Tablica integrala

$$\boxed{sh\,x = \frac{e^x - e^{-x}}{2} \qquad ch\,x = \frac{e^x + e^{-x}}{2}}$$ sinus hiperbolni

1. $\int 0\,dx = c$, c - konstanta
2. $\int dx = x + c$
3. $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c$, $\alpha \neq -1$
4. $\int \frac{dx}{x} = \ln|x| + c$
5. $\int a^x dx = \frac{a^x}{\ln a} + c$, $0 < a \neq 1$
6. $\int e^x dx = e^x + c$
7. $\int \sin x\,dx = -\cos x + c$
8. $\int \cos x\,dx = \sin x + c$
9. $\int \frac{dx}{\cos^2 x} = tg\,x + c$
10. $\int \frac{dx}{\sin^2 x} = -ctg\,x + c$
11. $\int sh\,x\,dx = ch\,x + c$
12. $\int ch\,x\,dx = sh\,x + c$
13. $\int \frac{dx}{ch^2 x} = th\,x + c$
14. $\int \frac{dx}{sh^2 x} = -cth\,x + c$
15. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$
16. $\int \frac{dx}{1+x^2} = arc\,tg\,x + c$
17. $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln|x + \sqrt{x^2 \pm 1}| + c$
18. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$

I $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$, $a > 0$

II $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\,arc\,tg\frac{x}{a} + c$, $a \neq 0$

III $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2+a^2}| + c$, $a \neq 0$

IV a) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$, $a \neq 0$

IV b) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

V $\int \frac{dx}{\sin x} = \ln\left|tg\frac{x}{2}\right| + c$

VI $\int \frac{dx}{\cos x} = \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c$

Određeni integrali

Određene integrale ćemo računati pomoću Newto-Leibnizove formule $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)$, gdje je $F'(x) = f(x)$

1. $$\int_{\pi/6}^{\pi/3} \frac{dx}{\cos^2 x} = tg\,x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = tg\frac{\pi}{3} - tg\frac{\pi}{6} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

2. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x\,dx = -\cos x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\right) = -\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = -\frac{1-\sqrt{2}}{2}$$

3. $$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \arcsin\frac{\sqrt{3}}{2} - \arcsin\frac{1}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

4. $$\int_a^b x^m\,dx = \frac{x^{m+1}}{m+1}\Big|_a^b = \frac{1}{m+1}\left(b^{m+1} - a^{m+1}\right)$$

5. $$\int_0^1 (e^x-1)^4 e^x\,dx = \left|\begin{array}{l} e^x - 1 = t \\ e^x dx = dt \\ x=0 \Rightarrow t=0 \\ x=1 \Rightarrow t=e-1 \end{array}\right| = \int_0^{e-1} t^4\,dt = \frac{t^5}{5}\Big|_0^{e-1} = \frac{1}{5}(e-1)^5$$

6. $$\int_2^9 \sqrt[3]{x-1}\,dx = \left|\begin{array}{l} x-1 = t^3 \\ dx = 3t^2 dt \\ x=2 \Rightarrow t=1 \\ x=9 \Rightarrow t=2 \end{array}\right| = \int_1^2 \sqrt[3]{t^3}\cdot 3t^2\,dt = 3\int_1^2 t^3\,dt = \frac{3}{4}t^4\Big|_1^2 =$$
$$= \frac{3}{4}(16-1) = \frac{45}{4}$$

7. $$\int_0^{\ln 5} \frac{\sqrt{e^x-1}}{1+3e^{-x}}\,dx \overset{\cdot e^x}{\underset{\cdot e^x}{=}} \int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx = \left|\begin{array}{l} e^x - 1 = t^2 \\ e^x dx = 2t\,dt \\ x=0 \Rightarrow t=0 \\ x=\ln 5 \Rightarrow t=2 \\ e^x = t^2+1 \end{array}\right| = \int_0^2 \frac{\sqrt{t^2}\cdot 2t}{t^2+1+3}\,dt$$

$$= 2\int_0^2 \frac{t^2 +4-4}{t^2+4}\,dt = 2\int_0^2 dt - 2\int_0^2 \frac{4}{t^2+4}\,dt = 2t\Big|_0^2 - 8\cdot\frac{1}{2}\,arctg\frac{t}{2}\Big|_0^2 =$$

$$= 4 - 4(arctg\,1 - arctg\,0) = 4 - 4\cdot\frac{\pi}{4} = 4 - \pi$$

Osobine određenih integrala

a) $\int_a^a f(x)\,dx = 0$

b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c) $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$

d) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

e) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall x$

8. $$\int_0^{\sqrt{7}} \frac{dx}{7+x^2} = \frac{1}{\sqrt{7}} \operatorname{arctg} \frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}\left(\operatorname{arctg}\frac{\sqrt{7}}{\sqrt{7}} - \operatorname{arctg}\frac{0}{\sqrt{7}}\right) = \frac{1}{\sqrt{7}}\cdot\frac{\pi}{4} = \frac{\sqrt{7}\pi}{28}$$

9. $$\int_0^{1/2} \sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \\ x=0 \Rightarrow \sin t = 0 \Rightarrow t=0 \\ x=\frac{1}{2} \Rightarrow \sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6} \end{array} \right| = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\,\cos t\,dt =$$

kako je $\sin^2 t + \cos^2 t = 1$

$$= \int_0^{\pi/6} \sqrt{\cos^2 t}\,\cos t\,dt = \int_0^{\pi/6} \cos^2 t\,dt = \frac{1}{2}\int_0^{\pi/6} (1+\cos 2t)\,dt = \frac{1}{2} t \Big|_0^{\frac{\pi}{6}} + \frac{1}{4}\sin 2t \Big|_0^{\frac{\pi}{6}}$$

$1 = \cos^2 t + \sin^2 t$
$\cos 2t = \cos^2 t - \sin^2 t$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{2\pi + 3\sqrt{3}}{24}$$

10. $$\int_0^4 \frac{dx}{1+\sqrt{2x+1}}$$

uputa: smjena $2x+1 = t^2$

rj. $2 - \ln 2$

11. $$\int_0^1 \frac{dx}{\sqrt{2-x^2+x}}$$

uputa: $-x^2+x+2 = \ldots$
$\ldots = \frac{9}{4} - (x-\frac{1}{2})^2$
$x - 1 = \frac{3}{2} t$

rj. $2\arcsin\frac{1}{3}$

12. $$\int_1^e x\ln x\,dx = \left| \begin{array}{ll} u = \ln x & dv = x\,dx \\ du = \frac{dx}{x} & v = \frac{x^2}{2} \end{array} \right| = \frac{1}{2}x^2 \ln x \Big|_1^e - \frac{1}{2}\int_1^e x^2 \frac{dx}{x} =$$

$$= \frac{1}{2}(e^2 \ln e - 1^2 \ln 1) - \frac{1}{2}\int_1^e x\,dx = \frac{1}{2}e^2 - \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_1^e =$$

$$= \frac{1}{2}e^2 - \frac{1}{4}(e^2-1) = \frac{1}{4}e^2 + \frac{1}{4}$$

Izračunati integral $\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5}\,dx$.

Rj. $x-4\sqrt{x}+5 = x-2\cdot\sqrt{x}\cdot 2+4+1 = (\sqrt{x}-2)^2+1$

$$\int_1^4 \frac{\sqrt{x}+2}{(\sqrt{x}-2)^2+1}\,dx = \left| \begin{array}{l} x=t^2 \\ dx=2t\,dt \\ x=1 \Rightarrow t=1 \\ x=4 \Rightarrow t=2 \end{array} \right| = \int_1^2 \frac{t+2}{(t-2)^2+1}\cdot 2t\,dt =$$

$$=2\int_1^2 \frac{t^2+2t}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2+2t-6t+6t+5-5}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2-4t+5}{t^2-4t+5}\,dt +$$

$$+2\int_1^2 \frac{6t-5}{t^2-4t+5}\,dt = 2\int_1^2 dt + 2\cdot 3\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}$$

$$\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4+4-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt + \frac{7}{3}\int_1^2 \frac{dt}{t^2-4t+5}$$

$$\int_1^2 dt = t\Big|_1^2 = 2-1 = 1, \quad \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt = \left| \begin{array}{l} t^2-4t+5=s \\ (2t-4)dt=ds \\ t=1 \Rightarrow s=2 \\ t=2 \Rightarrow s=1 \end{array} \right| = \int_2^1 \frac{ds}{s} = \ln|s|\Big|_2^1$$

$$=\ln 1-\ln 2 = -\ln 2$$

$$\int_1^2 \frac{dt}{t^2-4t+5} = \int_1^2 \frac{dt}{(t-2)^2+1} = \left| \begin{array}{l} t-2=s \\ dt=ds \\ t=1 \Rightarrow s=-1 \\ t=2 \Rightarrow s=0 \end{array} \right| = \int_{-1}^0 \frac{ds}{s^2+1} = \operatorname{arctg} s\Big|_{-1}^0 = 0-\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} = 2\cdot 1 + 6\left(-\ln 2 + \frac{7}{3}\cdot\frac{\pi}{4}\right) = 2-6\ln 2+\frac{7\pi}{2} \approx 8{,}8367$$

traženo rješenje

(#) Izračunati integral $\int_0^{\frac{\pi}{4}} \sin^5 x \cos^7 x \, dx$

Rj. $\int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^7 x \, dx = \int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^6 x \cdot \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \\ x = 0 \Rightarrow t = 0 \\ x = \frac{\pi}{4} \Rightarrow t = \frac{\sqrt{2}}{2} \\ \cos^6 x = (\cos^2 x)^3 = \\ = (1 - \sin^2 x)^3 = (1 - t^2)^3 \end{array} \right| =$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1 - t^2)^3 \, dt =$$

$$\begin{array}{c} 1 \\ 1 \; 1 \\ 1 \; 2 \; 1 \\ 1 \; 3 \; 3 \; 1 \end{array}$$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1 - 3t^2 + 3t^4 - t^6) \, dt = \int_0^{\frac{\sqrt{2}}{2}} (t^5 - 3t^7 + 3t^9 - t^{11}) \, dt =$$

$$= \frac{1}{6} t^6 \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{3}{8} t^8 \Big|_0^{\frac{\sqrt{2}}{2}} + \frac{3}{10} t^{10} \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{1}{12} t^{12} \Big|_0^{\frac{\sqrt{2}}{2}} =$$

$$= \frac{1}{\cancel{6}_3} \cdot \frac{\overset{\cancel{4}\,1}{\cancel{8}}}{\underset{16}{6\cancel{4}}} - \frac{3}{\underset{1}{\cancel{8}}} \cdot \frac{\overset{\cancel{2}\,1}{\cancel{16}}}{\underset{128}{2\cancel{56}}} + \frac{3}{\underset{5}{\cancel{10}}} \cdot \frac{\overset{\cancel{16}\,1}{\cancel{32}}}{\underset{64}{\cancel{1024}}} - \frac{1}{\underset{3}{\cancel{12}}} \cdot \frac{\overset{1}{\cancel{16}}\,\cancel{64}}{\underset{256}{\cancel{4096}}} =$$

$$= \frac{1}{\underbrace{3 \cdot 16}_{3 \cdot 2^4}} - \frac{3}{\underbrace{128}_{2^7}} + \frac{3}{\underbrace{5 \cdot 64}_{5 \cdot 2^6}} - \frac{1}{\underbrace{3 \cdot 256}_{3 \cdot 2^8}} = \frac{5 \cdot 2^4 - 3 \cdot 3 \cdot 5 \cdot 2 + 3 \cdot 3 \cdot 2^2 - 5}{3 \cdot 5 \cdot 2^8}$$

$$= \frac{80 - 90 + 36 - 5}{3 \cdot 5 \cdot 2^8} = \frac{2\overset{7}{\cancel{1}}}{\underset{1}{\cancel{3}} \cdot 5 \cdot 2^8} = \frac{7}{1280}$$ traženo rješenje

(#) Izračunati integral $\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx$

Rj.

Metoda Ostrogradskog:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = (ax^2+bx+c)\sqrt{x^2+3} + \lambda \int \frac{dx}{\sqrt{x^2+3}} \quad \Big/ \frac{d}{dx}$$

$$\frac{2x^3-7x+4}{\sqrt{x^2+3}} = (2ax+b)\sqrt{x^2+3} + (ax^2+bx+c)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}} + \frac{\lambda}{\sqrt{x^2+3}} \quad \Big/ \cdot \sqrt{x^2+3}$$

$$2x^3-7x+4 = (2ax+b)(x^2+3) + (ax^2+bx+c)x + \lambda$$

$$2x^3-7x+4 = 2ax^3+bx^2+6ax+3b+ax^3+bx^2+cx+\lambda$$

x^3: $2a+a=2 \Rightarrow 3a=2$, $a=\frac{2}{3}$

x: $6a+c=-7 \Rightarrow 6\cdot\frac{2}{3}+c=-7$

$4+c=-7$

$c=-11$

x^2: $b+b=0 \Rightarrow b=0$

x^0: $3b+\lambda=4$

$\lambda=4$

Prema tome:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \left(\frac{2}{3}x^2-11\right)\sqrt{x^2+3} + 4\int\frac{dx}{\sqrt{x^2+3}} =$$

$$= \frac{2}{3}x^2\sqrt{x^2+3} - 11\sqrt{x^2+3} + 4\ln\left|x+\sqrt{x^2+3}\right| + C$$

Prema tome

$$\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \frac{2}{3}x^2\sqrt{x^2+3}\Big|_{-1}^{1} - 11\sqrt{x^2+3}\Big|_{-1}^{1} + 4\ln\left|x+\sqrt{x^2+3}\right|\Big|_{-1}^{1} =$$

$$= \frac{2}{3}(2-2) - 11(2-2) + 4\left(\ln|1+2| - \ln|-1+2|\right) =$$

$$= 4(\ln 3 - \ln 1) = 4\ln 3$$

traženi rezultat

Izračunati: $\int_3^4 \frac{6x+8}{x^2+x-6}\,dx$

Rj.

$$\frac{6x+8}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3} \quad /(x-2)(x+3)$$

$D = 1+24 = 25$

$x_{1,2} = \frac{-1 \pm 5}{2}$

$x_1 = -3 \quad x_2 = 2$

$x^2+x-6 = (x-2)(x+3)$

$6x+8 = A(x+3) + B(x-2)$

$6x+8 = (A+B)x + (3A-2B)$

$$\begin{aligned} A+B &= 6 \quad /\cdot 2 \\ 3A-2B &= 8 \\ \hline 2A+2B &= 12 \\ + \; 3A-2B &= 8 \\ \hline 5A &= 20 \\ A &= 4 \end{aligned}$$

$A+B=6$

$4+B=6$

$B=2$

$$\int \frac{6x+8}{x^2+x-6}\,dx = \int\left(\frac{4}{x-2} + \frac{2}{x+3}\right)dx = 4\int\frac{dx}{x-2} + 2\int\frac{dx}{x+3} =$$

$$= 4\ln|x-2| + 2\ln|x+3| + C$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = 4\ln|x-2|\Big|_3^4 + 2\ln|x+3|\Big|_3^4 = 4(\ln 2 - \underbrace{\ln 1}_{=0}) +$$

$$+ 2(\ln 7 - \ln 6) = 4\ln 2 + 2\ln\frac{7}{6} = \ln 2^4 + \ln\left(\frac{7}{6}\right)^2$$

$$= \ln\frac{7^2}{2^2\cdot 3^2}\cdot 2^4 = \ln\frac{49\cdot 4}{9} = \ln\frac{196}{9}$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = \ln\frac{196}{9}$$ traženo rješenje

(#) Izračunati integral $$I=\int_{6-\sqrt{2}}^{7}\frac{(4x+2)}{\sqrt{-34+12x-x^2}}\,dx.$$

Rj.

$$I_1=\int\frac{4x+2}{\sqrt{-x^2+12x-34}}\,dx$$

Metoda Ostrogradskog:

$$\int\frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx=Q_{n-1}(x)\sqrt{ax^2+bx+c}+\lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int\frac{4x+2}{\sqrt{-x^2+12x-34}}\,dx=a\sqrt{-x^2+12x-34}+\lambda\int\frac{dx}{\sqrt{-x^2+12x-34}}\quad\Big/\frac{d}{dx}$$

$$\frac{4x+2}{\sqrt{-x^2+12x-34}}=a\cdot\frac{(-2x+12)}{2\sqrt{-x^2+12x-34}}+\lambda\cdot\frac{1}{\sqrt{-x^2+12x-34}}\quad\Big/\cdot\sqrt{-x^2+12x-34}$$

$$4x+2=a(-x+6)+\lambda$$
$$4x+2=-ax+6a+\lambda$$
$$-a=4 \qquad 6a+\lambda=2$$
$$a=-4 \qquad -24+\lambda=2$$
$$\lambda=26$$

$$\int\frac{4x+2}{\sqrt{-x^2+12x-34}}\,dx=-4\sqrt{-x^2+12x-34}+26\int\frac{dx}{\sqrt{-x^2+12x-34}}$$

$$-x^2+12x-34=-(x^2-12x+34)=-(x^2-2\cdot6x+36-36+34)$$
$$=-((x-6)^2-2)=2-(x-6)^2$$

$$\int\frac{dx}{\sqrt{-x^2+12x-34}}=\int\frac{dx}{\sqrt{2-(x-6)^2}}=\left|\begin{array}{l}x-6=\sqrt{2}t\\ dx=\sqrt{2}\,dt\\ t=\frac{x-6}{\sqrt{2}}\end{array}\right|=\int\frac{\sqrt{2}\,dt}{\sqrt{2-2t^2}}=\frac{\sqrt{2}}{\sqrt{2}}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\arcsin t+c=\arcsin\left(\frac{x-6}{\sqrt{2}}\right)+c$$

$$\int_{6-\sqrt{2}}^{7}\frac{4x+2}{\sqrt{-34+12x-x^2}}\,dx=-4\sqrt{-x^2+12x-34}\,\Big|_{6-\sqrt{2}}^{7}+26\arcsin\frac{x-6}{\sqrt{2}}\Big|_{6-\sqrt{2}}^{7}=$$

$$=-4\left(\sqrt{-49+84-34}-\sqrt{-(36-12\sqrt{2}+2)+72-12\sqrt{2}-34}\right)+26\left(\arcsin\frac{1}{\sqrt{2}}-\arcsin\left(-\frac{\sqrt{2}}{\sqrt{2}}\right)\right)$$

$$=-4\left(\sqrt{1}-\sqrt{0}\right)+26\left(\frac{\pi}{4}-\left(-\frac{\pi}{2}\right)\right)=-4+26\cdot\frac{3\pi}{4}=-4+\frac{39\pi}{2}$$

(#) Izračunati integral $I=\int_0^1 \sqrt{4-x^2}\,dx$.

Rj: Metoda Ostrogradskog

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = g_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int\sqrt{4-x^2}\,dx = \int\frac{4-x^2}{\sqrt{4-x^2}}dx = (ax+b)\sqrt{4-x^2} + \lambda\int\frac{dx}{\sqrt{4-x^2}} \quad \Big/ \frac{d}{dx}$$

$$\sqrt{4-x^2} = a\sqrt{4-x^2} + (ax+b)\frac{-\cancel{2}x}{\cancel{2}\sqrt{4-x^2}} + \lambda\cdot\frac{1}{\sqrt{4-x^2}} \quad \Big/ \sqrt{4-x^2}$$

$$4-x^2 = a(4-x^2) - ax^2 - bx + \lambda$$

$x^2:\ -a-a=-1$
$-2a=-1$
$a=\frac{1}{2}$

$x:\ -b=0$
$b=0$

$x^0:\ 4a+\lambda=4$
$4\cdot\frac{1}{2}+\lambda=4$
$2+\lambda=4$
$\lambda=2$

$$\int\sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2} + 2\int\frac{dx}{\sqrt{4-x^2}}$$

$$\int\frac{dx}{\sqrt{4-x^2}} = \left|\begin{array}{l} x=2t \\ dx=2dt \\ t=\frac{x}{2}\end{array}\right| = \int\frac{2dt}{\sqrt{4-4t^2}} = \frac{2}{\sqrt{4}}\int\frac{dt}{\sqrt{1-t^2}} = \arcsin t + c = \arcsin\frac{x}{2} + c$$

$$\int\sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + c$$

$$\int_0^1\sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2}\Big|_0^1 + 2\arcsin\frac{x}{2}\Big|_0^1 = \frac{1}{2}\sqrt{3} + 2\arcsin\frac{1}{2} -$$

$$-(0+2\arcsin 0) = \frac{\sqrt{3}}{2} + 2\cdot\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$ tražena vrijednost

Izračunati integral $I=\int_0^1 \arcsin\frac{x}{2}\,dx$.

Rj.

$$\int \arcsin\frac{x}{2}\,dx = \left| \begin{array}{l} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2dt \end{array} \right| = 2\int \arcsin t\,dt = \left| \begin{array}{ll} u=\arcsin t & dv=dt \\ du=\frac{dt}{\sqrt{1-t^2}} & v=t \end{array} \right| =$$

$$= 2\left(t\arcsin t - \int \frac{t}{\sqrt{1-t^2}}\,dt\right) = 2t\arcsin t - \int \frac{2t\,dt}{\sqrt{1-t^2}} \overset{(**)}{=}$$

$$\int \frac{-2t\,dt}{\sqrt{1-t^2}} = \left| \begin{array}{l} 1-t^2=s \\ -2t\,dt=ds \end{array} \right| = \int \frac{ds}{\sqrt{s}} = \int s^{-\frac{1}{2}}\,ds = \frac{s^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{s}+c =$$

$$= 2\sqrt{1-t^2}+c$$

$$\overset{(**)}{=} 2t\arcsin t + 2\sqrt{1-t^2}+c = x\arcsin\frac{x}{2} + 2\sqrt{1-\frac{x^2}{4}} + c$$

$$\int_0^1 \arcsin\frac{x}{2}\,dx = x\arcsin\frac{x}{2}\Big|_0^1 + 2\sqrt{1-\frac{x^2}{4}}\Big|_0^1 = \arcsin\frac{1}{2} + \left(2\sqrt{1-\frac{1}{4}} - 2\right) =$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{2} - 2 = \frac{\pi}{6} + \sqrt{3} - 2$$

Nepravi integrali

Nepravi integral u granicama od a do $+\infty$ je oblika

$$I = \int_a^{+\infty} f(x)\,dx$$

Rješavamo ga na sljedeći način:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

Ako postoji

$$\lim_{b\to\infty} \int_a f(x)\,dx$$

kažemo da integral konvergira ili da postoji nepravi integral, a ako limes ne postoji (kao realan broj), kažemo da integral divergira ili da nepravi integral ne postoji.

1. Izračunati:

a)
$$\int_1^{+\infty} \frac{dx}{x^4} = \lim_{a\to+\infty} \int_1^a \frac{dx}{x^4} = \lim_{a\to+\infty} \int_1^a x^{-4}\,dx = \lim_{a\to+\infty} \left.\frac{x^{-3}}{-3}\right|_1^a =$$

$$= -\frac{1}{3} \lim_{a\to\infty} \left.\frac{1}{x^3}\right|_1^a = -\frac{1}{3} \lim_{a\to\infty} \left(\overbrace{\frac{1}{a^3}}^{0} - 1\right) = \left(-\frac{1}{3}\right)(-1) = \frac{1}{3}$$

b)
$$\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b x^{-\frac{1}{2}}\,dx = \lim_{b\to\infty} \left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_1^b = 2 \lim \left.\sqrt{x}\right|_1^b$$

$$= 2 \lim_{b\to\infty} \left(\sqrt{b} - 1\right) = +\infty,$$ integral divergira

c)
$$\int_0^{+\infty} e^{-ax}\,dx = \lim_{t\to\infty} \int_0^t e^{-ax}\,dx = \left| \begin{matrix} -ax = s & x=0 \Rightarrow s=0 \\ -a\,dx = ds & x=t \Rightarrow s=-at \\ dx = -\frac{1}{a}ds & \end{matrix} \right| =$$

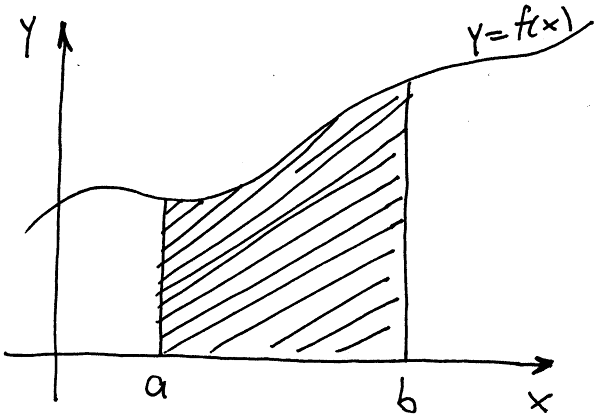
$(a>0)$

$$= \lim_{t\to\infty} \int_0^{-at} e^s \left(-\frac{1}{a}\right) ds = -\frac{1}{a} \lim_{t\to\infty} \left. e^s \right|_0^{-at} = -\frac{1}{a} \lim_{t\to\infty} \left(\overbrace{e^{-at}}^{0} - 1\right) = \frac{1}{a}$$

d) $\int_2^{+\infty} \frac{\ln x}{x}\,dx$ Rj. divergira. e) $\int_1^{+\infty} \frac{dx}{x^2(x+1)}$ Rj. $1-\ln 2$ f) $\int_0^{+\infty} x e^{-x^2}\,dx$ Rj. $\frac{1}{2}$

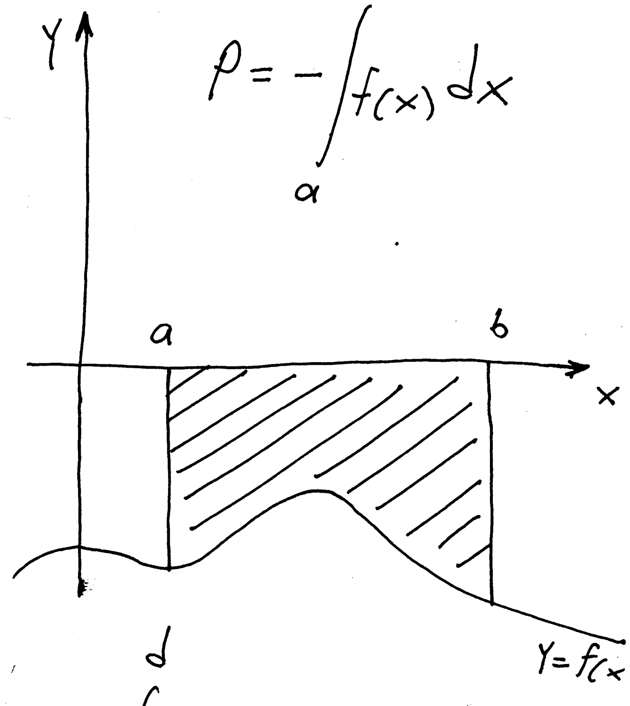
Primjena određenog integrala

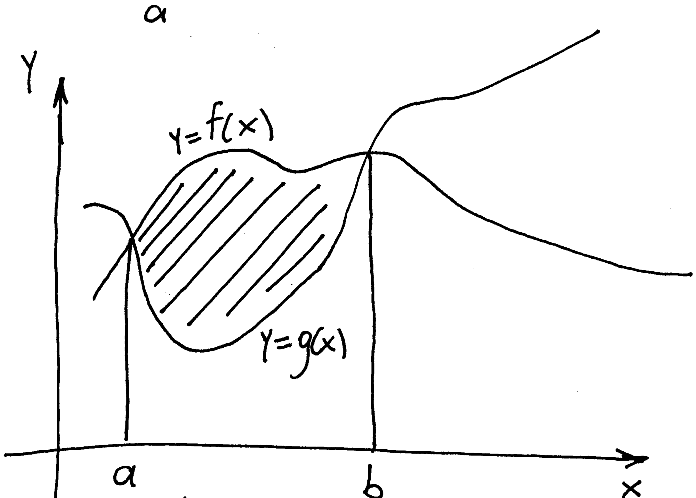
I Izračunavanje površine ravne figure



$$P=\int_a^b f(x)\,dx$$

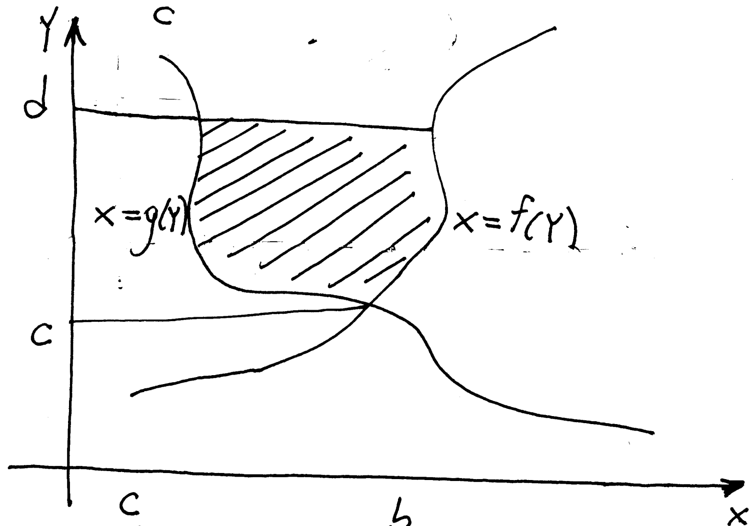
$$P=-\int_a^b f(x)\,dx$$

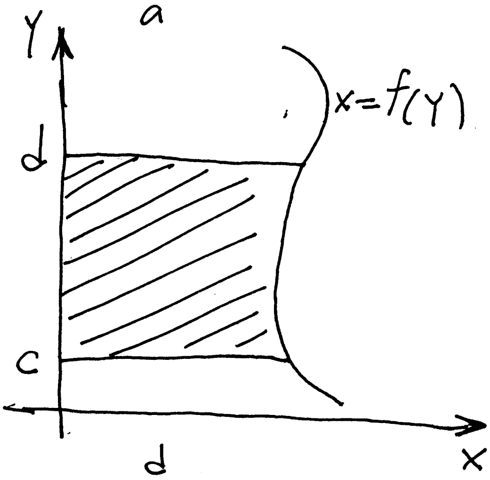




$$P=\int_a^b [f(x)-g(x)]\,dx$$

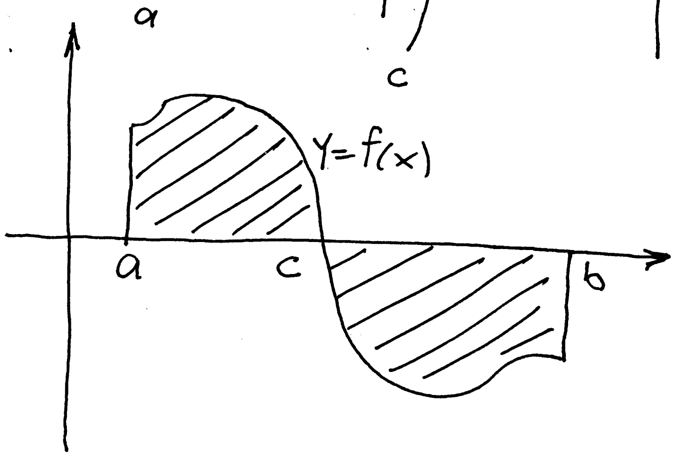
$$P=\int_c^d [f(y)-g(y)]\,dy$$





$$P=\int_c^d f(y)\,dy$$

$$P=\int_a^c f(x)\,dx+\left|\int_c^b f(x)\,dx\right|$$



1. Izračunati površinu ravne figure koja je ograničena linijama $y=4-(x-2)^2$ i $y=0$.

Rj.

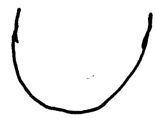
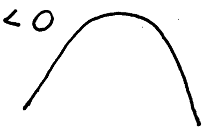
$y=4-(x-2)^2$

$y=4-(x^2-4x+4)$

$y=-x^2+4x$

$y=-x(x-4)$

Kriva $y=ax^2+bx+c$ ima grafik u obliku parabole.
Tjeme parabole $T\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

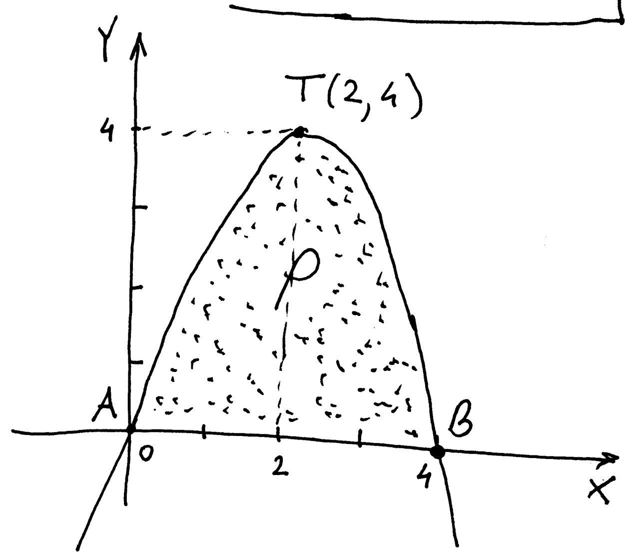
za $a>0$  za $a<0$ 

Nule $A(0,0)$ i $B(4,0)$

$$-\frac{b}{2a}=-\frac{4}{2\cdot(-1)}=2$$

$$\frac{4ac-b^2}{4a}=\frac{0-16}{-4}=4$$

Tjeme parabole $y=4-(x-2)^2$ je u tački $(2,4)$.



$$P=\int_0^4(-x^2+4x)\,dx=\int_0^4(-x^2)\,dx+\int_0^4 4x\,dx=-\frac{x^3}{3}\Big|_0^4+4\cdot\frac{x^2}{2}\Big|_0^4=-\frac{1}{3}(4^3-0^3)$$

$$+2(4^2-0^2)=-\frac{1}{3}\cdot 64+32=\frac{96}{3}-\frac{64}{3}=\frac{32}{3}$$

2. Izračunati površinu ravne figure koja je ograničena krivom $y=x^2-4x+3$ i pravama $y=0$, $x=0$ i $x=2$.

Rj. $y=x^2-4x+3$

$D=16-12=4$

$x_{1,2}=\frac{4\pm 2}{2}$

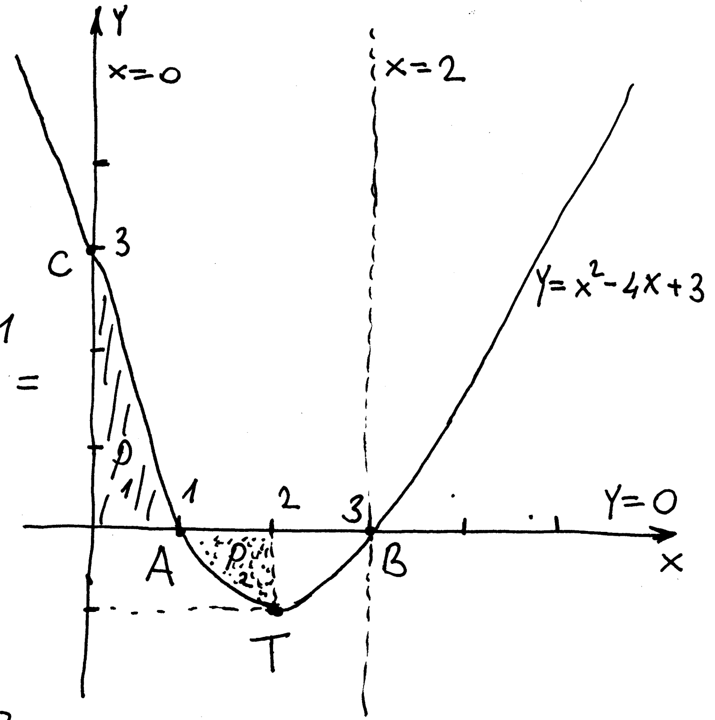
Nule krive $A(1,0)$ i $B(3,0)$

$$-\frac{b}{2a}=-\frac{-4}{2}=2$$

$$\frac{4ac-b^2}{4a}=\frac{12-16}{4}=-1$$

Tjeme krive $y=x^2-4x+3$ je u tački $T(2,-1)$.

C(0,3) je tačka presjeka krive sa Y-osom



$$P = P_1 + P_2$$

$$P_1 = \int_0^1 (x^2-4x+3)\,dx = \frac{x^3}{3}\Big|_0^1 - 4\cdot\frac{x^2}{2}\Big|_0^1 + 3x\Big|_0^1 =$$

$$= \frac{1}{3}(1-0) - 2(1-0) + 3(1-0) =$$

$$= \frac{1}{3} + 1 = \frac{4}{3}$$

$$P_2 = -\int_1^2 (x^2-4x+3)\,dx = -\left(\frac{x^3}{3}\Big|_1^2 - 4\cdot\frac{x^2}{2}\Big|_1^2 + 3x\Big|_1^2\right) = -\left(\frac{1}{3}(8-1) - 2(4-1) + 3\cdot 1\right)$$

$$= -\left(\frac{7}{3} - 6 + 3\right) = -\left(-\frac{2}{3}\right) = \frac{2}{3}$$

$P = \frac{4}{3} + \frac{2}{3} = 2$ tražena površina ravne figure.

3. Izračunati površinu ravne figure kojeg čine parabola $Y = x^2 - 2x + 2$ i prava $x + 2Y - 9 = 0$.

Rj.

prava $x + 2Y - 9 = 0$

$2Y = -x + 9$

$Y = -\frac{1}{2}x + \frac{9}{2}$

prava prolazi kroz tačke $A(0, \frac{9}{2})$ i $B(9, 0)$.

$Y = x^2 - 2x + 2$

$D = 4 - 8 = -4 < 0$

kriva ne siječe x-o

- nema nula

$x = 0 \Rightarrow Y = 2$

C(0,2) je presjek krive sa Y-osom

$$-\frac{b}{2a} = -\frac{-2}{2} = 1$$

$$\frac{4ac-b^2}{4a} = \frac{8-4}{4} = 1$$

T(1,1) je tjeme parabole

Trebamo naći još tačke presjeka prave i parabole.

$$\begin{array}{l} y=x^2-2x+2 \\ x+2y-9=0 \\ \hline y=x^2-2x+2 \\ x=-2y+9 \\ \hline y=(-2y+9)^2-2(-2y+9)+2 \end{array}$$

$$y=4y^2-36y+81+4y-18+2$$

$$4y^2-33y+65=0$$

$$D=33^2-16\cdot 65=49$$

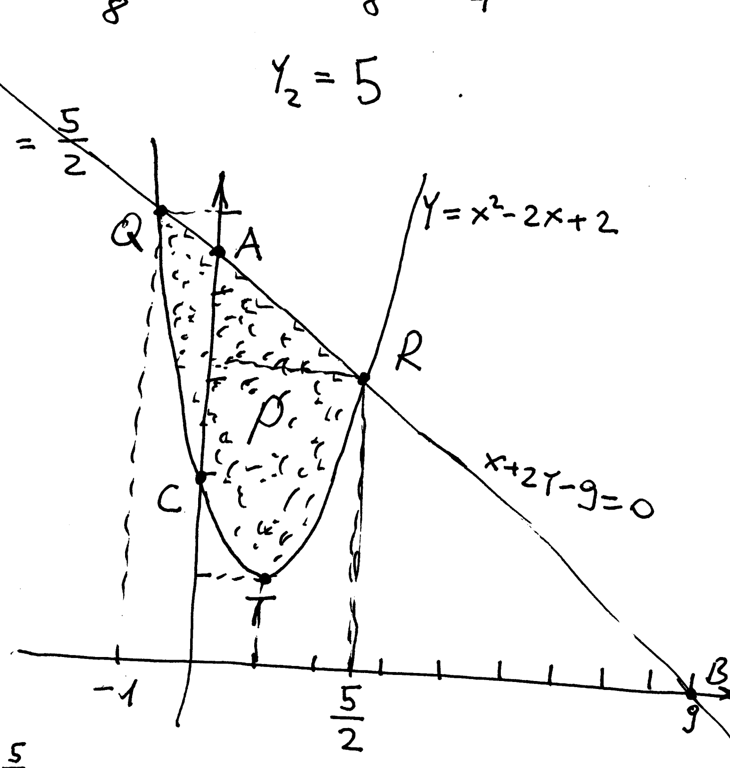
$$y_{1,2}=\frac{33\pm 7}{8} \qquad y_1=\frac{26}{8}=\frac{13}{4}$$

$$y_2=5$$

$$y_1=\frac{13}{4} \Rightarrow x=-2\cdot\frac{13}{4}+9=-\frac{13}{2}+\frac{18}{2}=\frac{5}{2}$$

$$y_2=5 \Rightarrow x=-2\cdot 5+9=-1$$

Tačke presjeka prave i parabole su $R\left(\frac{5}{2},\frac{13}{4}\right)$ i $Q(-1,5)$



$$P=\int_{-1}^{\frac{5}{2}}\left[\left(-\frac{1}{2}x+\frac{9}{2}\right)-(x^2-2x+2)\right]dx$$

$$\int_{-1}^{\frac{5}{2}}\left(-\frac{1}{2}x+\frac{9}{2}\right)dx=-\frac{1}{2}\cdot\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}}+\frac{9}{2}x\Big|_{-1}^{\frac{5}{2}}=-\frac{1}{4}\left(\frac{25}{4}-1\right)+\frac{9}{2}\left(\frac{5}{2}-(-1)\right)$$

$$=-\frac{1}{4}\cdot\frac{21}{4}+\frac{9}{2}\cdot\frac{7}{2}=\frac{231}{16}$$

$$\int_{-1}^{\frac{5}{2}}(x^2-2x+2)\,dx=\frac{x^3}{3}\Big|_{-1}^{\frac{5}{2}}-2\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}}+2x\Big|_{-1}^{\frac{5}{2}}=\frac{1}{3}\left(\frac{125}{8}-(-1)\right)-\left(\frac{25}{4}-1\right)+$$

$$+2\left(\frac{5}{2}-(-1)\right)=\frac{1}{3}\cdot\frac{133}{8}-\frac{21}{4}+2\cdot\frac{7}{2}=\frac{133}{24}+\frac{7}{4}=\frac{175}{24}$$

$$P=\frac{231}{16}-\frac{175}{24}=\frac{231}{4\cdot 4}-\frac{175}{6\cdot 4}=\frac{686}{96}=\frac{343}{48}$$

4. Izračunati površinu ravne figure koja je ograničena krivom $y^2=2x+1$ i pravom $y=2x-1$.

Rj. prava $y=2x-1$ prolazi kroz tačke $A(0,-1)$ i $B\left(\frac{1}{2},0\right)$.

$y^2 = 2x + 1$

$2x = y^2 - 1$

$x = \frac{1}{2}y^2 - \frac{1}{2}$

Kriva oblika $x = ay^2 + bx + c$ ima grafik u obliku parabole

⊃ ili ⊂

$a < 0$ $\qquad a > 0$

Tjeme krive se traži po formuli $T\left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$x = 0 \Rightarrow y^2 = 1$

$A(0,-1)$ i $B(0,1)$ su tačke presjeka f-je sa y-osom

$C(-\frac{1}{2}, 0)$ je nula f-je

$D = 1 \qquad -\frac{D}{4a} = -\frac{1}{4\cdot\frac{1}{2}}$

$-\frac{b}{2a} = -\frac{0}{2\cdot\frac{1}{2}} = 0$

$T(-\frac{1}{2}, 0)$ je tjeme parabole

Tražimo još tačke presjeka krive i prave

za $x = 0$

$\Downarrow$

$y = -1$

$y = 2x - 1$

$y^2 = 2x + 1$

$(2x-1)^2 = 2x + 1$

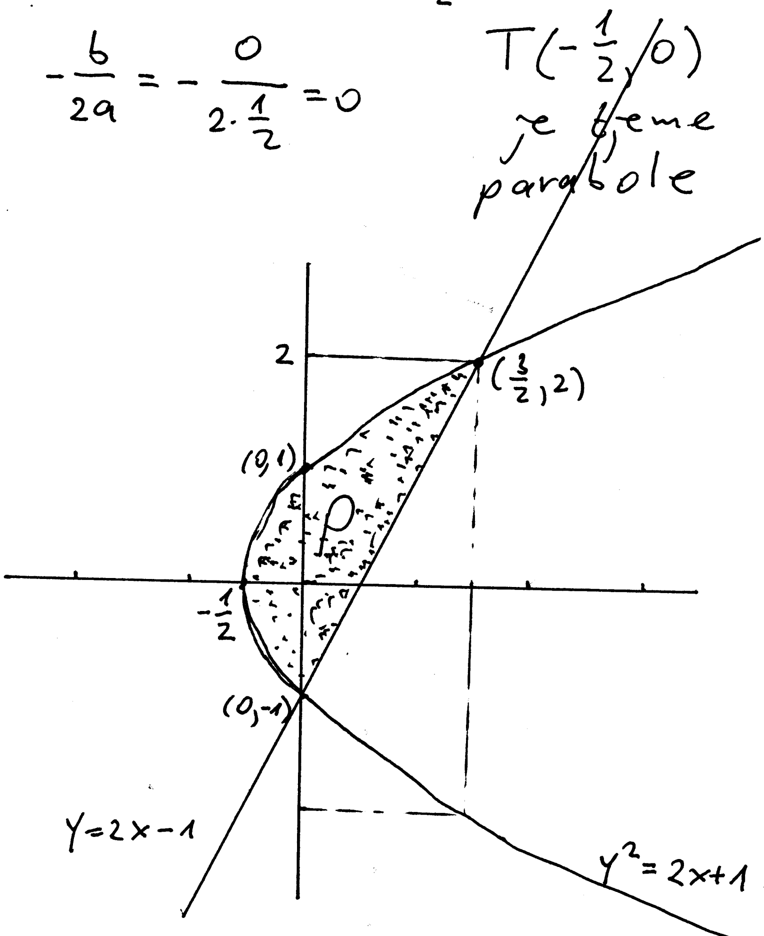
$4x^2 - 4x + 1 - 2x - 1 = 0$

$4x^2 - 6x = 0$

$2x(2x - 3) = 0$

$x = 0 \lor x = \frac{3}{2}$

$D(0,-1)$ i $E(\frac{3}{2}, 2)$ su tačke presjeka krive i prave



$y = 2x - 1 \Rightarrow x = \frac{1}{2}y + \frac{1}{2}$

$y^2 = 2x + 1 \Rightarrow x = \frac{1}{2}y^2 - \frac{1}{2}$

$$P = \int_{-1}^{2}\left[\frac{1}{2}y + \frac{1}{2} - \left(\frac{1}{2}y^2 - \frac{1}{2}\right)\right]dy = \frac{1}{2}\int_{-1}^{2}(y + 1 - y^2 + 1)\,dy = \frac{1}{2}\int_{-1}^{2}(-y^2 + y + 2)\,dy =$$

$$= \frac{1}{2}\cdot\left[\left(-\frac{y^3}{3}\right)\Big|_{-1}^{2} + \frac{y^2}{2}\Big|_{-1}^{2} + 2y\Big|_{-1}^{2}\right] = \frac{1}{2}\left[-\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2(2+1)\right] =$$

$$= \frac{1}{2}\left(-3 + \frac{3}{2} + 6\right) = \frac{1}{2}\cdot\frac{9}{2} = \frac{9}{4}$$ tražena površina

(#) Na parabolu $y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i y-osa.

Rj.

$y=1-x^2$

$y(0)=1$

$(0,1)$ je presjek sa y-osom

$1-x^2=0$

$x^2=1$

$x_{1,2}=\pm 1$

$(-1,0)$ i $(1,0)$ su nule f-je

$y=-x^2+1$

parabola izgleda

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$-\frac{b}{2a}=-\frac{0}{2\cdot(-1)}=0$

$D=0-4(-1)(1)=4$

$-\frac{D}{4a}=-\frac{4}{4\cdot(-1)}=1$

$T(0,1)$

$y-y_1=y'(x_1)(x-x_1)$

jednačina tangente u tački (x_1,y_1)

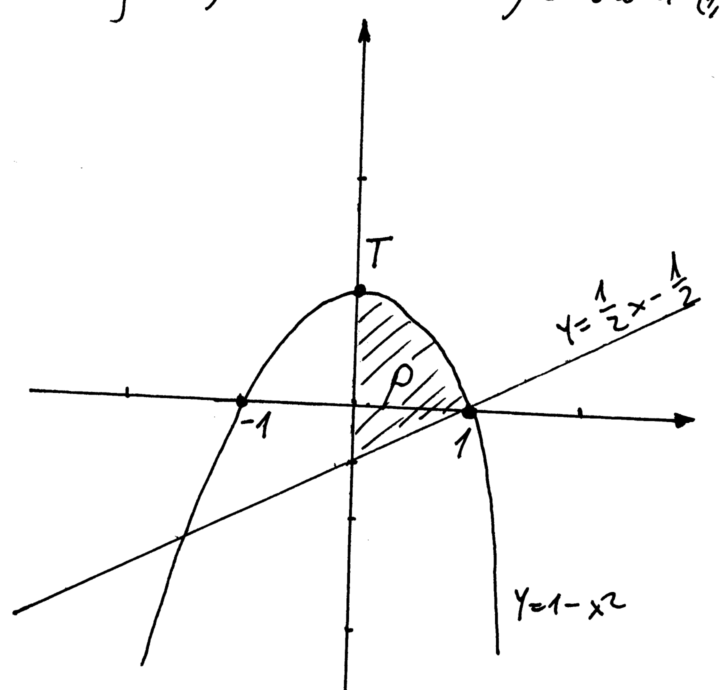
$y-y_1=-\frac{1}{y'(x_1)}(x-x_1)$ jednačina normale u tački (x_1,y_1)

$y'=-2x$ presjek parabole i pozitivnog dijela x-ose je tačka $(1,0)$

$y'(1)=-2$

$y-0=-\frac{1}{-2}(x-1)$

$y=\frac{1}{2}x-\frac{1}{2}$ jednačina normale u tački $(1,0)$



$$P=\int_0^1\left[(1-x^2)-\left(\frac{1}{2}x-\frac{1}{2}\right)\right]dx=$$

$$=\int_0^1\left(-x^2-\frac{1}{2}x+\frac{3}{2}\right)dx=-\frac{1}{3}x^3\Big|_0^1-\frac{1}{4}x^2\Big|_0^1+\frac{3}{2}x\Big|_0^1$$

$$=-\frac{1\cdot 4}{3\cdot 4}-\frac{1\cdot 3}{4\cdot 3}+\frac{3}{2}=\frac{3\cdot 6}{2\cdot 6}-\frac{7}{12}=\frac{18-7}{12}=\frac{11}{12}$$

$P=\frac{11}{12}$ tražena površina

(#) Izračunati površinu koju gradi kriva $Y=x^2+x-6$ zajedno sa svojim tangentama povučenim na tu krivu u nultačkama krive.

$k: \; y=x^2+x-6$

$D=1+24=25$

$Y=(x-2)(x+3)$

$x_1=2 \quad x_2=-3$

$(2,0)$ i $(-3,0)$ su nule f-je

$f(0)=-6$

$(0,-6)$ je tačka presjeka f-je sa y-osom

$Y'=2x+1$

$(-3,0), \; Y'(-3)=-5$

$Y-0=-5(x+3)$

$Y=-5x-15$ jednačina tangente na krivu Y u tački $(-3,0)$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ je tjeme f-je

$-\frac{b}{2a}=-\frac{1}{2}, \quad -\frac{D}{4a}=-\frac{25}{4}=-6\frac{1}{4}$

$T\left(-\frac{1}{2}, -6\frac{1}{4}\right)$

$a>0$

f-ja je $\cup$ oblika

$Y-Y_1=k(x-x_1)$ jednačina prave kroz tačku (x_1,Y_1) i koeficijentom k

u slučaju tangente $k=Y'(x_1)$

$(2,0), \; Y'(2)=5$

$Y-0=5(x-2)$

$Y=5x-10$ jednačina tangente na krivu Y u tački $(2,0)$

presjek pravih:

$Y=-5x-15 \quad (1)$

$Y=5x-10 \quad (2)$

$(1)+(2): \; 2Y=-25$

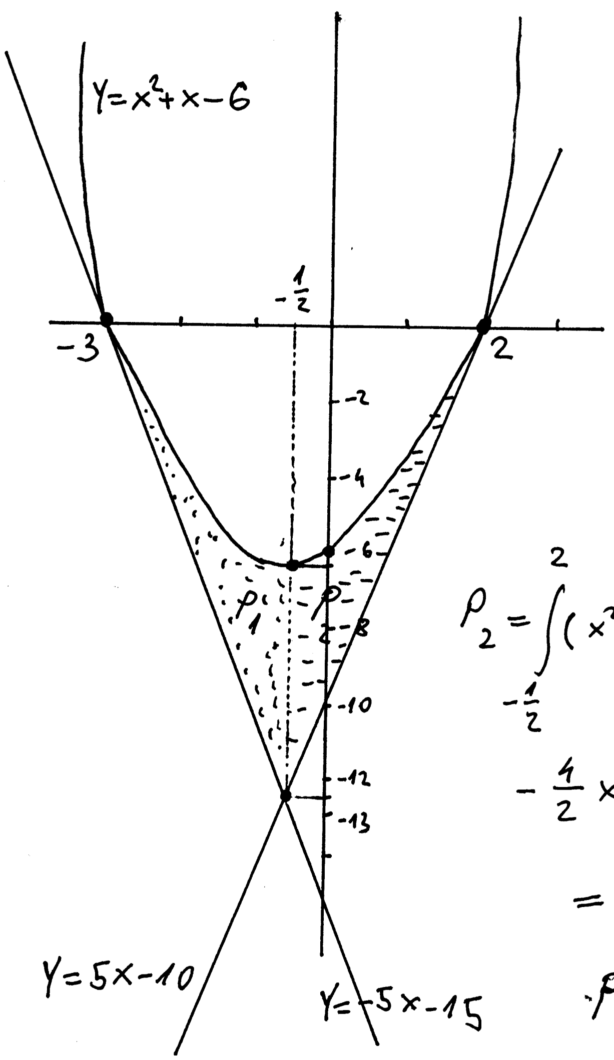
$Y=-\frac{25}{2}=-12\frac{1}{2}$

$(1)-(2): \; -10x-5=0$

$-10x=5$

$x=-\frac{1}{2}$

$\left(-\frac{1}{2}, -12\frac{1}{2}\right)$ je tačka presjeka pravih



$P=P_1+P_2$

$$P_1=\int_{-3}^{-\frac{1}{2}}\left(x^2+x-6-(-5x-15)\right)dx=\int_{-3}^{-\frac{1}{2}}(x^2+6x+9)dx=$$

$$=\frac{1}{3}x^3\Big|_{-3}^{-\frac{1}{2}}+\frac{6}{2}x^2\Big|_{-3}^{-\frac{1}{2}}+9x\Big|_{-3}^{-\frac{1}{2}}=\frac{1}{3}\left(-\frac{1}{8}+27\right)$$

$$+3\left(\frac{1}{4}-9\right)+9\left(-\frac{1}{2}+3\right)=\frac{1}{3}\cdot\frac{215}{8}+3\cdot\frac{-35}{4}$$

$$+9\cdot\frac{5}{2}=\frac{215}{24}-\frac{630}{24}+\frac{540}{24}=\frac{125}{24}$$

$$P_2=\int_{-\frac{1}{2}}^{2}\left(x^2+x-6-(5x-10)\right)dx=\int_{-\frac{1}{2}}^{2}(x^2-4x+4)\,dx=\frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{2}-$$

$$-\frac{4}{2}x^2\Big|_{-\frac{1}{2}}^{2}+4x\Big|_{-\frac{1}{2}}^{2}=\frac{1}{3}\left(8+\frac{1}{8}\right)-2\left(4-\frac{1}{4}\right)+4\left(2+\frac{1}{2}\right)=$$

$$=\frac{1}{3}\cdot\frac{65}{8}-2\cdot\frac{15}{4}+4\cdot\frac{5}{2}=\frac{65}{24}-\frac{180}{24}+\frac{240}{24}=\frac{125}{24}$$

$P=P_1+P_2=\frac{125}{24}+\frac{125}{24}=\frac{125}{12}$ tražena površina