \cdot \frac{(x\,dx+y\,dy)^2}{x^2+y^2} + f' \frac{(y\,dx-x\,dy)}{\sqrt{(x^2+y^2)^3}}$$ potpuni diferencijal drugog reda

(#) Naći potpuni diferencijal prvog i drugog reda f-je $u=f(\xi,\eta)$, gdje je $\xi=x+y$, $\eta=x-y$. (x i y su nezavisne promjenjive).

Rj:

$$du=\frac{\partial f}{\partial \xi}d\xi+\frac{\partial f}{\partial \eta}d\eta$$

ξ KSI, η ETA

$$d\xi=\frac{\partial \xi}{\partial x}dx+\frac{\partial \xi}{\partial y}dy=dx+dy$$

$$d\eta=\frac{\partial \eta}{\partial x}dx+\frac{\partial \eta}{\partial y}dy=dx-dy$$

$$du=\frac{\partial f}{\partial \xi}(dx+dy)+\frac{\partial f}{\partial \eta}(dx-dy)=\left(\frac{\partial f}{\partial \xi}+\frac{\partial f}{\partial \eta}\right)dx+\left(\frac{\partial f}{\partial \xi}-\frac{\partial f}{\partial \eta}\right)dy$$

potpuni diferencijal prvog reda

$$d^2u=d\left(\frac{\partial f}{\partial \xi}+\frac{\partial f}{\partial \eta}\right)dx+d\left(\frac{\partial f}{\partial \xi}-\frac{\partial f}{\partial \eta}\right)dy$$

$$d\left(\frac{\partial f}{\partial \xi}+\frac{\partial f}{\partial \eta}\right)=\frac{\partial^2 f}{\partial \xi^2}d\xi+\frac{\partial^2 f}{\partial \xi\partial \eta}d\eta+\frac{\partial^2 f}{\partial \eta\partial \xi}d\xi+\frac{\partial^2 f}{\partial \eta^2}d\eta$$

$$=\frac{\partial^2 f}{\partial \xi^2}(dx+dy)+\frac{\partial^2 f}{\partial \xi\partial \eta}(dx-\underline{\underline{dy}})+\frac{\partial^2 f}{\partial \eta\partial \xi}(dx+\underline{\underline{dy}})+$$

$$+\frac{\partial^2 f}{\partial \eta^2}(dx-dy)=$$

$$=\left(\frac{\partial^2 f}{\partial \xi^2}+2\frac{\partial^2 f}{\partial \xi\partial \eta}+\frac{\partial^2 f}{\partial \eta^2}\right)dx+\left(\frac{\partial^2 f}{\partial \xi^2}-\frac{\partial^2 f}{\partial \eta^2}\right)dy$$

$$d\left(\frac{\partial f}{\partial \xi}-\frac{\partial f}{\partial \eta}\right)=\frac{\partial^2 f}{\partial \xi^2}\underset{(dx+dy)}{d\xi}+\frac{\partial^2 f}{\partial \xi\partial \eta}\underset{\underline{\underline{(dx-dy)}}}{d\eta}-\frac{\partial^2 f}{\partial \xi\partial \eta}\underset{\substack{(dx+dy)\\ \underline{\underline{-dx}}-dy}}{d\xi}-\frac{\partial^2 f}{\partial \eta^2}\underset{\substack{(dx-dy)\\ -dx+dy}}{d\eta}=$$

$$=\left(\frac{\partial^2 f}{\partial \xi^2}-\frac{\partial^2 f}{\partial \eta^2}\right)dx+\left(\frac{\partial^2 f}{\partial \xi^2}-2\frac{\partial^2 f}{\partial \xi\partial \eta}+\frac{\partial^2 f}{\partial \eta^2}\right)dy$$

$$d^2u=\left(\frac{\partial f}{\partial \xi}+\frac{\partial f}{\partial \eta}\right)^2dx^2+\left(\frac{\partial^2 f}{\partial \xi^2}-\frac{\partial^2 f}{\partial \eta^2}\right)dx\,dy+\left(\frac{\partial^2 f}{\partial \xi^2}-\frac{\partial^2 f}{\partial \eta^2}\right)dx\,dy+$$

$$+\left(\frac{\partial f}{\partial \xi}-\frac{\partial f}{\partial \eta}\right)^2dy^2=\left[\left(\frac{\partial f}{\partial \xi}+\frac{\partial f}{\partial \eta}\right)dx+\left(\frac{\partial f}{\partial \xi}-\frac{\partial f}{\partial \eta}\right)dy\right]^2$$

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