Izračunati površinu figure koju ograničavaju linije

$x = y^2 - 2y - 3$ i $y = 3 - 3x$

$$y = 3 - 3x$$
$$3x = 3 - y$$
$$x = 1 - \frac{1}{3}y$$

Rj. Nađimo presječnu tačku ovih linija

$$x = y^2 - 2y - 3$$
$$y = 3 - 3x$$

$$x = (3-3x)^2 - 2(3-3x) - 3$$
$$x = 9 - 18x + 9x^2 - 6 + 6x - 3$$
$$9x^2 - 13x = 0$$
$$x(9x - 13) = 0$$
$$x = 0 \text{ ili } 9x = 13$$
$$x = \frac{13}{9}$$

$$x = 0 \Rightarrow y = 3$$

$$x = \frac{13}{9} \Rightarrow y = 3 - \cancel{3} \cdot \frac{13}{\cancel{9}_3} = \frac{9}{3} - \frac{13}{3} = -\frac{4}{3}$$

$A(0,3)$ i $B(\frac{13}{9}, -\frac{4}{3})$ su presječne tačke linija

$x = y^2 - 2y - 3$ je kriva oblika parabole C čije je tjeme $T(-\frac{D}{4a}, -\frac{b}{2a})$

$$-\frac{b}{2a} = -\frac{-2}{2} = 1, \quad D = 4 + 12 = 16 \qquad -\frac{D}{4a} = -\frac{16}{4} = -4$$

$$T(1, -4)$$

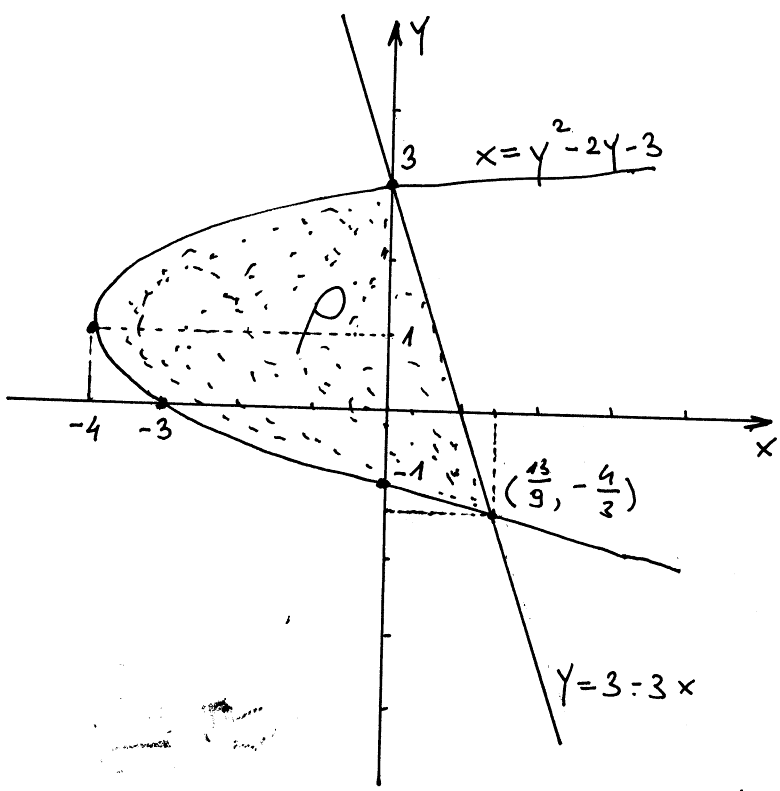
$$y_{1,2} = \frac{2 \pm 4}{2}$$
$$y_1 = \frac{-2}{2} = -1 \quad y_2 = \frac{6}{2} = 3$$

$M_1(0,-1)$ i $M_2(0,3)$ su presjek parabole sa y-osom

$x = y^2 - 2y - 3$

$y = 0 \Rightarrow x = -3$

$(-3,0)$ je presjek krive sa x-osom



$$P = \int_{-\frac{4}{3}}^{3} \left[\left(1 - \frac{1}{3}y\right) - (y^2 - 2y - 3)\right] dy =$$

$$= \int_{-\frac{4}{3}}^{3} \left(-y^2 + \frac{5}{3}y + 4\right) dy =$$

$$= -\frac{1}{3}y^3\Big|_{-\frac{4}{3}}^{3} + \frac{5}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{4}{3}}^{3} + 4y\Big|_{-\frac{4}{3}}^{3} =$$

$$= -\frac{1}{3}\left(27 + \frac{64}{27}\right) + \frac{5}{6}\left(9 - \frac{16}{9}\right) + 4\left(3 + \frac{4}{3}\right)$$

$$= -\frac{1}{3} \cdot \frac{793}{27} + \frac{5}{6} \cdot \frac{65}{9} + 4 \cdot \frac{13}{3} =$$

$$= -\frac{793}{81} + \frac{325}{54} + \frac{52}{3} = \frac{-793 \cdot 2 + 325 \cdot 3 + 52 \cdot 54}{162} = \frac{-1586 + 975 + 2808}{162}$$

$$P = \frac{2197}{162} = 13\frac{91}{162}$$

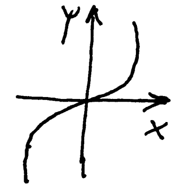
tražena površina

Izračunati površinu figure koja je određena linijama $y=-x$, $y=\sqrt[3]{x}$, $y=3x-2$.

Rj:

Grafički nije teško predstaviti prave $y=-x$ i $y=3x-2$. Problem predstavlja kriva $y=\sqrt[3]{x}$.

Ako znamo da kriva $y=x^3$ izgleda ovako



Onda nije teško nacrtati krivu $x=y^3$ što je ekvivalentno sa $y=\sqrt[3]{x}$.

Pronađimo tačke presjeka datih krivih.

$$\begin{aligned} y&=-x\\ y&=3x-2\\ \hline -x&=3x-2\\ -4x&=-2\\ x=\tfrac{1}{2} &\Rightarrow y=-\tfrac{1}{2}\end{aligned}$$

$$\begin{aligned} y&=-x\\ y&=\sqrt[3]{x}\\ \hline y&=-x\\ y^3&=x\\ \hline -x^3&=x\\ x^3+x&=0\\ x(x^2+1)&=0\\ x=0 &\Rightarrow y=0\end{aligned}$$

$$\begin{aligned} y&=3x-2\\ y&=\sqrt[3]{x}\\ \hline \sqrt[3]{x}&=3x-2\\ (3x-2)^3&=x\end{aligned}$$

$$27x^3-3\cdot(3x)^2\cdot 2+3\cdot 3x\cdot(-2)^2+(-2)^3=x$$

$$27x^3-54x^2+36x-8=x$$

$$27x^3-54x^2+35x-8=0$$

pokušajmo riješiti sistem na drugi način

$$\sqrt[3]{x}=3x-2$$

$$\boxed{3x=y+2}$$

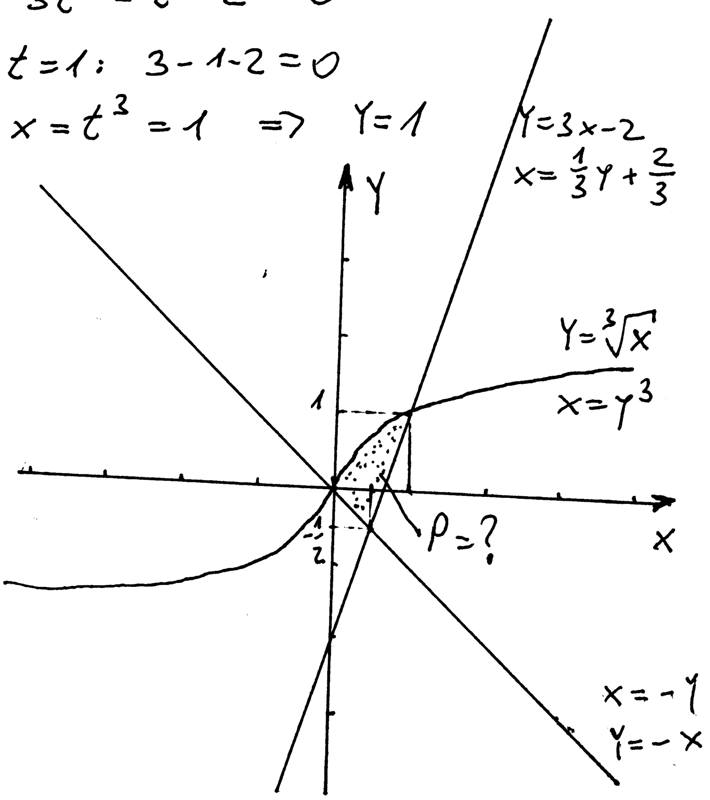
$$x=t^3$$

$$3t^3-2=t$$

$$3t^3-t-2=0$$

$$t=1:\ 3-1-2=0$$

$$x=t^3=1 \Rightarrow y=1$$



$$P=\int_{-\frac{1}{2}}^{0}\left[\left(\tfrac{1}{3}y+\tfrac{2}{3}\right)-(-y)\right]dy+\int_{0}^{1}\left[\left(\tfrac{1}{3}y+\tfrac{2}{3}\right)-y^3\right]dy=$$

$$=\int_{-\frac{1}{2}}^{0}\left(\tfrac{4}{3}y+\tfrac{2}{3}\right)dy+\int_{0}^{1}\left(-y^3+\tfrac{1}{3}y+\tfrac{2}{3}\right)dy=$$

$$=\frac{4}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{1}{2}}^{0}+\frac{2}{3}y\Big|_{-\frac{1}{2}}^{0}-\frac{1}{4}y^4\Big|_{0}^{1}+\frac{1}{3}\cdot\frac{1}{2}y^2\Big|_{0}^{1}$$

$$+\frac{2}{3}y\Big|_{0}^{1}=\frac{\cancel{2}}{3}\cdot\left(-\frac{1}{\cancel{4}_2}\right)+\frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}-\frac{1}{4}+\frac{1}{6}+$$

$$+\frac{2}{3}=-\cancel{\frac{1}{6}}+\frac{1}{3}-\frac{1}{4}+\cancel{\frac{1}{6}}+\frac{2}{3}=\frac{3}{4}$$

(#) Izračunati površinu figure koja je određena linijama $y=-2$, $y=x^3+x$, $x+y=3$.

Rj. $y=-2$, $x+y=3$ su prave linije i njih nije teško nacrtati. Problem za crtanje predstavlja kriva $y=x^3+x$.

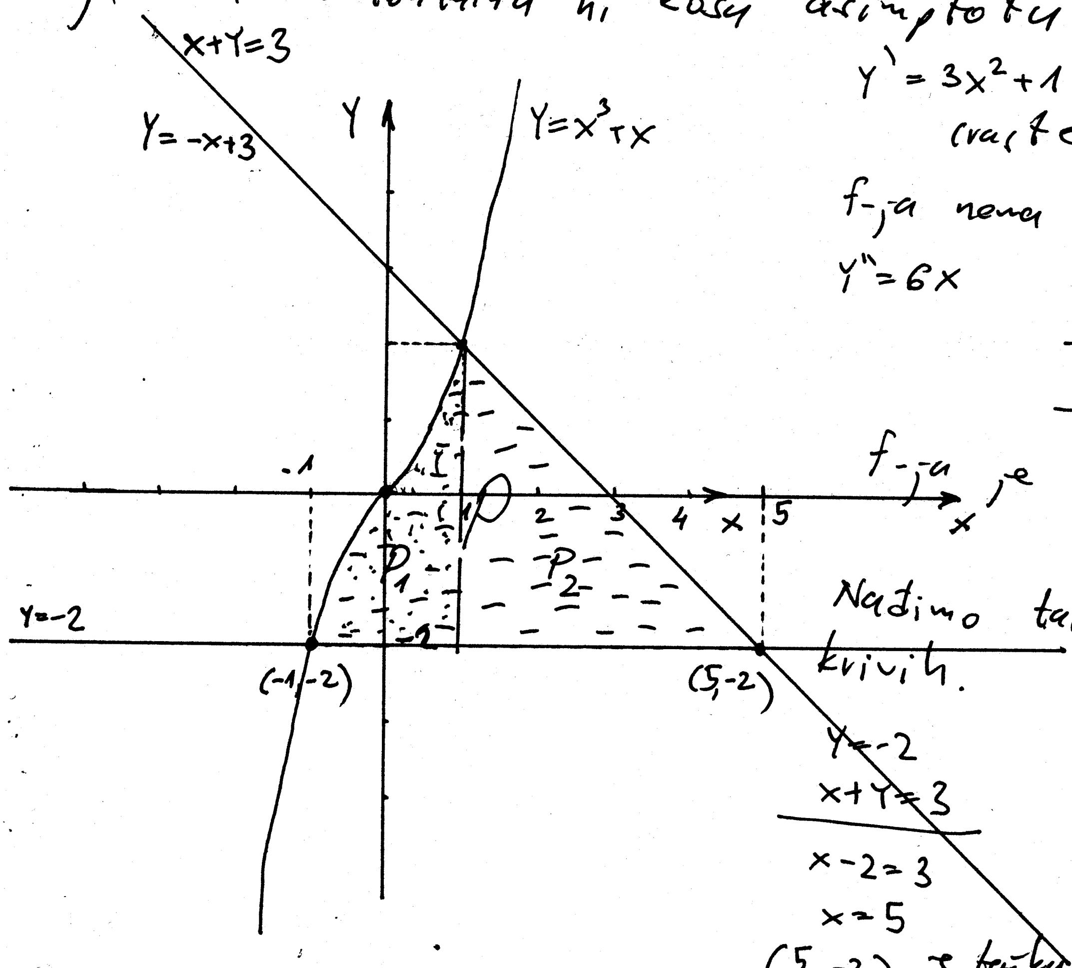
Ispitajmo f-ju $y=x^3+x$. $\mathcal{D}: x\in\mathbb{R}$

$f(-x)=-x^3-x=-(x^3+x)$ f-ja je neparna

$A(0,0)$ je nula f-je i presjek sa y-osom

f-ja nema prekida $\Rightarrow$ f-ja nema vertikalnu asimptotu

f-ja nema horizontalnu ni kosu asimptotu



$y'=3x^2+1$ f-ja je uvijek pozitivna (raste za svako x)

f-ja nema ekstrem

$y''=6x$

x	$(0,+\infty)$
y''	$+$
y	$\cup$

$(0,0)$ je prevojna tačka

f-ja je ovog oblika

Nađimo tačke presjeka datih krivih.

$$\begin{array}{c} y=-2 \\ x+y=3 \\ \hline x-2=3 \\ x=5 \end{array}$$

$(5,-2)$ je tačka presjeka

$$\begin{array}{c} y=-2 \\ y=x^3+x \\ \hline -2=x^3+x \\ x^3+x+2=0 \end{array}$$

$x=-1$: $-1-1+2=0$

$$\begin{array}{l} (x^3+x+2):(x+1)=x^2-x+2 \\ \underline{-\ x^3+x^2} \\ \quad -x^2+x+2 \\ \underline{-\ -x^2-x} \\ \qquad 2x+2 \\ \qquad \underline{2x+2} \\ \qquad // \ // \end{array}$$

$x^3+x+2=(x+1)\underbrace{(x^2-x+2)}_{>0\ \forall x}$

Rješenje jednačine $x^3+x+2=0$ je $x=-1$.

$(-1,-2)$ je tačka presjeka datih krivih

$$Y = x^3 + x$$
$$x + Y = 3$$

$$Y = x^3 + x$$
$$Y = -x + 3$$

$$-x + 3 = x^3 + x$$
$$x^3 + 2x - 3 = 0$$
$$x = 1: \quad 1^3 + 2\cdot 1 - 3 = 3 - 3 = 0$$
$$(x^3 + 2x - 3) : (x - 1) = x^2 + x + 3$$
$$\begin{array}{l} x^3 - x^2 \\ \hline x^2 + 2x - 3 \\ -\ x^2 - x \\ \hline 3x - 3 \\ -\ 3x - 3 \\ \hline = \quad = \end{array}$$

$$x^3 + 2x - 3 = \underbrace{(x^2 + x + 3)}_{>0\ \forall x}(x - 1)$$

$(1, 2)$ je presečna tačka krivih

$$P_1 = \int_{-1}^{1} \left[(x^3 + x) - (-2)\right] dx = \int_{-1}^{1} (x^3 + x + 2)\, dx = \frac{1}{4}x^4\Big|_{-1}^{1} + \frac{1}{2}x^2\Big|_{-1}^{1} + 2x\Big|_{-1}^{1} = 4$$

$$P_2 = \int_{1}^{5} \left[(-x + 3) - (-2)\right] dx = \int_{1}^{5} (-x + 5)\, dx = -\frac{x^2}{2}\Big|_{1}^{5} + 5x\Big|_{1}^{5} =$$

$$= -\frac{1}{2}(25 - 1) + 5\cdot 4 = -\frac{1}{2}\cdot 24 + 20 = 20 - 12 = 8$$

$P = P_1 + P_2 = 8 + 4 = 12$ površina figure

(#) Izračunati površinu figure koju čine linije

$y=(x-1)^2$, $\frac{x^2}{1}-\frac{y^2}{2}=1$.

Rj. Da bi odredili granice za računanje površine potrebno je grafički predstaviti ove dvije linije.

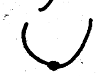
ispitajmo f-ju $y=(x-1)^2$

$D: x \in \mathbb{R}$

f-ja nije ni parna ni neparna

$f(0)=1$, $(0,1)$ je presjek sa y-osom

$(x-1)^2=0 \Rightarrow x=1$ $(1,0)$ je nula f-je

$y=(x-1)^2=x^2-2x+1 \Rightarrow$ f-ja je  oblika

Nađimo još tjeme f-je

$y'=2x-2$

$y'=0 \Rightarrow 2x-2=0 \Rightarrow x=1$

$T(1,0)$

Prije nego grafički predstavimo liniju $y=(x-1)^2$ pronađimo u kojim tačkama siječe liniju $\frac{x^2}{1}-\frac{y^2}{2}=1$.

$y=x^2-2x+1=(x-1)^2$

$2x^2-y^2=2$

$y^2=(x^2-2x+1)^2=$

$=(x^2-2x+1)(x^2-2x+1)=$

$=x^4-2x^3+x^2-2x^3+4x^2-2x+x^2-2x+1$

$=x^4-4x^3+6x^2-4x+1$.

$2x^2-y^2=2$

$2x^2-x^4+4x^3-6x^2+4x-1-2=0 \quad /\cdot(-1)$

$x^4-4x^3+4x^2-4x+3=0$

Označimo ovo sa $g(x)=x^4-4x^3+4x^2-4x+3$

Kako je $g(1)=0 \Rightarrow g(x)$ je djeljivo sa $(x-1)$

$(x^4-4x^3+4x^2-4x+3):(x-1)=x^3-3x^2+x-3$

$\underline{-\ x^4-x^3}$

$-3x^3+4x^2-4x+3$

$\underline{-\ -3x^3+3x^2}$

x^2-4x+3

$\underline{-\ x^2-x}$

$-3x+3$

$\underline{-3x+3}$

$= \ =$

$g(x)=\underbrace{(x^3-3x^2+x-3)}_{g_1(x)}(x-1)$

$g_1(0)=-3$

$g_1(1)=1-3+1-3=-4$

$g_1(2)=8-12+2-3=-5$

$g_1(3)=27-27+3-3 \Rightarrow g_1(x)$ je djeljivo sa $x-3$

$g_1(-2)=-8-12-2-3=-25$

$g_1(-1)=-1-3-1-3=-8$

$g_1(-3)=-27-27-3-3<0$

$(x^3-3x^2+x-3):(x-3)=x^2+1$

$\underline{-\ x^3-3x^2}$

$x-3$

$\underline{-\ x-3}$

$= \ =$

Prema tome $g(x)=(x^2+1)(x-3)(x-1)$

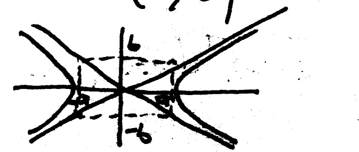
Za $x=3 \Rightarrow y=4$

Za $x=1 \Rightarrow y=0$

Presječne tačke krivih su $(3,4)$ i $(1,0)$

Krive oblika

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo HIPERBOLE i one su oblika

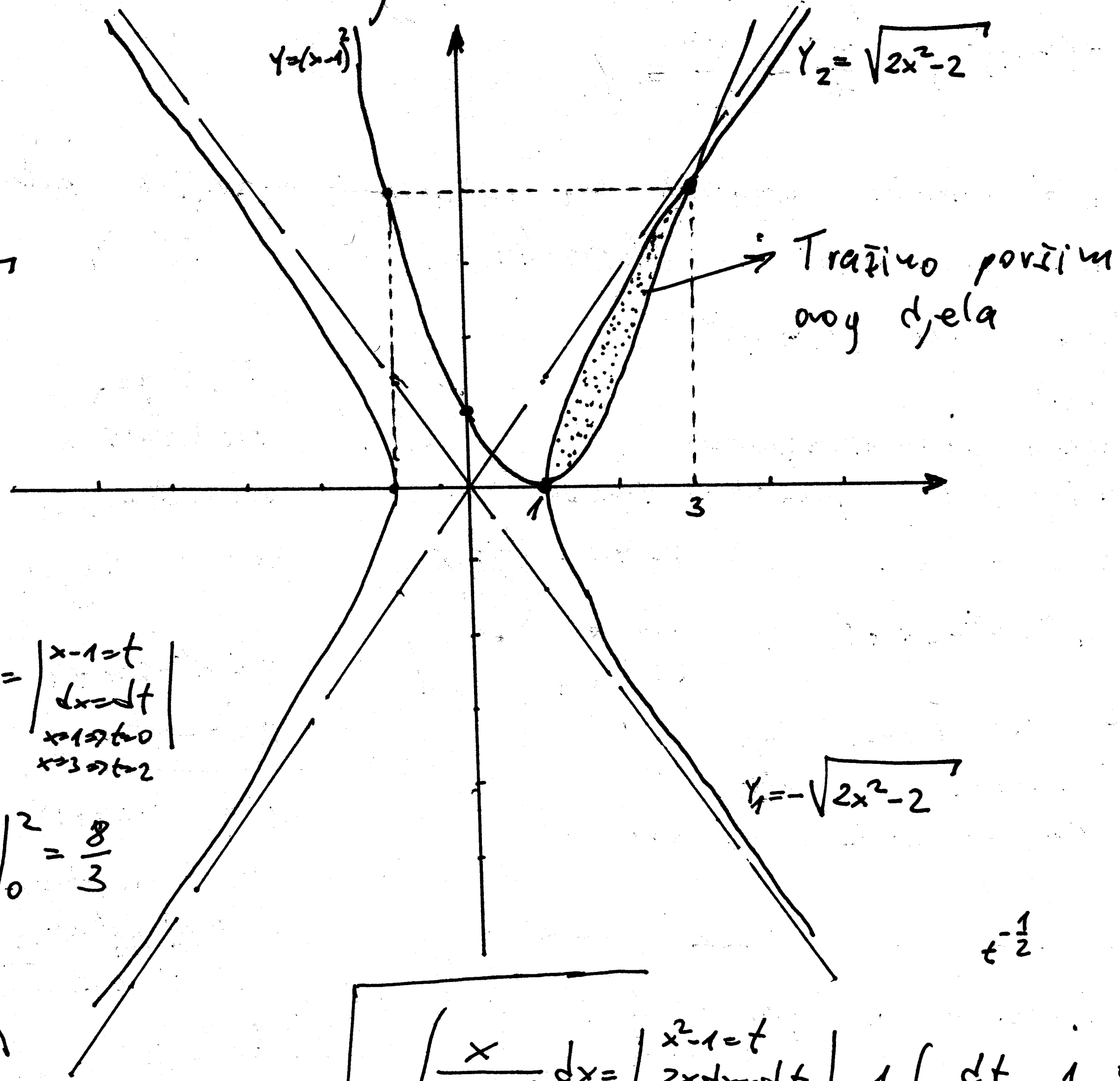


Nacrtajmo naše krive linije

$\sqrt{2} \approx 1{,}41$

$$y^2 = 2x^2 - 2$$

$$y_{1,2} = \pm\sqrt{2x^2-2}$$



$$P_2 = \int_1^3 (x-1)^2\,dx = \left| \begin{matrix} x-1=t \\ dx=dt \\ x=1 \Rightarrow t=0 \\ x=3 \Rightarrow t=2 \end{matrix} \right|$$

$$= \int_0^2 t^2\,dt = \frac{1}{3}t^3\Big|_0^2 = \frac{8}{3}$$

$t^{-\frac{1}{2}}$

$$\boxed{\int \frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} x^2-1=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{t}+C = \sqrt{x^2-1}+C}$$

$$P = \int_1^3 \Big(\underbrace{\sqrt{2x^2-2}}_{P_1} - \underbrace{(x-1)^2}_{P_2} \Big)\,dx$$

$$P_1 = \int_1^3 \sqrt{2}\cdot\sqrt{x^2-1}\,dx = \sqrt{2}\int_1^3 \frac{x^2-1}{\sqrt{x^2-1}}\,dx = \sqrt{2}\left(\int_1^3 \frac{x^2}{\sqrt{x^2-1}}\,dx - \int_1^3 \frac{dx}{\sqrt{x^2-1}} \right)$$

$$\int_1^3 x\cdot\frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} u=x & dv=\frac{x}{\sqrt{x^2-1}}\,dx \\ du=dx & v=\sqrt{x^2-1} \end{matrix} \right| = \underbrace{x\sqrt{x^2-1}\Big|_1^3}_{3\sqrt{8}-0} - \int_1^3 \sqrt{x^2-1}\,dx$$

$$\int_1^3 \frac{dx}{\sqrt{x^2-1}} = \ln\left|x+\sqrt{x^2-1}\right|\Big|_1^3 = \ln|3+\sqrt{8}|$$

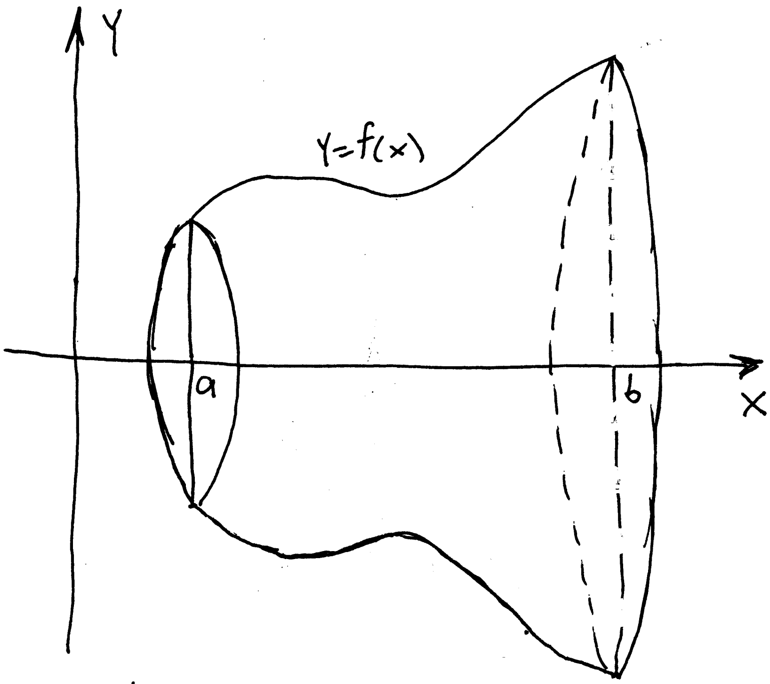
$$\sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx = \sqrt{2}\cdot\underbrace{3\sqrt{8}}_{6\sqrt{2}} - \sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx - \sqrt{2}\ln(3+2\sqrt{2})$$

$$\int_1^3 \sqrt{2x^2-2}\,dx = 6 - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

$$P = P_1 - P_2 = \frac{10}{3} - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

tražena površina

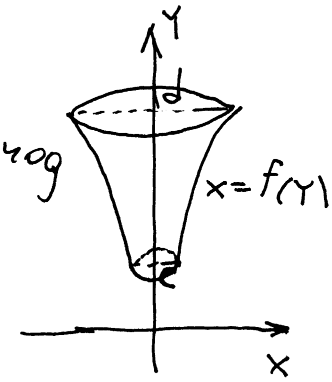
II Zapremina rotacionog tijela



zapremina tijela dobijenog rotacijom dijela krive $y=f(x)$ oko x-ose

$$V_x = \pi \cdot \int_a^b [f(x)]^2 dx$$

$$V_y = \pi \int_c^d [f(y)]^2 dy$$ - zapremina tijela dobijenog rotacijom dijela krive $x=f(y)$ oko y-ose



Ako je kriva data u parametarskom obliku:

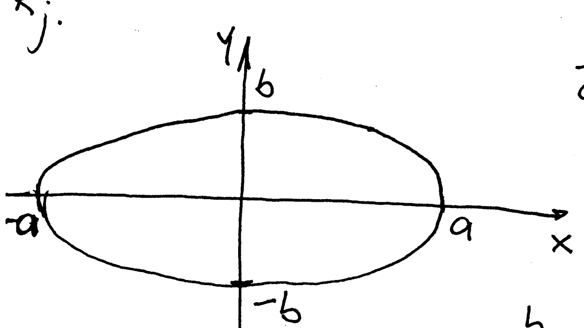
$x = \alpha(t)$
$y = \beta(t)$
$t_1 \le t \le t_2$

$$V_x = \pi \int_{t_1}^{t_2} [\beta(t)]^2 \, |\alpha'(t)| \, dt$$

$$V_y = \pi \int_{t_1}^{t_2} [\alpha(t)]^2 \, |\beta'(t)| \, dt$$

1. Izračunati zapreminu tijela koje nastaje rotacijom krive $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oko y-ose.

Rj.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2 \cdot \left(1 - \frac{y^2}{b^2}\right)$$

$$x^2 = a^2 \cdot \frac{b^2 - y^2}{b^2}$$

$$x^2 = \frac{a^2}{b^2} \cdot (b^2 - y^2)$$

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$$

$$V_y = \pi \int_{-b}^{b} [f(y)]^2 dy = \pi \int_{-b}^{b} \frac{a^2}{b^2} (b^2 - y^2) \, dy = 2\pi \frac{a^2}{b^2} \int_0^b (b^2 - y^2) \, dy =$$

Parna f-ja (simetrična u odnosu na Y-osu)

$$= 2\pi \frac{a^2}{b^2}\left(b^2 y\Big|_0^b - \frac{y^3}{3}\Big|_0^b\right) = 2\pi \frac{a^2}{b^2}\left(b^3 - \frac{1}{3}b^3\right) = 2\pi \frac{a^2}{b^2}\cdot\frac{2}{3}b^3 = \frac{4\pi a^2 b}{3}.$$

2. Figura u ravni ograničena parabolom $y = 4 - x^2$ i pravama $y \geq 3x$, $y \geq 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

Rj.

$y = 4 - x^2$
$y = 3x$

$3x = 4 - x^2$

$x^2 + 3x - 4 = 0$

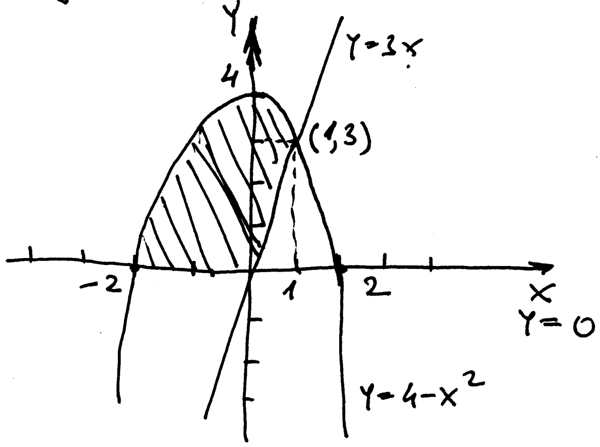
$D = 9 + 16 = 25$

$x_{1,2} = \frac{-3 \pm 5}{2}$

$x_1 = -4 \Rightarrow y_1 = -12$

$x_2 = 1 \Rightarrow y_2 = 3$

$A(-4, -12)$ i $B(1,3)$ su tačke presjeka prave i parabole



$$V_x = V_1 - V_2, \quad V_1 = \pi\int_{-2}^{1}(4-x^2)^2\,dx, \quad V_2 = \pi\int_0^1 (3x)^2\,dx$$

$$V_1 = \pi\int_{-2}^{1}(16 - 8x^2 + x^4)\,dx = \pi\left(16x\Big|_{-2}^{1} - 8\frac{x^3}{3}\Big|_{-2}^{1} + \frac{x^5}{5}\Big|_{-2}^{1}\right) = \pi\left(16\cdot 3 - \frac{8}{3}\cdot 9 + \frac{1}{5}\cdot 33\right)$$

$1-(-2)$ $\quad 1^3-(-2)^3 \quad 1^5-(-2)^5$

$$= \pi\left(48 - 24 + \frac{33}{5}\right) = \frac{158}{5}\pi$$

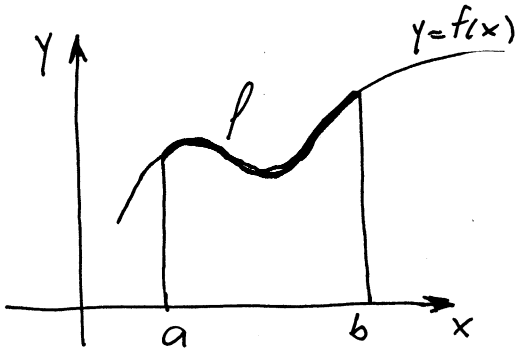
$$V_2 = \pi\cdot 9\int_0^1 x^2\,dx = 9\pi\,\frac{x^3}{3}\Big|_0^1 = 3\pi(1-0) = 3\pi$$

$$V = V_1 - V_2 = \frac{158}{5}\pi - 3\pi = \frac{158\pi - 15\pi}{5} = \frac{138}{5}\pi$$

3. ✓ Izračunati zapreminu tijela nastalog obrtanjem oko x-ose figure omeđenu krivom $y = \arcsin x$ i pravama $x = 1$ i $y = 0$. Uputa: parcijalna integracija 2x

4. ✓ Izračunati zapreminu tijela koje nastaje rotacijom ravne figure ograničene parabolom $y = 6 - x - x^2$ i prave $y = 0$ oko x-ose.

III Dužina luka krive



$$l=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

dužina luka krive $y=f(x)$

$$l=\int_c^d \sqrt{1+[f'(y)]^2}\,dy$$

dužina luka krive $x=f(y)$

Ako je kriva data u parametarskom obliku:

$x=x(t)$
$y=y(t)$
$t_1\le t\le t_2$ $\Rightarrow$ $$l=\int_{t_1}^{t_2}\sqrt{(\dot{x})^2+(\dot{y})^2}\,dt$$

gdje je $\dot{x}=\dfrac{dx}{dt}$

i $\dot{y}=\dfrac{dy}{dt}$ (izvod po t)

1. Izračunati dužinu luka krive $y=\dfrac{x^2}{2}-\dfrac{\ln x}{4}$ ako je $1\le x\le 3$.

Rj. $l=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$, $y=f(x)=\dfrac{x^2}{2}-\dfrac{\ln x}{4}=\dfrac{1}{2}x^2-\dfrac{1}{4}\ln x$

$$y'=\frac{1}{2}\cdot 2x-\frac{1}{4}\cdot\frac{1}{x}=x-\frac{1}{4x}$$

$$l=\int_1^3\sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx=\int_1^3\sqrt{1+x^2-\cancel{2x}\cdot\frac{1}{\cancel{4x}_{2}}+\frac{1}{16x^2}}\,dx=\int_1^3\sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}$$

$$=\int_1^3\sqrt{\left(x+\frac{1}{4x}\right)^2}\,dx=\int_1^3\left(x+\frac{1}{4x}\right)dx=\frac{x^2}{2}\Big|_1^3+\frac{1}{4}\ln x\Big|_1^3=\frac{1}{2}(9-1)+\frac{1}{4}(\ln 3-\ln 1)$$

$$=4+\frac{1}{4}\ln 3=4+\ln\sqrt[4]{3}$$

$$\boxed{\frac{1}{4}\ln 3=\ln 3^{\frac{1}{4}}}$$

2. Naći dužinu luka kojeg na paraboli $y^2=2x+1$ odsjeca prava $x-y=1$.

Rj.

$$\frac{\begin{array}{c} y^2=2x+1 \\ x-y=1 \end{array}}{\begin{array}{c} y^2=2x+1 \\ y=x-1 \end{array}}$$

$$(x-1)^2=2x+1$$

$$x^2-2x+1=2x+1$$

$$x^2-4x=0$$

$$x(x-4)=0$$

$$x_1=0 \Rightarrow y_1=-1$$

$$x_2=4 \Rightarrow y_2=3$$

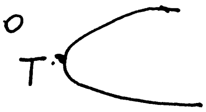
$A(0,-1)$; $B(4,3)$ su tačke presjeka parabole i prave

$$y^2=2x+1 \qquad x=0 \Rightarrow y=\pm 1$$

$$2x=y^2-1$$

$$x=\frac{1}{2}y^2-\frac{1}{2}$$

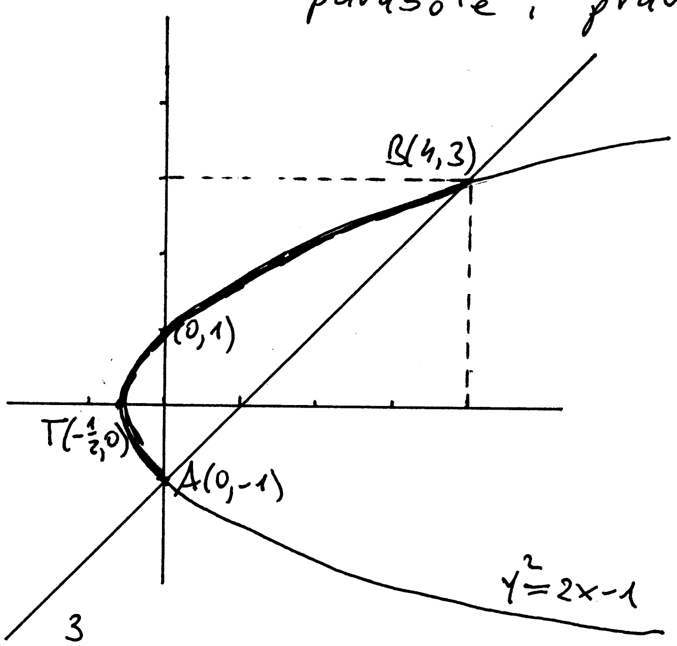
$a>0$



$T\left(-\frac{1}{2},0\right)$

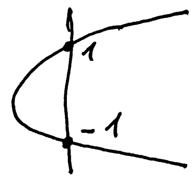
$D=1$

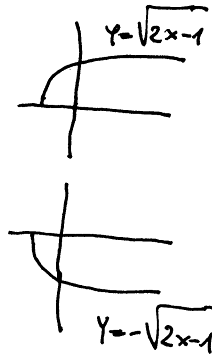
$$-\frac{D}{4a}=-\frac{1}{4\cdot\frac{1}{2}}=-\frac{1}{2}, \quad -\frac{b}{2a}=0$$



$$y^2=2x+1$$

$$y=\pm\sqrt{2x-1}$$





$$y^2=2x+1$$

$$x=\frac{1}{2}y^2-\frac{1}{2}$$

$$x'=\frac{1}{2}\cdot 2y=y \qquad \text{tj.} \quad x'_y=y$$

$$l=\int_{-1}^{3}\sqrt{1+y^2}\,dy$$

integral $\int\sqrt{1+y^2}\,dy$ smo uradili Metodom Ostrogradskog, 3 zadatak na 65 strani u skripti (umjesto y imali smo x)

$$l=\int_{-1}^{3}\sqrt{1+y^2}\,dy=\frac{1}{2}y\sqrt{y^2+1}\Big|_{-1}^{3}+\frac{1}{2}\ln\left|y+\sqrt{y^2+1}\right|\Big|_{-1}^{3}=$$

$$=\frac{1}{2}\left(3\sqrt{10}-(-1)\sqrt{2}\right)+\frac{1}{2}\left(\ln|3+\sqrt{10}|-\ln|-1+\sqrt{2}|\right)=$$

$$=\frac{3\sqrt{10}+\sqrt{2}}{2}+\frac{1}{2}\ln\left|\frac{3+\sqrt{10}}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}\right|=\frac{3\sqrt{10}+\sqrt{2}}{2}+\frac{1}{2}\ln\left|(3+\sqrt{10})(\sqrt{2}+1)\right|$$

$$=\frac{3\sqrt{10}-\sqrt{2}}{2}+\ln\sqrt{(3+\sqrt{10})(\sqrt{2}+1)}$$

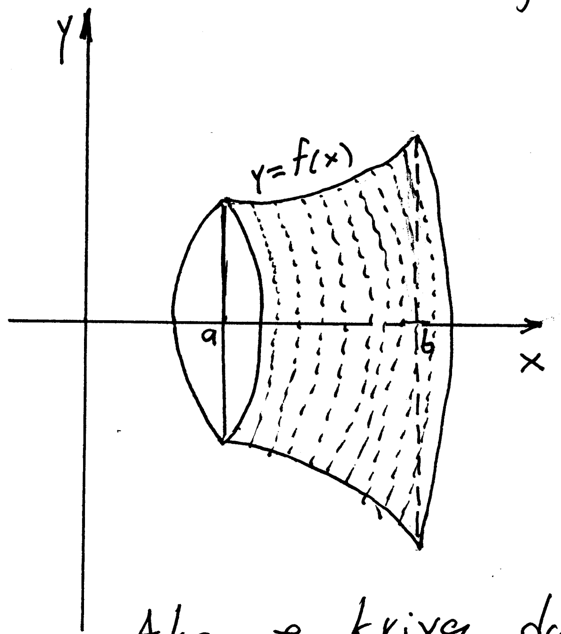
3. Izračunati dužinu luka krive

a) $y=\sqrt{2x-x^2}-1$, ako je $\frac{1}{4}\le x\le 1$

b) $x=\frac{1}{4}y^2-\frac{1}{2}\ln y$, ako je $1\le y\le e$

IV Komplanacija obrtne površi

komplanacija lat. postupak za izračunavanje površina dijelova zakrivljenih ploha



površina omotača tijela dobijenog rotacijom dijela krive $y=f(x)$ oko x-ose

$$P=2\pi\int_a^b |f(x)|\cdot\sqrt{1+(f'(x))^2}\,dx$$

Ako je kriva data u parametarskom obliku

$x=\alpha(t)$

$y=\beta(t)$

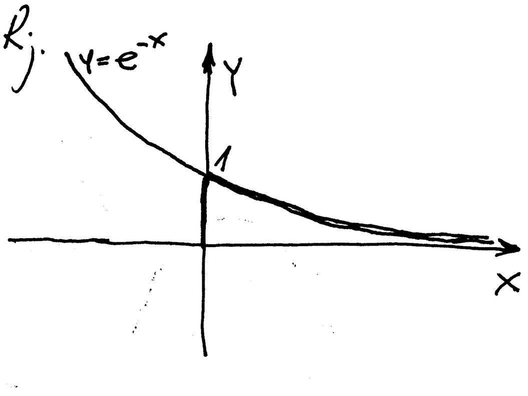
$t_1\le t\le t_2$

$$\Rightarrow\quad P=2\pi\int_{t_1}^{t_2}|\beta(t)|\cdot\sqrt{(\dot{x})^2+(\dot{y})^2}\,dt$$

gdje je $\dot{x}=\frac{dx}{dt}$, $\dot{y}=\frac{dy}{dt}$

1. Izračunati površinu omotača tijela koje nastaje rotacijom krive $y=e^{-x}$ oko x-ose za $x\ge 0$.

Rj.



$y=e^{-x}$, $y'=e^{-x}\cdot(-1)=-e^{-x}$

$$P=2\pi\int_0^{+\infty}e^{-x}\cdot\sqrt{1+e^{-2x}}\,dx$$

$$=2\pi\lim_{R\to+\infty}\int_0^R e^{-x}\sqrt{1+e^{-2x}}\,dx$$

$$\int_0^R e^{-x}\sqrt{1+e^{-2x}}\,dx = \left|\begin{array}{ll} u=e^{-x} & x=0 \Rightarrow u=1 \\ du=-e^{-x}dx & x=R \Rightarrow u=e^{-R} \\ -du=e^{-x}dx & \end{array}\right| = \int_1^{e^{-R}}\sqrt{1+u^2}\cdot(-du)$$

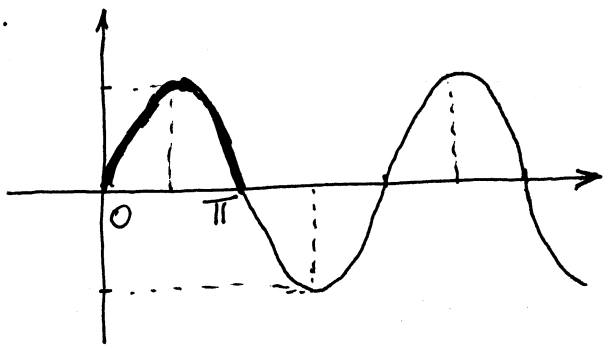
$$= -\int_1^{e^{-R}}\sqrt{1+u^2}\,du = \int_{e^{-R}}^1\sqrt{1+u^2}\,du \xrightarrow[\text{(metoda Ostrograd.)}]{\text{rađeno ranije}} \frac{1}{2}u\sqrt{1+u^2}\Big|_{e^{-R}}^1 + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\Big|_{e^{-R}}^1$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2}e^{-R}\sqrt{1+e^{-2R}} + \frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{2}\ln\left(e^{-R}+\sqrt{1+e^{-2R}}\right),\qquad \begin{array}{l} e^{-2R}\to 0,\ R\to\infty \\ e^{-R}\to 0,\ R\to\infty \end{array}$$

$$P = 2\pi\lim_{R\to+\infty}\int_0^R e^{-x}\sqrt{1+e^{-2x}}\,dx = 2\pi\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}\ln(1+\sqrt{2})\right) = \pi\left(\sqrt{2}+\ln(1+\sqrt{2})\right)$$

2. Izračunati površinu omotača tijela koje nastaje rotacijom jednog svoda sinusoide $y=\sin x$ oko x-ose.

Rj.



$$y=\sin x$$
$$y'=\cos x$$
$$P = 2\pi\int_0^\pi \sin x\cdot\sqrt{1+\cos^2 x}\,dx =$$

$$= \left|\begin{array}{ll} \cos x = t & x=0 \Rightarrow t=1 \\ -\sin x\,dx = dt & x=\pi \Rightarrow t=-1 \\ \sin x\,dx = -dt & \end{array}\right| = 2\pi\int_1^{-1}\sqrt{1+t^2}\,(-dt) = -2\pi\int_1^{-1}\sqrt{1+t^2}\,dt$$

$$= 2\pi\int_{-1}^1\sqrt{1+t^2}\,dt = 4\pi\int_0^1\sqrt{1+t^2}\,dt \xrightarrow[\text{(metoda Ostrogradskog)}]{\text{rađeno ranije}} 4\pi\left[\frac{1}{2}t\sqrt{1+t^2}\Big|_0^1 + \frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right|\Big|_0^1\right]$$

parna f-ja

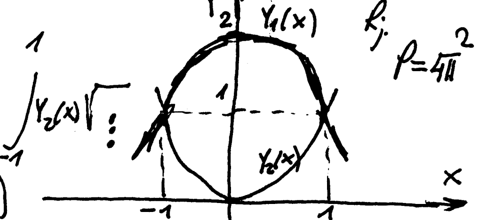
$$= 4\pi\cdot\frac{\sqrt{2}}{2} + 4\pi\cdot\frac{1}{2}\left(\ln(1+\sqrt{2}) - \underbrace{\ln 1}_{0}\right) = 2\sqrt{2}\pi + 2\pi\ln(1+\sqrt{2})$$

3. Izračunati površinu tijela koje nastaje rotacijom kružnice $x^2+(y-1)^2=1$ oko x-ose.

Uputa: $y_1(x) = 1+\sqrt{1-x^2}$
$y_2(x) = 1-\sqrt{1-x^2}$

$$P = P_1 + P_2 = 2\pi\int_{-1}^1 y_1(x)\sqrt{1+(y_1')^2}\,dx + \int_{-1}^1 y_2(x)\sqrt{\vdots}$$

tijelo koje nastaje rotacijom → TORUS (ŠLAUF)



Rj. $P=4\pi^2$